Approximate Capacity of the Two-User MISO Broadcast Channel with Delayed CSIT

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Abstract—We consider the problem of the two-user multiple-input single-output complex Gaussian Broadcast Channel where the transmitter has access to delayed knowledge of the channel state information. We characterize the capacity region of this channel to within a constant number of bits for all values of the transmit power. The proposed signaling strategy utilizes the delayed knowledge of the channel state information and the previously transmitted signals, in order to create a signal of common interest for both receivers. This signal is the quantized version of the summation of the previously transmitted signals.

To guarantee the independence of quantization noise and signal, we extend the framework of lattice quantizers with dither, together with an interleaving step. For converse, we use the fact that the capacity region of this problem is upper-bounded by the capacity region of a physically degraded broadcast channel with no channel state information where one receiver has two antennas. We then derive an outer-bound on the capacity region of this degraded broadcast channel which in turn provides an outer-bound on the capacity region of the two-user multiple-input single-output complex Gaussian broadcast channel with delayed knowledge of the channel state information. By careful examination, we show that the achievable rate region and the outer-bound are within 1.81 bits/sec/Hz per user.

I. INTRODUCTION

In fast-fading scenarios, the coherence time of the channel is smaller than the delay of the feedback channel, and as a result, providing the transmitters with up-to-date channel state information is practically infeasible. Thus, we are left with no choice but to try and understand the behavior of wireless networks under such constraint.

In the context of multiple-input single-output (MISO) broadcast channels (BCs), it has been shown that even completely stale CSIT (also known as delayed CSIT) can still be very useful and can change the scale of the capacity, measured by the degrees of freedom (DoF) [1].

The impact of delayed CSIT has also been studied in wireless networks with distributed transmitters. This includes studying the DoF region of multiple-antenna interference channels (ICs) and X channels with delayed CSIT [2]–[6], the capacity region of two-user IC with binary fading with delayed CSIT [7]–[9], the DoF region of $K$-user Gaussian IC and X channels with delayed CSIT [5], [6], and multi-antenna two-user Gaussian IC with delayed CSIT and Shan-

non feedback [10], [11]. Also, in multi-hop networks with delayed CSIT, it is shown that the DoF can still scale with the number of users [12]. Moreover, researchers have considered variations of the assumption on the available CSIT at the transmitter (see [13]–[16]), these variations assume that in addition to the delayed CSIT, the transmitter has an imperfect estimate of the current channel realization.

The degrees of freedom by definition provides a first order approximation of the capacity. While DoF has been found useful in understanding the behavior of the capacity in high power regimes, it is not suitable for practical settings with finite signal-to-noise ratio (SNR). In this work, we consider the two-user MISO BC and focus on the impact of delayed CSIT at finite SNR regime, as opposed to the asymptotic DoF analysis. There are some prior works in the literature (for example [17]) that have proposed and analyzed several achievability strategies at finite SNR regime, however, characterizing the capacity region of the two-user MISO BC with delayed CSIT, even to within a constant gap, has been an open problem. We propose new inner-bound and outer-bound on the capacity region of this network that are to within 1.81 bits/sec/Hz per user of each other at any SNR. Hence, we obtain a constant-gap characterization of the capacity region of the two-user MISO BC with delayed CSIT.

The proposed achievability scheme has three phases, which are briefly explained here. In Phase 1 and Phase 2, the transmitter respectively sends messages intended for receivers one and two. In each of these phases, the unintended receiver overhears and saves some signal (interference), which is only useful for the other receiver. In the third phase, the transmitter will swap the overheard signals between the receivers. Note that at this time, the transmitter can evaluate the overheard signals using delayed CSIT. The swapping is performed exploiting the overheard signals as available side-information at receivers side. The overall information that each receiver collects in the three phases is enough to decode the original intended message.

The main idea of outer-bound is similar to [1], however, here our objective is capacity result rather than DoF characterization. We create a physically degraded BC by providing the received signal of user one to user two. Then, since we know that feedback does not enlarge the capacity region of a physically degraded BC [18], we ignore the delayed knowledge of the channel state information at the transmitter (i.e. no CSIT assumption). We derive an outer-bound on the capacity region for this channel. This outer-bound in turn provides an outer-bound on the capacity region of the two-
user MISO complex Gaussian BC with delayed CSIT. We can take similar steps and provide the received signal of user two to user one. The capacity region of the two-user multiple-input single-output complex Gaussian broadcast channel with delayed CSIT is included in the intersection of the outer-bounds we obtain. We show that the achievable rate region and the outer-bound are within 1.81 bits/sec/Hz per user for all values of the transmit power.

The rest of the paper is organized as follows. In Section [II] we formulate our problem. In Section [III] we present our main results. We describe our achievability strategy in Section [IV]. Section [V] is dedicated to deriving the outer-bound. In Section [VI] we show that our inner-bound and outer-bound are within a constant number of bits. Section [VII] concludes the paper and describes interesting future directions.

II. PROBLEM SETTING

We consider the two-user multiple-input single-output (MISO) complex Gaussian broadcast channel (BC) with Rayleigh fading as depicted in Fig. 1. The channel gains from the transmitter to receivers one and two are denoted by \( h[t], g[t] \in \mathbb{C}^{2 \times 1} \), respectively, where the entries of \( h[t] \) and \( g[t] \) are distributed i.i.d. across time, antenna, and users, and are distributed as \( \mathcal{CN}(0, 1) \). At each receiver, the received signal can be expressed as follows.

\[
\begin{align*}
y_1[t] &= h^\top[t] x[t] + z_1[t], \\
y_2[t] &= g^\top[t] x[t] + z_2[t],
\end{align*}
\]

where \( x[t] \in \mathbb{C}^{2 \times 1} \) is the transmit signal subject to power constraint \( P \), i.e., \( \mathbb{E} \left[ |x[t]|^2 \right] \leq P \) for \( P > 0 \). The noise processes are independent from the transmit signal and are distributed i.i.d. as \( z_k[t] \sim \mathcal{CN}(0, 1) \). Furthermore, we define

\[
\begin{align*}
s_1[t] &= h^\top[t] x[t], \\
s_2[t] &= g^\top[t] x[t],
\end{align*}
\]

to be the noiseless versions of the received signals.

![Two-user MISO Broadcast Channel](image)

The transmitter has access to the delayed (outdated) channel state information, meaning that at time instant \( t \), the transmitter has access to

\[
(h[t], g[t])_{t=1}^{n-1}, \quad t = 1, 2, \ldots, n.
\]

Due to the delayed knowledge of the channel state information, the encoded signal \( x[t] \) is a function of both the messages and the previous channel realizations.

Each receiver \( k, k = 1, 2 \), uses a decoding function \( \varphi_{k,n} \) to get the estimate \( \hat{w}_k \) from the channel outputs \( \{y_k[t] : t = 1, \ldots, n\} \). An error occurs whenever \( \hat{w}_k \neq w_k \). The average probability of error is given by

\[
\lambda_{k,n} = \mathbb{E}[P(\hat{w}_k \neq w_k)], \quad k = 1, 2,
\]

where the expectation is taken with respect to the random choice of the transmitted messages \( w_1 \) and \( w_2 \).

We say that a rate pair \( (R_1, R_2) \) is achievable, if there exists a block encoder at the transmitter, and a block decoder at each receiver, such that \( \lambda_{k,n} \) goes to zero as the block length \( n \) goes to infinity, \( k = 1, 2 \). The capacity region \( \mathcal{C} \) is the closure of the set of the achievable rate pairs.

III. STATEMENT OF MAIN RESULT

Our main contribution is the characterization of the capacity region of the two-user MISO complex Gaussian BC to within a constant number of bits. The achievability scheme has three phases. In Phase 1 and Phase 2, the transmitter respectively sends messages intended for receiver one and receiver two. In each of these phases, the unintended receiver overhears and saves some signal (interference), which is only useful for the other receiver. Later, the transmitter will provide these overheard signals to the intended receivers with some distortion during Phase 3. In fact, in the third phase, the transmitter evaluates what each receiver overheard about the other receiver’s message using the delayed knowledge of the channel state information and provides these overheard signals efficiently to both receivers exploiting available side information at each receiver.

The outer-bound is derived based on creating a physically degraded broadcast channel where one receiver is enhanced by having two antennas. In this channel, feedback and in particular delayed knowledge of the channel state information, does not increase the capacity region. Thus, we can ignore the delayed knowledge of the channel state information and consider a degraded BC with no CSIT. This would provide us with the outer-bound. Before stating our main result, we need to define what we mean by constant gap approximation of the capacity region.

**Definition 1:** For a region \( \mathcal{C}_0 \subseteq \mathbb{R}^2 \), we define

\[
\mathcal{C}_0 \ominus (\tau, \tau) \triangleq \{(R_1, R_2) \mid R_1, R_2 \geq 0, \ (R_1 + \tau, R_2 + \tau) \in \mathcal{C}_0 \}.
\]

**Definition 2:** The capacity region of the two-user MISO BC with delayed CSIT, \( \mathcal{C} \), is said to be within \( \tau \in \mathbb{R}^+ \) bits/sec/Hz per user of \( \mathcal{C}_0 \), if

\[
\mathcal{C}_0 \ominus (\tau, \tau) \subseteq \mathcal{C} \subseteq \mathcal{C}_0.
\]
In order to state our main result, we also need to define the following region.

**Definition 3**: Rate region $C_k$, $k = 1, 2$, is defined as

$$C_k = \{R_1, R_2 \geq 0 | R_k + 2R_{\bar{k}} \leq 2C_{2 \times 1}\} \quad k = 1, 2,$$

where $\bar{k} = 3 - k$, and

$$C_{2 \times 1} = \mathbb{E} \log_2 \left[1 + \frac{P}{2} g^H g\right],$$

where $g$ is a 2 by 1 vector where entries are i.i.d. $\mathcal{CN}(0, 1)$.

**Remark 1**: $C_{2 \times 1}$ is in fact the ergodic capacity of a complex Gaussian point-to-point channel with 2 transmit antennas and 1 receive antenna where only the receiver has access to the channel state information (no CSIT assumption).

This following Theorem 1 states our main contribution.

**Theorem 1**: The capacity region of the two-user MISO BC with delayed CSIT, $C$, is within 1.81 bits/sec/Hz per user of $C_2 = C_1 \cap C_2$ where $C_1$ and $C_2$ are defined in Definition 3.

Rest of the paper is dedicated to the proof of Theorem 1. In Section IV, we provide the achievability proof of Theorem 1. In particular, we describe how the transmitter can utilize the outdated channel state information and the previously transmitted signal to reduce future communication time. Then, we provide the converse proof in Section V.

**IV. ACHIEVABILITY PROOF OF THEOREM 1**

In this section, we describe the achievability strategy of Theorem 1. First, we need to introduce some notations.

As mentioned before, $C_{2 \times 1}$ denotes the ergodic capacity of a complex Gaussian point-to-point channel with average transmit power of $P$ and with 2 transmit antennas and 1 receive antenna where only the receiver has access to the channel state information. Furthermore, let $C_{2 \times 2}(D)$ denote the ergodic capacity of a complex Gaussian point-to-point channel with average transmit power of $P$ and with 2 transmit antennas and 2 receive antennas where only the receiver has access to the channel state information, moreover, the noise process at one receiver has variance 1 while the noise process at the other receiver has variance $(1 + D)$ for some positive constant $D$, see Fig. 3.

![Fig. 3](image3.png)

We are now ready to describe our achievable rate region. For parameter $D \geq 4$, and for $R_1, R_2 \in \mathbb{R}^+$, we show that a rate region $\mathcal{R}(D)$ given by

$$\mathcal{R}(D) = \left\{ R_1 + \frac{3C_{2 \times 1}(D) - 1}{2C_{2 \times 2}(D)} R_2 \leq C_{2 \times 1}, \right\}$$

for $k = 1, 2$, is achievable.

Later in Section VI, we show that for $D = 4$ the achievable rate region is within 1.81 bits/sec/Hz per user of the outer-bound, i.e. for any rate tuple $(R_1, R_2)$ on the boundary of it, $(R_1 + 1.81, R_2 + 1.81)$ is beyond the outer-bound stated in Theorem 1.

Fig. 3 depicts our achievable rate region $\mathcal{R}(D)$ for $D \geq 4$. The achievable rate region, $\mathcal{R}(D)$, is obtained by time sharing corner points $A, B$, and $C$. The corner points $B$ and $C$ of the rate region are achievable using the work of Telatar on the ergodic capacity of multiple-antenna point-to-point channels with no CSIT [19]. Therefore, we only need to describe the achievability strategy for corner point $A$.

**A. Transmission Strategy for Corner Point A**

Our achievability strategy is carried on over $n$ blocks, each block consisting of 3 phases. As we shall prove by the end of
this section, upon completion of all n blocks, each receiver will be able to decode all the messages intended for it with \( \lambda_{1,n}, \lambda_{2,n} \to 0 \) as \( n \to \infty \).

Denote by \( w_k^b \) the message of user \( k \) in block \( b \), \( k \in \{1,2\}, b = 1,2,\ldots,n \). We assume that \( w_k^b \in \{1,2,\ldots,2^{nR}\} \) and that the messages are distributed uniformly and independently.

- **Encoding:** At the transmitter, message \( w_k^b \) is mapped to a Gaussian codeword \( u_k^b \) picked from a codebook of size \( 2^{nR} \) where any element of this codebook is drawn i.i.d. from \( \mathcal{CN}(0,P/2I_2) \). Here, \( I_2 \) is the \( 2 \times 2 \) identity matrix.

We set

\[
R = C_{2\times2}(D) - \epsilon,
\]

where \( C_{2\times2}(D) \) is defined before and \( \epsilon \in \mathbb{R}^+ \).

We note that given message \( w_k^b \), \( u_k^b[t_1] \) and \( u_k^b[t_2] \) are not independent anymore, \( k = 1,2, \) \( t_1, t_2 = 1,2,\ldots,n, t_1 \neq t_2 \), which is an important property we require later in our transmission strategy. Hence, we create the transmit signal intended for user \( k \) during block \( b \) and at time instant \( t \), \( x_k^b[t] \), using interleaving as depicted in Fig. 5 according to the following mapping

\[
x_k^b[t] = u_k^b[b],
\]

where \( k = 1,2, b = 1,2,\ldots,n \) and \( t = 1,2,\ldots,n \). It is important to notice that with this interleaving, the transmit signals at different time instants of a given phase at a given block are independent from each other. This is due to the fact that these signals are created from independent messages.

We now describe the transmission during each phase of communication block \( b \).

- **Communication during Phase 1 of block \( b \):** During this phase, the transmitter communicates \( x_k^{b,n} \) from its two transmit antennas, \( j = 1,2 \). Receiver one obtains \( y_{1j}^{b,n} \) and receiver two obtains \( y_{2j}^{b,n} \).

- **Communication during Phase 3 of block \( b \):** Using the delayed CSIT, the transmitter has access to \( s_{21}^{b,n} \) which is the received signal at \( \text{Rx}_2 \) during Phase 1 of block \( b \), \( y_{21}^{b,n} \), minus the noise term, and \( s_{12}^{b,n} \) which is the received signal at \( \text{Rx}_1 \) during Phase 2 of block \( b \), \( y_{12}^{b,n} \), minus the noise term.

Transmitter then creates the summation of the two aforementioned signals, i.e.

\[
s_{21}^{b,n} + s_{12}^{b,n}.
\]

From (14), we conclude that we can quantize \( s_{21}^{b,n} + s_{12}^{b,n} \) with rate greater than or equal to

\[
\min_{D_2: \mathbb{E}[\sigma^2] \leq D} \mathbb{E} \left[ \log_2 \frac{\sigma^2}{D_2} \right] + \sigma^2,
\]

is achievable at distortion \( D \) (per sample), where the expectation is with respect to the distribution of \( \sigma^2 \).

It is easy to see that any rate than or equal to

\[
\mathbb{E} \left[ \log_2 \left( \frac{1 + \frac{\sigma^2}{D}}{D} \right) \right],
\]

is also achievable at distortion \( D \) (per sample). Basically, we have ignored the optimization over \( D \) and added a 1 to remove \( \max\{0,0\} \) (or \( + \)). This helps us avoid the hassle of the optimization involved in (13). In order to have a distortion that is independent of the signal and is uncorrelated across time, we can incorporate lattice quantization with “dither” as described in [20].

From (14), we conclude that we can quantize \( s_{21}^{b,n} + s_{12}^{b,n} \) with rate \( R_Q(D) \) (per sample), defined as

\[
R_Q(D) \triangleq \mathbb{E} \left[ \log_2 \left( 1 + \frac{P}{2D} \left( \|g\|^2 + \|h\|^2 \right) \right) \right],
\]

where the channel gains are distributed as described in Section III.

Transmitter encodes this quantized signal at rate \( R_Q(D) \) using the coding strategy of [19], and communicates it from the two transmit antennas during Phase 3. Phase 3 has

\[
\left[ \frac{nR_Q(D)}{2} - \delta \right]_{2\times1}
\]
time instants for some $\delta > 0$. Next, we need to show that given the appropriate choice of parameters, receivers can recover the corresponding messages with vanishing error probability as $n \to \infty$.

B. Decoding

In order to obtain a constant-gap approximation of the capacity region, we need the quantization rate $R_Q(D)$ to be less than or equal to $C_{2 \times 1}$. In Appendix I we show that for $D = 4$, we have $R_Q(D)/C_{2 \times 1} \leq 1$ for all values of $P$. However, we have plotted $R_Q(D)$ and $C_{2 \times 1}$ in Fig. 6 for $D = 3$ and as we can see, we have $R_Q(3)/C_{2 \times 1} \leq 1$.

Therefore, our choice of $D = 4$ is not the minimum value of $D$ to satisfy $R_Q(D)/C_{2 \times 1} \leq 1$. For the rest of this section, we assume $R_Q(D)/C_{2 \times 1} \leq 1$.

Upon completion of Phase 3 of block $b$, each receiver decodes the quantized signal. We know that as $\delta \to 0$ and $n \to \infty$, this could be done with arbitrary small decoding error probability. Therefore, each receiver has access to

$$s^{b,n}_{21} + s^{b,n}_{12} + z^b_Q,$$

where $z^b_Q$ is the quantization noise with variance $D$ which is independent of the transmit signals. Note that $z^b_Q[t_1]$ and $z^b_Q[t_2]$ are uncorrelated but not necessarily independent, $t_1, t_2 = 1, 2, \ldots, n, t_1 \neq t_2$.

Receiver 1 at the end of the $n^{th}$ communication block, reconstructs signals by reversing the interleaving procedure described above as depicted in Fig. 7 and removes $y^{b,n}_{12}$ to obtain

$$\tilde{y}^{b,n}_{21} = y^{b,n}_{21} + z^b_Q,$$

here $z^b_Q$ is the quantization noise with variance $D$ which is independent of the transmit signals. Moreover, $z^b_Q[t_1]$ and $z^b_Q[t_2]$ are independent, $t_1, t_2 = 1, 2, \ldots, n, t_1 \neq t_2$.

Note that since the messages are encoded at rate $C_{2 \times 2}(D) - \epsilon$ for $\epsilon > 0$, if receiver one has access to $y^{b,n}_{21}$ up to distortion $D$, it can recover $w^b_t$ with arbitrary small decoding error probability as $\epsilon \to 0$ and $n \to \infty$. Thus, from $y^{b,n}_{11}$ and $\tilde{y}^{b,n}_{21}$, receiver one can decode $w^b_t$, $b = 1, 2, \ldots, n$. Similar argument holds for receiver two.

An error may occur in either of the following steps: (1) if an error occurs in decoding message $w^b_k$ provided required signals to the receiver, $k = 1, 2$; (2) if an error occurs in quantizing $s^{b,n}_{21} + s^{b,n}_{12}$; and (3) if an error occurs in decoding $s^{b,n}_{21} + s^{b,n}_{12} + z^b_Q$ at either of the receivers, $b = 1, 2, \ldots, n$. The probability of each one of such errors decreases exponentially in $n$ (see [19], [21] and references therein). Using union bound and given that we have $O(n^2)$ terms, the total error probability goes to zero as $n \to \infty$.

C. Achievable Rate

Using the achievable strategy described above, provided that $R_Q(D)/C_{2 \times 1} \leq 1$, as $\epsilon, \delta \to 0$ and $n \to \infty$, we can achieve a (symmetric) sum-rate point of

$$(R_1, R_2) = \left(\frac{C_{2 \times 2}(D)}{2 + R_Q(D)/C_{2 \times 1}}, \frac{C_{2 \times 2}(D)}{2 + R_Q(D)/C_{2 \times 1}}\right).$$

(19)

Given $R_Q(D)/C_{2 \times 1} \leq 1$, a (symmetric) sum-rate point of

$$(R_1, R_2) = \left(\frac{C_{2 \times 2}(D)}{3}, \frac{C_{2 \times 2}(D)}{3}\right),$$

(20)

is achievable.

V. CONVERSE

The main idea behind the converse proof is to create two physically degraded BCs such that the capacity region of the two-user MISO BC with delayed CSIT, is included in the intersection of the capacity region of these two physically degraded BCs.

We create a new channel by providing $y^{n}_{21}$ to $Rx_1$. This channel is physically degraded and from [18], we know that feedback does not enlarge the capacity region. Therefore, we ignore the delayed knowledge of the channel state information at the transmitter (i.e. no CSIT assumption), see Fig. 8.

We derive an outer-bound on the capacity region of this new channel denoted by $C_1$. Similarly, we define $C_2$ by providing $y^{n}_{12}$ to $Rx_2$. We have $C \subseteq C_1 \cap C_2$. In what follows, we derive $C_1$ and the derivation of $C_2$ would be similar.

Suppose there exists encoders and decoders at the transmitter and receivers such that each message can be decoded
at its corresponding receiver with arbitrary small decoding error probability.

\[ n (R_1 + 2R_2 - 3\epsilon_n) \]

\[ \leq I (w_1; y_{11}^n, y_{12}^n | H^n, g^n) + I (w_2; y_{11}^n | H^n, g^n) \]

\[ + I (w_2; y_{12}^n | H^n, g^n) = h (y_{11}^n, y_{12}^n | w_2, H^n, g^n) - h (y_{11}^n, y_{12}^n | w_1, H^n, g^n) \]

\[ + h (y_{11}^n | H^n, g^n) - h (y_{11}^n | w_2, H^n, g^n) \]

\[ + h (y_{12}^n | H^n, g^n) - h (y_{12}^n | w_2, H^n, g^n) \]

\[ = h (y_{11}^n | H^n, g^n) - h (z_{11}^n | H^n, g^n) + h (y_{12}^n | H^n, g^n) - h (z_{12}^n | H^n, g^n) \]

\[ + h (y_{11}^n, y_{12}^n | w_2, H^n, g^n) - h (y_{11}^n | w_2, H^n, g^n) \]

\[ - h (y_{12}^n | w_2, H^n, g^n) \]

\[ \leq 2E \log_2 \left[ 1 + \frac{P}{2} g_1^2 \right] - I (y_{11}^n; y_{12}^n | H^n, g^n) \]

\[ \leq 2E \log_2 \left[ 1 + \frac{P}{2} g_1^2 \right], \quad (21) \]

where (a) follows from Fano’s inequality, the mutual independence of the messages and the channel realizations, and the fact that due to no CSIT assumption, we have

\[ h (w_2|y_{11}^n, H^n, g^n) \leq n\epsilon_n, \]

\[ h (w_2|y_{12}^n, H^n, g^n) \leq n\epsilon_n; \quad (22) \]

(b) holds since

\[ h (y_1^n | w_1, w_2, H^n, g^n) = h (y_1^n | w_1, w_2, x^n, H^n, g^n) = h (z_{11}^n | w_1, w_2, x^n, H^n, g^n) = h (z_{11}^n | H^n, g^n) + h (z_{12}^n | H^n, g^n); \quad (23) \]

(c) follows from the results in [19]; and (d) follows from fact that mutual information is always positive. Dividing both sides by \( n \) and letting \( n \to \infty \), we obtain the desired result. This completes the derivation of \( C_1 \). Similarly, we can derive \( C_2 \), and we have \( C \subseteq C_1 \cap C_2 \) which completes the converse proof for Theorem [1].

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**VI. GAP ANALYSIS**

In this section, we evaluate the gap between our achievable rate-region and the outer-bound. Since the inner-bound and outer-bound are defined by straight lines, the gap has its maximum value at the symmetric sum-rate point. Thus, we evaluate the gap between the inner-bound in (20), i.e.

\[ (R_1, R_2) = \left( \frac{C_{2\times 2}(D)}{3}, \frac{C_{2\times 2}(D)}{3} \right), \quad (24) \]

and the sum-rate outer-bound obtained from Theorem [1] i.e.

\[ (R_1, R_2) = \left( \frac{2C_{2\times 1}}{3}, \frac{2C_{2\times 1}}{3} \right). \quad (25) \]

A numerical evaluation of the gap between the sum-rate inner-bound and outer-bound is plotted in Fig. [9]. The gap between the two bounds results from the gap between \( C_{2\times 2}(D) \) and \( 2C_{2\times 1} \). We first study this gap.

![Fig. 9. Numerical evaluation of the gap between the sum-rate inner-bound and outer-bound.](image)

**Corollary 1:** Consider a MIMO point-to-point channel with 2 transmit antennas and 2 receive antennas as described in [19]. The only difference is that the additive noise at one antenna has variance 1 while the additive noise at the other antenna has variance \( (1 + D) \). The ergodic capacity of this channel, \( C_{2\times 2}(D) \), satisfies

\[ C_{2\times 2}(D) \geq \max \left\{ \log_2 \text{det} \left[ I_2 + \frac{P}{2} HH^T \right] \right\}. \quad (27) \]

The proof is provided at the top of the page in [26]. We have

\[ (a) \text{ holds since the right hand side is obtained by evaluating the mutual information between the input and output, for a complex Gaussian input with covariance matrix } E \{ xx^T \} = P/2I_2. \]

**Corollary I:** For the MIMO channel with 2 transmit antennas and 2 receive antennas as described in [19], the only difference is that the additive noise at one antenna has variance 1 while the additive noise at the other antenna has variance \( (1 + D) \). The ergodic capacity of this channel, \( C_{2\times 2}(D) \), satisfies

\[ C_{2\times 2}(D) \geq \frac{4C_{21}}{3} - \frac{2C_{2\times 2}(D)}{3}. \quad (28) \]

Therefore, the gap between the sum-rate inner-bound and outer-bound can be upper-bounded by

\[ \frac{4C_{21}}{3} - \frac{2C_{2\times 2}(D)}{3} \leq \frac{2(2C_{2\times 1} - C_{2\times 2} + \log_2 (1 + D))}{3}. \]
For $P \leq 2$, the sum-rate outer-bound is smaller than 2 bits (smaller than the gap itself). So, we assume $P > 2$. We have

$$2C_{2 \times 1} - C_{2 \times 2} = 2\mathbb{E} \log_2 \left( 1 + \frac{P}{2} \mathbf{g}^\dagger \mathbf{g} \right) - \mathbb{E} \log_2 \left( \mathbf{I}_2 + \frac{P}{2} \mathbf{H} \mathbf{H}^\dagger \right)$$

$$= 2\mathbb{E} \log_2 \left( \frac{2}{P} \mathbf{I}_2 + \mathbf{g}^\dagger \mathbf{g} \right) - \mathbb{E} \log_2 \left( \frac{2}{P} \mathbf{I}_2 + \mathbf{H} \mathbf{H}^\dagger \right)$$

$$= \mathbb{E} \log_2 \left( \frac{2}{P} \mathbf{I}_2 + \mathbf{g}^\dagger \mathbf{g} \right) - \mathbb{E} \log_2 \left[ \mathbf{H} \mathbf{H}^\dagger \right].$$

Thus, we have

$$\frac{4C_{2 \times 1}}{3} - \frac{2C_{2 \times 2}(D)}{3} \leq \frac{2}{3} \left( 2\mathbb{E} \log_2 \left[ 1 + \mathbf{g}^\dagger \mathbf{g} \right] \right) + \mathbb{E} \log_2 \left( 1 + (1 + D) \right) \approx 3.62,$$

or 1.81 bits per user. Note that there is no closed form solution for the expectations in the inequality above, see [19].

We mention that in [30], we have shown the gap is a constant independent of power $P$. However, we could use numerical analysis of the gap to see that the sum-rate inner-bound and outer-bound are at most 2.6 bits (or 1.3) away from each other, see Fig. 9.

VII. CONCLUSION AND FUTURE DIRECTIONS

We studied the capacity region of the two-user multiple-input single-output complex Gaussian Broadcast Channel with delayed CSIT. We described our transmission strategy for finite SNR regime that incorporates delayed CSIT to obtain an inner-bound which is within a constant number of bits of the outer-bound. Currently, the gap is 1.81 bits/sec/Hz/user. When talking about constant gap approximation of the capacity region of wireless networks, from a practical point of view, it is very important that the gap is small. Thus, one future direction would be to improve the outer-bound and the inner-bound to tighten the gap.

This result can be extended to the case of $K$-user MISO Gaussian Broadcast Channel with delayed CSIT. An important consideration there would be the issue of noise accumulation. Another interesting direction is to consider a two-user MISO BC with delayed CSIT where the noise processes and the channel gains are not distributed as i.i.d. random variables. For example, consider the scenario where the noise processes have different variances. This model captures the scenario where users are located at different distances to the base station. For this setting, even the (generalized) DoF region is not known.

APPENDIX I

DETERMINING $D$ SUCH THAT $R_Q(D)/C_{2 \times 1} \leq 1$

As mentioned in Section VI, we are interested in $P > 2$. Using Jensen’s inequality, we have

$$R_Q(D) = \mathbb{E} \left[ \log_2 \left( 1 + \frac{D}{2P} \left( ||\mathbf{g}||^2_2 + ||\mathbf{h}||^2_2 \right) \right) \right]$$

$$\leq \log_2 \left( 1 + \frac{D}{2P} \mathbb{E} \left[ ||\mathbf{g}||^2_2 + ||\mathbf{h}||^2_2 \right] \right)$$

$$= \log_2 \left( 1 + \frac{D}{2P} \right).$$

Moreover, from [19], we have

$$C_{2 \times 1} = \int_0^\infty \log_2 \left( 1 + P\lambda /2 \right) \lambda e^{-\lambda} d\lambda$$

$$= \int_0^1 \log_2 \left( 1 + P\lambda /2 \right) \lambda e^{-\lambda} d\lambda$$

$$+ \int_1^\infty \log_2 \left( 1 + P\lambda /2 \right) \lambda e^{-\lambda} d\lambda$$

$$= \sum_{m=1}^\infty \int_{2^{m-1}}^{2^m} \log_2 \left( 1 + P\lambda /2 \right) \lambda e^{-\lambda} d\lambda$$
and
\[
\sum_{j=1}^{45} \int_{2^j+1(1-j-1)}^{2^{j+1}+1} \log_2 \left( 1 + \frac{P \lambda_2}{2} \right) \lambda e^{-\lambda} d\lambda
\]
\[
\geq \log_2 \left( 1 + \frac{P \lambda_2}{2} \right) \int_0^1 \lambda e^{-\lambda} d\lambda - \sum_{m=1}^{\infty} m \int_{2^{1-m}}^{2^{-m}} \lambda e^{-\lambda} d\lambda
\]
\[
\geq \log_2 \left( 1 + \frac{P \lambda_2}{2} \right) \int_0^1 \lambda e^{-\lambda} d\lambda
\]
\[
+ \log_2 \left( 1 + \frac{P \lambda_2}{2} \right) \int_{2^{m}}^{2^{m+1}} \lambda e^{-\lambda} d\lambda
\]
\[
\geq \log_2 \left( 1 + \frac{P \lambda_2}{2} \right) \int_0^1 \lambda e^{-\lambda} d\lambda
\]
\[
+ \log_2 \left( 1 + \frac{P \lambda_2}{2} \right) \int_{2^{m}}^{2^{m+1}} \lambda e^{-\lambda} d\lambda
\]
\[
\geq \log_2 \left( 1 + \frac{P \lambda_2}{2} \right) \int_0^1 \lambda e^{-\lambda} d\lambda
\]
\[
+ \log_2 \left( 1 + \frac{P \lambda_2}{2} \right) \int_{2^{m}}^{2^{m+1}} \lambda e^{-\lambda} d\lambda
\]
\[
> \log_2 \left( 1 + \frac{P \lambda_2}{2} \right) \int_0^\infty \lambda e^{-\lambda} d\lambda
\]
\[
= \log_2 \left( 1 + \frac{P \lambda_2}{2} \right) .
\]
where (a) holds since
\[
\sum_{m=1}^{\infty} m \int_{2^{1-m}}^{2^{-m}} \log_2 \left( 1 + \frac{P \lambda_2}{2} \right) \lambda e^{-\lambda} d\lambda
\]
\[
\geq \sum_{m=1}^{\infty} m \int_{2^{1-m}}^{2^{-m}} \log_2 \left( 1 + \frac{P \lambda_2}{2} \right) \lambda e^{-\lambda} d\lambda
\]
\[
\geq \sum_{m=1}^{\infty} m \int_{2^{1-m}}^{2^{-m}} \log_2 \left( 1 + \frac{P \lambda_2}{2} \right) - \log_2 \left( 2^m \right) \lambda e^{-\lambda} d\lambda
\]
\[
= \log_2 \left( 1 + \frac{P \lambda_2}{2} \right) \int_0^1 \lambda e^{-\lambda} d\lambda - \sum_{m=1}^{\infty} m \int_{2^{1-m}}^{2^{-m}} \lambda e^{-\lambda} d\lambda,
\]
and \(\sum_{m=1}^{\infty} m \int_{2^{1-m}}^{2^{-m}} \lambda e^{-\lambda} d\lambda\) converges since
\[
\left\{ m \int_{2^{1-m}}^{2^{-m}} \lambda e^{-\lambda} d\lambda \right\}_{m=1}^{\infty}.
\]
is a Cauchy sequence.
Thus, for \(D = 4\), we have \(R_0(D)/C_{2 \times 1} < 1\).

**REFERENCES**


