An iterative scheme for a class of fractional optimal control problems

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Abstract. This paper presents a hybrid scheme based on Dinkelbach approach and wavelet collocation method to extract approximate solutions of fractional optimal control problems (FOCP)’s. First Dinkelbach approach is considered to linearize the problem, then it is tried by combination of collocation wavelet approach and a numerical scheme of solving nonlinear equations, an iterative approach be proposed to obtain approximate optimal trajectory and control functions. Finally, numerical examples are listed to show the efficiency of the given approach.

Keywords: Optimal control; Fractional programming; Haar wavelet; Nonlinear programming.

Mathematics Subject Classification 2010: 49M30, 49M37.

1 Introduction

Recently, optimal control problem and its various branches has been much attention. FOCP as a form of such problems in various classes has been studied. FOCP’s in a particular form where the objective functional is given by the ratio of two integrals, is considered by Stancu-Minasian [11], Bhatt [1], in a more general framework, by Miele [7] and on problems with affine integrands and linear dynamics with respect to state and control by Bykadorov et. al. [2]. Despite the simplicity of these problems, the standard optimal control theory cannot directly be used to solve them. Stancu-Minasian [11] suggested to face the general fractional optimal control problem applying the Dinkelbachs method [4, 8], which is used in fractional programming to remove the denominator in the objective function.
Among the direct and indirect numerical methods of solving optimal control problems the efficiency of wavelet collocation methods for achievement to approximate solutions is very salient. Because unlike other numerical approaches, this method can present good approximate solutions for an extensive category of optimal control problems as time-optimal problems, problems with unknown initial and final conditions and etc, [3]. This approach uses the preliminaries of the Haar wavelets theory and properties including the Haar wavelets basis and its integral operational matrix. In this paper we combine wavelet collocation technique and Dinkelbach approach to create an iterative scheme for solving a class of fractional optimal control problem (see [11]),

\[(FP) \quad \max \frac{\int_0^T f(t, x(t), u(t))dt}{\int_0^T g(t, x(t), u(t))dt}, \quad (1.1)\]

subject to
\[\dot{x} = h(t, x(t), u(t)), \quad (1.2)\]
with boundary conditions
\[x(0) = x_0, \quad x(T) = x_T, \quad (1.3)\]
where \(x(t) \in \mathbb{R}^n, u(t) \in U \subset \mathbb{R}^m, \ t \in [0, T]\) and it is assumed that the denominator of the fraction is strictly positive, i.e.
\[\int_0^T g(t, x(t), u(t))dt > 0, \quad \text{for all} \ u(t) \in U.\]

2 Dinkelbach approach

**Definition 2.1.** Pair \(p = (x(\cdot), u(\cdot))\) is admissible if the following conditions hold:
- \(i)\) The state function \(x(\cdot) \in \mathbb{R}^n\) be differentiable on \([0, T]\).
- \(ii)\) The control function \(u(\cdot)\) takes its value in the compact set \(U \subset \mathbb{R}^m\) and their components are measurable on \([0, T]\).
- \(iii)\) The pair \(p\) satisfies the differential equation (1.2) on \([0, T]\) and the boundary conditions (1.3).

Assume set of all admissible pairs denoted by \(\mathcal{P}\) and it is nonempty. The Dinkelbach approach consists in transforming the original fractional problem into an equivalent family of parametric optimal control problems, which become nonfractional. More precisely, let us define an auxiliary function \(F\) such that, for each \(q \in \mathbb{R}\), \(F(q)\) is the maximum value of the following (nonfractional) optimal control problem:

\[(P_q) \quad \max_{(x,u) \in \mathcal{P}} \left\{ \int_0^T \zeta_q(t, x(t), u(t))dt \right\}, \quad (2.1)\]
where
\[ q(t, x(t), u(t)) = f(t, x(t), u(t)) - qg(t, x(t), u(t)). \]

If for each \( q \in \mathbb{R} \) there exists an optimal control function \( u(t) \in U \) for problem \( P_{q_i} \), then the function \( F \) is strictly decreasing and convex and has a (unique) zero \( q^* \) (see [11]).
The nice property that relates the original fractional optimal control problem (FP) with the family of auxiliary problems \( P_{q_i} \), is that, if \( F(q^*) = 0 \), then \( q^* \) is the optimal value of (FP) and the optimal control and the optimal trajectory of \( P_{q} \) are optimal also for problem (FP) (see [11], Theorem 4.6.1, p. 157). It follows that solving problem (FP) is equivalent to determining the root of the equation \( F(q) = 0 \).

3 Iterative scheme using wavelet collocation method

The interesting property of the function \( F \) is an intensive motivation to create numerical iterative scheme to obtain near optimal admissible pair of FOCP’s. For this purpose we only need to consider two different parameters \( q_l \) and \( q_u \), where the product of \( F(q_l) \) and \( F(q_u) \) is negative, to establish an interval restriction method, as bisection or false position method, for producing a sequence \( \{q_i\} \) which is convergent to the root of equation \( F(q) = 0 \). To obtain the value of function \( F \) for each \( q \in \mathbb{R} \) we need to solve the optimal control problem (2.1). For this purpose, it is tried to use the wavelet collocation method as a numerical scheme for detecting the approximate solution of (2.1) for each \( q \).

By a simple linear transformation, the interval \([0, T]\) can be converted to the interval \([0, 1]\). The continuous solution to a problem will be represented by state and control variables in terms of Haar series and its operational matrix to satisfy the differential equations. The standard interval considered here is denoted as \( \tau \in [0, 1] \) with collocation points \( \tau_k \) set as
\[ \tau_k = (k - 0.5)/M, \quad k = 1, \ldots, M, \]
where \( M \) is the number of nodes used in the discretization and also is the maximum wavelet index number. Note that the magnitude of \( M \) is in the power of 2, so that the number of collocation points is also increasing in that power. All the collocation points are equally distributed over the entire time interval \([0, 1]\) with \( 1/M \) as the time distance to adjacent nodes. We assume that the derivative of the state variables \( \dot{x}(\tau) \) and control variables \( u(\tau) \) can be approximated by Haar wavelets with \( M \) collocation points, i.e.,
\[ \dot{x}(\tau) \approx C^T_x \Psi_M(\tau), \]
\[ u(\tau) \approx C^T_u \Psi_M(\tau), \]
where \( \Psi_M \)'s are vector of orthogonal Haar wavelet [3] and
\[ C^T_x = [C_{x1}, C_{x2}, \ldots, C_{xM}], C^T_u = [C_{u1}, C_{u2}, \ldots, C_{uM}]. \]
By using the Haar operational integration matrix, where we denote by $P$ [5], the state variables $x(\tau)$ can be expressed as

$$x(\tau) = \int_0^\tau \dot{x}(\tau')d\tau' + x_0 = \int_0^\tau C_x^T \Psi_M(\tau')d\tau' + x_0 = C_x^T P \Psi_M(\tau) + x_0.$$ 

The expansion of the matrix $\Psi_M(\tau)$ at the $M$ collocation points will yield the $M \times M$ Haar matrix $H = [\Psi_M(\tau_1), \Psi_M(\tau_2), \ldots, \Psi_M(\tau_M)]$. It follows that

$$\dot{x}(\tau_k) = C_x^T \Psi_M(\tau_k), \quad u(\tau_k) = C_u^T \Psi_M(\tau_k), \quad x(\tau_k) = C_x^T P \Psi_M(\tau_k) + x_0, \quad k = 1, \ldots, M. \quad (3.1)$$

From the above expression, we can evaluate the variables at any collocation point by using the product of its coefficients vector and the corresponding column vector in the Haar matrix.

When the Haar collocation method is applied in the optimal control problems, the nonlinear programming variables can be set as the unknown coefficients vector of the derivative of the state variables and control variables together with final time, that is,

$$\mathcal{X} = [C_{x1}, C_{x2}, \ldots, C_{xM}, C_{u1}, C_{u2}, \ldots, C_{uM}, T].$$

The objective function in (4) is then restated as

$$T \int_0^1 \zeta_q(\tau, (C_x^T P \Psi_M(\tau) + x_0), C_u^T \Psi_M(\tau))d\tau.$$ 

Since the Haar wavelets are expected to be constant steps at each time interval, the above equation can be simplified as

$$J_q^* = J_q(\mathcal{X}) = \frac{T}{M} \sum_{k=1}^M \zeta_q(\tau_k, (C_x^T P \Psi_M(\tau_k) + x_0), C_u^T \Psi_M(\tau_k)). \quad (3.2)$$

Substituting $\dot{x}, u$ and $x$ in (1.2) with the Haar wavelets expression (3.1) separately yields

$$C_x^T \Psi_M(\tau_k) = Th(\tau_k, (C_x^T P \Psi_M(\tau_k) + x_0), C_u^T \Psi_M(\tau_k)). \quad (3.3)$$

The system equation constraints are all treated as nonlinear constraints in nonlinear programming solver. The boundary constraints need to be paid more attention. Since the first and last collocation points are not set as the initial and final time, the initial and final state variables are calculated according to

$$x_0 = x(1) - \dot{x}(1)/2M, \quad x_T = x(M) + \dot{x}(M)/2M. \quad (3.4)$$

In this way, the FOCP’s are transformed into nonlinear programming problems in a structured form.
4 Algorithm of the approach

It should be note that, we need to obtain the root of equation \( F(q) = 0 \), thus we consider the optimal objective value of problem (3.2)-(3.4) i.e. \( J_q^* \) as an approximate value for \( F(q) \). An basis of the above discussions, we present an algorithm in two stages, initialization step and main steps.

**Initialization step:**
Determine \( \varepsilon > 0 \) and an initial interval \([q_l, q_u]\) where \( J_{q_l}^* < J_{q_u}^* < 0 \) and put \( \text{iter} = 1 \).

**Main steps:**
1. Compute \( J_{q_m}^* \). If \( |J_{q_m}^*| < \varepsilon \) or \( \text{iter} > \log_2 \frac{q_u - q_l}{\varepsilon} \) stop, otherwise goto step 3.
2. If \( J_{q_m}^* < 0 \) then \( q_l = q_m \) otherwise \( q_u = q_m \), put \( \text{iter} = \text{iter} + 1 \) and goto Step 1.

5 Numerical examples

In this section, we present two numerical examples for deliberation of the proposed algorithm efficiency.

**Example 5.1.** Consider the problem
\[
\begin{align*}
\max & \quad x(1) + 3 \int_0^1 (x(t) + 0.1)dt \frac{1 + \int_0^1 u(t)dt}{1}, \\
\text{s.t.} & \quad \dot{x} = -4x(t) + 2u(t)), \\
& \quad x(0) = 0.1, \\
& \quad u \in [0, 30],
\end{align*}
\]

where one can find the discussion on the analytical solution of this problem in [2]. Equivalently we try to obtain the solution of equation \( F(q) = 0 \) where

\[
F(q) = x(1) + 3 \int_0^1 (x(t) + 0.1)dt - q(1 + \int_0^1 u(t)dt).
\]

Considering \( M = 8 \), the result of applying iterative approach of the given algorithm can be seen in Table 1.

The exact and approximate optimal control and state functions can be seen in Fig. 1 and Fig. 2, respectively.

**Example 5.2.** Consider the following nonlinear FOCP
\[
\begin{align*}
\max & \quad \frac{\int_0^1 [1 - (x(t) - e^t)^2 - (u(t) - t)^2]dt}{\int_0^1 [(tx(t) - e^t)^2 + 1]dt}, \\
\text{s.t.} & \quad \dot{x} = x(t)[u(t)^2 e^t - t^2 x(t) + 1], \\
& \quad x(0) = 1, \\
& \quad u \in [0, 1],
\end{align*}
\]
Table 1: Results of applying iterative approach for Example 5.1.

<table>
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<tr>
<th>Iteration</th>
<th>( q_l )</th>
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<th>( q_m )</th>
<th>( J^*_m )</th>
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Figure 1: Exact and approximate optimal controls for Example 5.1.

where exact optimal trajectory and control functions are \( x(t) = e^t \) and \( u(t) = t \), respectively. Equivalently we try to obtain the solution of equation \( F(q) = 0 \) where

\[
F(q) = \int_0^1 \left( [1 - (x(t) - e^t)^2 - (u(t) - t)^2] - q[(tx(t) - e^t)^2 + 1] \right) dt.
\]

In Table 2 the results of applying iterative approach for this FOCP is shown where the number of collocation point is considered as \( M = 4 \). The exact and approximate optimal control and state functions can be seen in Fig. 3 and Fig. 4, respectively.
6 Conclusions

In this paper a hybrid scheme is presented to obtain approximate solutions for a class of fractional optimal control problems. By using the benefits of Dinkelbach approach and applying a successful collocation technique an algorithm for extracting near optimal numerical solutions is given. It seems that can be claimed to choose a large number of collocation points may be given rise to optimal exact solution.

Table 2: Results of applying iterative approach for Example 5.2.

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![Graph](image)

Figure 4: Exact and approximate optimal trajectories for Example 5.2.

References