POLYTOPIC INVARIANT VERIFICATION AND SYNTHESIS FOR POLYNOMIAL DYNAMICAL SYSTEMS VIA LINEAR PROGRAMMING

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Abstract. This paper deals with the verification and the synthesis of polytopic invariant sets for polynomial dynamical systems. An invariant set of a dynamical system is a subset of the state space such that if the state of the system belongs to the set at a given instant, it will remain in the set forever in the future. Polytopic invariants can be verified by solving a set of optimization problems involving multivariate polynomials on bounded polytopes. Using the blossoming principle for polynomials together with properties of multi-affine functions and Lagrangian duality, we show that certified lower bounds of the optimal values of such optimization problems can be computed effectively using linear programs. This allows us to propose a method based on linear programming for verifying polytopic invariant sets of polynomial dynamical systems. Additionally, we show that using sensitivity analysis of linear programs, we can synthesize a polytopic invariant set. Finally, we show using a set of examples borrowed from engineering or biological applications, that our approach is effective in practice.

1. Introduction

An invariant set of a dynamical system is a subset of the state space such that if the state of the system belongs to the set at a given instant, it will remain in the set forever in the future. Invariant sets have played an important role in control theory, e.g. for the analysis of performance, robustness or practical stability (see [Bla99] for a survey). They are also of great interest in the field of formal analysis of dynamical systems, especially for verification of safety properties. In such problems, the goal is to prove that trajectories of a dynamical system starting from a given set of initial states will never reach a specified set of unsafe states. Direct approaches compute certified over-approximations of the set of reachable states (see [DS09, DLM09, NNY09] for the most recent advances on nonlinear systems). Despite the recent progress from the point of view of computational complexity, these approaches can usually only certify that the set of unsafe states will not be reached on a bounded time interval. Indirect approaches consist in exhibiting an invariant set, containing the set of initial states, and whose intersection with the set of unsafe states is empty. For polynomial dynamical systems, these invariants are usually given by semi-algebraic sets [PJP07, San10] or for specific subclasses such as multi-affine or quasi multi-affine systems by rectangles [BH06, ATS09].

In this paper, we deal with verification and synthesis of polytopic invariant sets of polynomial dynamical systems. Let us remark that rectangles form a subclass of polytopic invariants and in some sense, our work extends the work of [BH06, ATS09]. Also, polytopes form a subclass of semi-algebraic sets and therefore there is an opportunity to design specific algorithmic procedures that may be more efficient and effective than those presented in [PJP07, San10]. More precisely, we shall consider a dynamical system of the form:

\[ \dot{x}(t) = f(x(t)), \quad x(t) \in \mathbb{R}^n \]

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where \( f : \mathbb{R}^n \to \mathbb{R}^n \) is a polynomial vector field. Given a bounded polytope \( P \) with a set of facets \( \{ F_k \mid k \in K \} \), it follows from the standard characterization of invariant sets (see e.g. [Aub91]) that \( P \) is invariant for the dynamical system (1.1) if and only if
\[
(1.2) \quad \forall k \in K, \forall x \in F_k, a_k \cdot f(x) \leq 0
\]
where \( a_k \) is the normal vector to \( F_k \) pointing outside \( P \). As pointed out in [ATS09] and by application of Tarski’s Theorem [Tar48], this a decidable problem. However, the complexity of the decision procedure gives little hope for practical application. Let us remark that (1.2) can be reformulated as follows:
\[
(1.3) \quad \forall k \in K, \min_{x \in F_k} -a_k \cdot f(x) \geq 0.
\]
This consists in showing that the minimal values of the multivariate polynomials \(-a_k \cdot f\) on the bounded polytopes \( F_k \) are positive. Hence, if we are able to compute non-negative certified lower bounds of these minimal values, it is sufficient to prove that the polytope \( P \) is invariant for the dynamical system (1.1). This is the approach followed in this paper.

The paper is organized as follows. In section 2, we show that a certified lower bound of the minimal value of a multivariate polynomial on bounded polytopes can be computed effectively by solving a simple linear program. Using the blossoming principle [Ram89], we first show that the problem of optimizing a multivariate polynomial on a bounded polytope can be translated to an equivalent problem of optimizing a multi-affine function on a bounded polytope. Then, using Lagrangian duality, we show that a lower bound of the optimal value of this latter problem can be computed using linear programming. Our approach is conservative (the lower bound is not tight) but it is effective and may be sufficient for proving (1.3). In section 3 we apply our approach for the verification of polytopic invariant sets for polynomial dynamical systems. Additionally, we show that using sensitivity analysis of linear programs, it allows us to synthesize a polytopic invariant set. Finally, we show using a set of examples borrowed from engineering or biological applications, that our approach is effective in practice.

2. Linear Relaxations for Optimization of Polynomials on Polytopes

As stated in the introduction, the verification of polytopic invariants for polynomial dynamical systems can be handled by solving a set of problems of optimization of multivariate polynomials on bounded polytopes. Therefore, in this section, we consider the following problem:
\[
\begin{align*}
\text{minimize} & \quad p(x) \\
\text{over} & \quad x \in \mathbb{R}, \\
\text{subject to} & \quad a_i \cdot x \leq b_i, \quad i \in I, \\
& \quad c_j \cdot x = d_j, \quad j \in J.
\end{align*}
\]
where \( p : \mathbb{R}^n \to \mathbb{R} \) is a multivariate polynomial, \( R = [x_1, \overline{x_1}] \times \cdots \times [x_n, \overline{x_n}] \), with \( x_k < \overline{x_k} \) for all \( k \in \{1, \ldots, n\} \), is a rectangle of \( \mathbb{R}^n \); \( I = \{1, \ldots, m_I\} \) and \( J = \{1, \ldots, m_J\} \) are sets of indices; \( a_i \in \mathbb{R}^n \), \( b_i \in \mathbb{R} \), for all \( i \in I \) and \( c_j \in \mathbb{R}^n \), \( d_j \in \mathbb{R} \), for all \( j \in J \). Let us remark that even though the polytope defined by the constraints indexed by \( I \) and \( J \) is unbounded in \( \mathbb{R}^n \), the fact that we consider \( x \in R \) which is a bounded rectangle of \( \mathbb{R}^n \) results in an optimization problem on a bounded (not necessarily full dimensional) polytope of \( \mathbb{R}^n \). We will assume that the problem is feasible: there exists \( x \in R \) satisfying all the constraints.

As the function \( p \) is usually non-convex, this may be a non-trivial problem to solve. Let us remark that as far as verification of polytopic invariants is concerned, we are not interested in computing
the solution of problem (2.1) (i.e. $x^* \in \mathbb{R}$ satisfying the constraints and minimizing $p$). Indeed, it is sufficient to compute the optimal value of (2.1), that is $p^* = p(x^*)$, or at least a certified lower bound of the optimal value. We will first show how this can be done for a particular class of polynomials, namely multi-affine functions. Then, we will extend these results to arbitrary polynomial functions.

2.1. Optimization of multi-affine functions. Multi-affine functions form a particular class of multivariate polynomials. Essentially, a multi-affine function is a function which is affine in each of its variables when the other variables are regarded as constant. For instance, a bivariate multi-affine function is of the form

$$p(x_1, x_2) = p_{00} + p_{10}x_1 + p_{01}x_2 + p_{11}x_1x_2.$$  

The general definition for an arbitrary number of variables is as follows:

**Definition 2.1.** A multi-affine function $p : \mathbb{R}^n \to \mathbb{R}$ is a multivariate polynomial in the variables $x_1, \ldots, x_n$ where the degree of $p$ in each of the variable is at most 1:

$$p(x) = p(x_1, \ldots, x_n) = \sum_{k_1, \ldots, k_n \in \{0, 1\}} p_{k_1, \ldots, k_n} x_1^{k_1} \cdots x_n^{k_n}$$

where $p_{k_1, \ldots, k_n} \in \mathbb{R}$ for all $k_1, \ldots, k_n \in \{0, 1\}$.

Let $R = [x_1, \overline{a}] \times \cdots \times [x_n, \overline{b}]$, with $x_k < \overline{x}_k$, for all $k \in \{1, \ldots, n\}$, be a rectangle of $\mathbb{R}^n$, the set of vertices of $\overline{R}$ is given by

$$V = \prod_{k \in \{1, \ldots, n\}} \{x_k, \overline{x}_k\}.$$  

Thus, a rectangle of $\mathbb{R}^n$ has $2^n$ vertices. It is shown in [BH06] that a multi-affine function $p$ is uniquely determined by its values at the vertices of a rectangle $R$ of $\mathbb{R}^n$. Moreover, for all $x \in R$, $p(x)$ is a convex combination of the values at the vertices, that is $p(R) \subseteq CH(\{p(v) \mid v \in V\})$ where $CH(S)$ denotes the convex hull of the set $S$. Let us remark that generally $p(R)$ is not convex and therefore $P(R) \neq CH(\{p(v) \mid v \in V\})$. From the previous discussion, we have the following result:

**Lemma 2.2.** Let $p : \mathbb{R}^n \to \mathbb{R}$ be a multi-affine function and $R$ a rectangle of $\mathbb{R}^n$ with set of vertices $V$, then $\min_{x \in R} p(x) = \min_{v \in V} p(v)$.

In the following, we show how to compute, using linear programming, a certified lower bounded of the optimal value $p^*$ of (2.1) where $p$ is a multi-affine function. The linear program is derived through Lagrangian duality. We start by writing the Lagrangian of problem (2.1):

$$L(x, \lambda, \mu) = p(x) + \sum_{i \in I} \lambda_i (a_i \cdot x - b_i) + \sum_{j \in J} \mu_j (c_j \cdot x - d_j).$$

where $x \in R$, $\lambda_i \geq 0$ for all $i \in I$, and $\mu_j \in \mathbb{R}$ for all $j \in J$. Then, the dual formulation of problem (2.1) is

$$\begin{align*}
\text{maximize} & \quad \min_{x \in R} L(x, \lambda, \mu) \\
\text{over} & \quad \lambda \in \mathbb{R}^m, \mu \in \mathbb{R}^m, \\
\text{subject to} & \quad \lambda_i \geq 0, \quad i \in I.
\end{align*}$$

(2.2)

Since (2.1) is feasible, the optimal value of (2.2) is bounded, it is denoted $d^* \in \mathbb{R}$. It is well-known from duality theory (see e.g. [BV04]) that we have $d^* \leq p^*$. Let us remark that a multi-affine function is generally non-convex. Therefore, we cannot expect strong duality (i.e. $d^* = p^*$) in general. The following result shows that the problem (2.2) can be recasted as a linear program:
Proposition 2.3. Let $p : \mathbb{R}^n \to \mathbb{R}$ be a multi-affine function, the dual of problem (2.1) is equivalent to the linear program:

$$
\begin{align*}
\max & \quad t \\
\text{subject to} & \quad \lambda_i \geq 0, \quad i \in I, \\
& \quad t \leq p(v) + \sum_{i \in I} \lambda_i (a_i \cdot v - b_i) + \sum_{j \in J} \mu_j (c_j \cdot v - d_j), \quad v \in V.
\end{align*}
$$

Proof. Let us remark that the Lagrangian $L(x, \lambda, \mu)$ is a multi-affine function of $x$. Then, it follows from Lemma 2.2 that

$$
\min_{x \in \mathbb{R}} L(x, \lambda, \mu) = \min_{v \in V} \left( p(v) + \sum_{i \in I} \lambda_i (a_i \cdot v - b_i) + \sum_{j \in J} \mu_j (c_j \cdot v - d_j) \right).
$$

Then, problem (2.2) consist in optimizing a piecewise linear function under a set of linear constraints which it is straightforward (see [BV04], pp 150-151) to formulate as the linear program (2.3). 

Thus, we can see that a certified lower bound $d^*$ of the optimal value $p^*$ of (2.1) can be effectively computed by solving a linear program with $m_I + m_J + 1$ variables and $2n + m_I$ inequality constraints.

Example 2.4. Let us consider the following simple optimization problem:

$$
\begin{align*}
\min & \quad -x_1 - x_2 - x_1 x_2 \\
\text{over} & \quad (x_1, x_2) \in [0, 1]^2, \\
\text{subject to} & \quad x_1 + x_2 \leq 1.
\end{align*}
$$

It is easy to check that the optimal value of this problem is $p^* = -5/4$ which is obtained for $(x_1, x_2) = (0.5, 0.5)$. The dual of this problem as expressed in Proposition 2.3 is

$$
\begin{align*}
\max & \quad t \\
\text{subject to} & \quad \lambda \geq 0, \\
& \quad t \leq -\lambda, \quad t \leq -1, \quad t \leq -1, \quad t \leq -3 + \lambda.
\end{align*}
$$

The optimal value of this problem is $d^* = -3/2$ obtained for $\lambda = 3/2$. We can check that we indeed have $d^* \leq p^*$ and that strong duality does not hold.

We have presented a simple approach to compute a certified lower bound of the solution $p^*$ of (2.1). Though conservative, our approach relies on linear programming and is therefore effective and efficient. More insight on the lower bound $d^*$ can be gained by looking at the dual of the dual problem (2.3).
Proposition 2.5. Let \( p : \mathbb{R}^n \to \mathbb{R} \) be a multi-affine function, the dual of problem (2.3) is the linear program:

\[
\begin{align*}
\text{minimize} \quad & \sum_{v \in V} y_v p(v) \\
\text{over} \quad & y \in \mathbb{R}^{2n}, \\
\text{subject to} \quad & y_v \geq 0, \quad v \in V, \\
& \sum_{v \in V} y_v = 1, \\
& a_i \cdot \sum_{v \in V} y_v z_i \leq b_i, \quad i \in I, \\
& c_j \cdot \sum_{v \in V} y_v z_j = d_j, \quad j \in J.
\end{align*}
\]

Moreover, problems (2.3) and (2.4) have the same optimal value \( d^* \).

The proof is stated in appendix. Proposition 2.5 can be interpreted as follows: the lower bound \( d^* \) is the minimal value over all convex combinations, satisfying the inequality and equality constraints of the problem, of the values of the function \( p \) at the vertices of \( R \), whereas \( p^* \) is the minimal value over the convex combinations corresponding to the multi-affine function \( p \). Then, it appears clearly that we must have \( d^* \leq p^* \) and that generally \( d^* \neq p^* \). Moreover, since \( d^* \) is a convex combination of the value of \( p \) at the vertices of \( R \), we always have \( d^* \geq \min_{x \in R} p(x) \). In the following, we extend our approach to arbitrary multivariate polynomial functions.

2.2. Optimization of polynomial functions. We consider problem 2.1 where \( p : \mathbb{R}^n \to \mathbb{R} \) is now an arbitrary multivariate polynomial function. Let \( \delta_1, \ldots, \delta_n \) denote the degree of \( p \) in the variables \( x_1, \ldots, x_n \) respectively. Let \( \Delta = \{0, \ldots, \delta_1\} \times \cdots \times \{0, \ldots, \delta_n\} \), then \( p(x) \) can be written under the form:

\[
p(x) = p(x_1, \ldots, x_n) = \sum_{(k_1, \ldots, k_n) \in \Delta} p_{k_1, \ldots, k_n} x_1^{k_1} \cdots x_n^{k_n}.
\]

In order to reduce our polynomial optimization problem to a multi-affine optimization problem, we shall use the blossoming principle (see e.g. [Ram89]).

Definition 2.6. The blossom or polar form of the multivariate polynomial \( p : \mathbb{R}^n \to \mathbb{R} \) is the function \( q : \mathbb{R}^{\delta_1 + \cdots + \delta_n} \to \mathbb{R} \) given for \( z = (z_1, 1, 1, \ldots, z_{n,1}, \ldots, z_{n,\delta_n}) \in \mathbb{R}^{\delta_1 + \cdots + \delta_n} \) by

\[
q(z) = \sum_{(k_1, \ldots, k_n) \in \Delta} p_{k_1, \ldots, k_n} B_{k_1, \delta_1}(z_1, 1, 1, \ldots, z_{n,1}, \ldots, z_{n,\delta_n}) \cdots B_{k_n, \delta_n}(z_{n,1}, \ldots, z_{n,\delta_n})
\]

with

\[
B_{k, \delta}(z_1, \ldots, z_\delta) = \frac{1}{k!} \sum_{\sigma \in C(k, \delta)} z_{\sigma_1} \cdots z_{\sigma_k}
\]

where \( C(k, \delta) \) denotes the set of combinations of \( k \) elements in \( \{1, \ldots, \delta\} \).

An example may help to understand the previous definition:

Example 2.7. The blossom of the bivariate polynomial \( p(x_1, x_2) = 3x_1 + x_1^2 x_2^2 + 2x_2^3 \) is

\[
q(z_1, 1, z_2, 1, z_2, 2, z_2, 3) = \frac{3}{2} (z_1, 1 + z_1, 2) + \frac{1}{3} z_1, 1, 1, 2, 1 (z_1, 2, 2, 2, 2, 2, 2, 3) + 2z_1, 1, 2, 2, 2, 2, 3.
\]
From Definition 2.6, it follows that the blossom \( q \) of the polynomial \( p \) satisfies the following properties:

1. It is a multi-affine function.
2. It is a symmetric function of its arguments:
   \[
   q(z_1, \ldots, z_1, \delta_1, \ldots, z_n, \delta_n) = q(\pi_1(z_1, \ldots, z_1, \delta_1), \ldots, \pi_n(z_n, \ldots, z_n, \delta_n)),
   \]
   where \( \pi_i \) is any permutation of its \( \delta_i \) arguments.
3. It satisfies the diagonal property:
   \[
   q(z_1, \ldots, z_1, \ldots, z_n) = p(z_1, \ldots, z_n).
   \]

The third property allows us to recast problem (2.1) for a multivariate polynomial function \( p \) as a problem of minimization of its blossom \( q \) on a polytope. The following result is straightforward and is therefore stated without proof.

**Proposition 2.8.** Let \( p : \mathbb{R}^n \rightarrow \mathbb{R} \) be a multivariate polynomial and \( q : \mathbb{R}^{\delta_1+\cdots+\delta_n} \rightarrow \mathbb{R} \) its blossom. Problem (2.1) is equivalent to:

\[
\begin{align*}
\text{minimize} & \quad q(z) \\
\text{over} & \quad z \in R', \\
\text{subject to} & \quad a'_i \cdot z \leq b_i, \quad i \in I, \\
& \quad c'_j \cdot z = d_j, \quad j \in J, \\
& \quad z_{k,l} = z_{k,l+1}, \quad k \in \{1, \ldots, n\}, \quad l \in \{1, \ldots, \delta_k - 1\},
\end{align*}
\]

where \( R' = [\underline{x}_1, \overline{x}_1]^{\delta_1} \times \cdots \times [\underline{x}_n, \overline{x}_n]^{\delta_n} \) and the vectors \( a'_i, c'_j \) are given for \( i \in I, j \in J \) by

\[
\begin{align*}
a'_i &= (a_{i,1}, a_{i,1}, a_{i,n}, a_{i,n}), \\
c'_j &= (c_{j,1}, c_{j,1}, c_{j,n}, c_{j,n}).
\end{align*}
\]

Now, since the blossom of a multivariate polynomial is a multi-affine function, we can remark that problem (2.5) is similar to those considered in Section 2.1. Then, we can use Proposition 2.3 to obtain its dual, which is given by the following linear program:

\[
\begin{align*}
\text{maximize} & \quad t \\
\text{over} & \quad t \in \mathbb{R}, \ \lambda \in \mathbb{R}^{m_I}, \ \mu \in \mathbb{R}^{m_J}, \ \alpha \in \mathbb{R}^{(\delta_1 - 1)+\cdots+(\delta_n - 1)}, \\
\text{subject to} & \quad \lambda_i \geq 0, \quad i \in I, \\
& \quad t \leq q(v) + \sum_{i \in I} \lambda_i (a'_i \cdot v - b_i) + \sum_{j \in J} \mu_j (c'_j \cdot v - d_j) \\
& \quad + \sum_{k \in \{1, \ldots, n\}} \sum_{l \in \{1, \ldots, \delta_k - 1\}} \alpha_{k,l} (e_{k,l} \cdot v), \quad v \in V'.
\end{align*}
\]

where \( V' \) is the set of vertices of the rectangle \( R' \) of \( \mathbb{R}^{\delta_1+\cdots+\delta_n} \) and \( e_{k,l} \in \mathbb{R}^{\delta_1+\cdots+\delta_n} \) are the vectors such that for all \( z = (z_1, \ldots, z_1, \delta_1, \ldots, z_n, \delta_n) \) in \( \mathbb{R}^{\delta_1+\cdots+\delta_n} \):

\[
\forall k \in \{1, \ldots, n\}, \ l \in \{1, \ldots, \delta_k - 1\}, \ e_{k,l} \cdot z = z_{k,l} - z_{k,l+1}.
\]

By Propositions 2.3 and 2.8, the optimal value of this linear program gives a certified lower bound of the optimal value of the polynomial optimization problem (2.1). However, we shall not solve directly the linear program (2.6) as the symmetries of the problem can be used to further reduce the computational complexity.
2.3. Symmetries and complexity reduction. To exploit the symmetries of \((2.6)\), we define an equivalence relation \(\cong\) on \(\mathbb{R}^{\delta_1+\cdots+\delta_n}\). Let \(z, z' \in \mathbb{R}^{\delta_1+\cdots+\delta_n}\), \(z = (z_{1,1}, \ldots, z_{1,\delta_1}, \ldots, z_{n,1}, \ldots, z_{n,\delta_n})\) and \(z' = (z'_{1,1}, \ldots, z'_{1,\delta_1}, \ldots, z'_{n,1}, \ldots, z'_{n,\delta_n})\), then \(z \cong z'\) if and only if, for all \(i \in \{1, \ldots, n\}\), there exists a permutation \(\pi_i\) such that \((z_{i,1}, \ldots, z_{i,\delta_i}) = \pi_i(z'_{i,1}, \ldots, z'_{i,\delta_i})\). It is easy to see that \(\cong\) is an equivalence relation. Furthermore, from the symmetry property of blossoms it follows that \(q(z) = q(z')\) for all \(z \cong z'\). Also, from the definition of the vectors \(a'_i\) and \(c'_j\) in Proposition 2.8, it is clear that for all \(z \cong z'\), \(a'_i \cdot z = a'_i \cdot z'\) and \(c'_j \cdot z = c'_j \cdot z'\), for all \(i \in I, j \in J\). This can be used to achieve some complexity reduction in \((2.6)\).

**Theorem 2.9.** Let \(p : \mathbb{R}^n \to \mathbb{R}\) be a multivariate polynomial and \(q : \mathbb{R}^{\delta_1+\cdots+\delta_n} \to \mathbb{R}\) its blossom. The optimal value of the linear program \((2.6)\) is equal to the optimal value \(d^*\) of:

\[
\begin{align*}
\text{maximize} \quad & t \\
\text{over} \quad & t \in \mathbb{R}, \; \lambda \in \mathbb{R}^{m_I}, \; \mu \in \mathbb{R}^{m_J}, \\
\text{subject to} \quad & \lambda_i \geq 0, \\
& t \leq q(\mathbf{v}) + \sum_{i \in I} \lambda_i (a'_i \cdot \mathbf{v} - b_i) + \sum_{j \in J} \mu_j (c'_j \cdot \mathbf{v} - d_j), \quad \mathbf{v} \in V'.
\end{align*}
\]

where \(V' = (V' / \cong)\). Moreover, \(d^* \leq p^*\) where \(p^*\) is the optimal value of problem \((2.1)\).

The proof is stated in appendix. Let us highlight the gain of solving \((2.7)\) in place of \((2.6)\). Firstly, the decision variables \(\alpha_{k,l}\) in problem \((2.6)\) do not appear anymore in \((2.7)\). Secondly, the number of constraints indexed by \(v' \in V'\) in \((2.6)\) is \(2^{\delta_1+\cdots+\delta_n}\) whereas the number of constraints indexed by \(\mathbf{v} \in V'\) is \((\delta_1 + 1) \times \cdots \times (\delta_n + 1)\). Therefore, the linear program \((2.6)\) has \(m_I + m_J + (\delta_1 - 1) + \cdots + (\delta_n - 1) + 1\) variables and \(2^{\delta_1+\cdots+\delta_n} + m_I\) inequality constraints whereas the linear program \((2.7)\) has only \(m_I + m_J + 1\) variables and \((\delta_1 + 1) \times \cdots \times (\delta_n + 1) + m_I\) inequality constraints. In particular, the number of constraints in \((2.6)\) grows exponentially in the degrees of \(p\) whereas the number of constraints in \((2.7)\) grows only polynomially in the degrees of \(p\).

**Remark 2.10.** We would like to sketch the relation of the linear program \((2.7)\) to the notion of Bernstein basis for polynomials. It is actually possible to show (though out of the scope of this paper) that the numbers \(q(\mathbf{v})\) for \(\mathbf{v} \in V'\) are the so-called Bernstein coefficients (i.e. the coordinates in the Bernstein basis) of the polynomial \(p(x)\). Using problem \((2.7)\) to compute a lower bound of the optimal value of polynomial optimization problem \((2.1)\) is consistent with the fact that for all \(x \in R\) the value of \(p(x)\) is in the range of the Bernstein coefficients (see e.g. [Riv70]).

**Example 2.11.** Let us consider the following simple example:

\[
\begin{align*}
\text{minimize} \quad & p(x_1, x_2) = x_1^2 + x_2 \\
\text{over} \quad & (x_1, x_2) \in [0, 1] \times [0, 1], \\
\text{subject to} \quad & -x_1 - x_2 \leq -1.
\end{align*}
\]

The optimal value of this problem is \(p^* = 3/4\) which is obtained for \((x_1, x_2) = (0.5, 0.5)\). The blossom of \(p\) is the multi-affine function \(q(z_{1,1}, z_{1,2}, z_{2,1}) = z_{1,1}z_{1,2} + z_{2,1}\). Theorem 2.9 leads to the following linear program:

\[
\begin{align*}
\text{maximize} \quad & t \\
\text{over} \quad & t \in \mathbb{R}, \; \lambda \in \mathbb{R}, \\
\text{subject to} \quad & \lambda \geq 0, \quad i \in I, \\
& t \leq q(\mathbf{v}) + \lambda (a'_i \cdot \mathbf{v} + 1), \quad \mathbf{v} \in V'.
\end{align*}
\]
where \( a' = (\frac{-1}{2}, \frac{-1}{2}, -1) \) and \( V' = \{(0, 0, 0), (0, 0, 1), (1, 0, 0), (1, 0, 1), (1, 1, 0), (1, 1, 1)\} \) has 6 elements. The optimal value of this problem is \( d^* = 2/3 \) which is obtained for \( \lambda = 4/3 \). We can check that \( d^* \leq p^* \).

2.4. Sensitivity analysis. An interesting feature of Lagrangian duality is that it enables sensitivity analysis (see e.g. \([BV04]\)). In this section, we are interested in analyzing the variations of the optimal value of \((2.1)\), or of its lower bound given by the optimal value of \((2.7)\), under modifications of the polytope. This will be used in the next section for the synthesis of polytopic invariants for polynomial dynamical systems. More precisely, we consider the following variation of problem \((2.1)\):

\[
\begin{align*}
\text{minimize} & \quad p(x) \\
\text{over} & \quad x \in \mathbb{R}, \\
\text{subject to} & \quad a_i \cdot x \leq b_i + \alpha_i, \quad i \in I, \\
& \quad c_j \cdot x = d_j + \beta_j, \quad j \in J,
\end{align*}
\]

(2.8)

where \( \alpha_i \in \mathbb{R} \), for all \( i \in I \) and \( \beta_j \in \mathbb{R} \) for all \( j \in J \). This problem coincides with the original problem \((2.1)\) when \( \alpha = 0 \) and \( \beta = 0 \). We assume that problem \((2.8)\) is feasible as well. Let \( p^* \) and \( p^*(\alpha, \beta) \) denote the optimal values of problems \((2.1)\) and \((2.8)\), respectively. Let \( d^* \) and \( d^*(\alpha, \beta) \) be the lower bounds of \( p^* \) and \( p^*(\alpha, \beta) \) obtained by application of Theorem 2.9. The following result shows how the solution of \((2.7)\) allows us to compute a lower bound of \( d^*(\alpha, \beta) \) and thus of \( p^*(\alpha, \beta) \).

**Theorem 2.12.** Let \( d^* \) and \((t^*, \lambda^*, \mu^*)\) be the optimal value and an optimal solution of the linear program \((2.7)\). Then, for all \( \alpha \in \mathbb{R}^m_I \) and \( \beta \in \mathbb{R}^m_J \), such that \((2.8)\) is feasible we have:

\[
p^*(\alpha, \beta) \geq d^*(\alpha, \beta) \geq d^* - \lambda^* \cdot \alpha - \mu^* \cdot \beta.
\]

**Proof.** By applying Theorem 2.9 to the perturbed problem \((2.1)\), we have that \( d^*(\alpha, \beta) \) is the optimal value of

\[
\begin{align*}
\text{maximize} & \quad t \\
\text{over} & \quad t \in \mathbb{R}, \ \lambda \in \mathbb{R}^{m_I}, \ \mu \in \mathbb{R}^{m_J}, \\
\text{subject to} & \quad \lambda_i \geq 0, \\
& \quad t \leq q(\bar{v}) + \sum_{i \in I} \lambda_i (a_i \cdot \bar{v} - b_i - \alpha_i) + \sum_{j \in J} \mu_j (c_j \cdot \bar{v} - d_j - \beta_j), \ \bar{v} \in V'.
\end{align*}
\]

(2.9)

The fact that \( p^*(\alpha, \beta) \geq d^*(\alpha, \beta) \) is a consequence of Theorem 2.9. Let \((t^*, \lambda^*, \mu^*)\) be an optimal solution of the problem \((2.7)\), let us show that \((t^* - \lambda^* \cdot \alpha - \mu^* \cdot \beta, \lambda^*, \mu^*)\) is feasible for \((2.9)\). It is clear that \( \lambda^*_i \geq 0 \), for all \( i \in I \). Also, for all \( \bar{v} \in V' \),

\[
q(\bar{v}) + \sum_{i \in I} \lambda^*_i (a_i \cdot \bar{v} - b_i - \alpha_i) + \sum_{j \in J} \mu^*_j (c_j \cdot \bar{v} - d_j - \beta_j)
\]

\[
= q(\bar{v}) + \sum_{i \in I} \lambda^*_i (a_i \cdot \bar{v} - b_i) + \sum_{j \in J} \mu^*_j (c_j \cdot \bar{v} - d_j) - \lambda^* \cdot \alpha - \mu^* \cdot \beta \\
\geq t^* - \lambda^* \cdot \alpha - \mu^* \cdot \beta
\]

Then, \((t^* - \lambda^* \cdot \alpha - \mu^* \cdot \beta, \lambda^*, \mu^*)\) is feasible for \((2.9)\). It follows that \( d^*(\alpha, \beta) \geq t^* - \lambda^* \cdot \alpha - \mu^* \cdot \beta \) which leads to the expected inequality since \( d^* = t^* \). \(\Box\)
3. Invariant Verification and Synthesis for Polynomial Systems

In the following, we show how the results developed in the previous section can be used for verification and synthesis of polytopic invariants for polynomial dynamical systems. Let us consider a dynamical system of the form:

\[
\dot{x}(t) = f(x(t)), \quad x(t) \in \mathbb{R}^n
\]

where \( f : \mathbb{R}^n \to \mathbb{R}^n \) is a polynomial vector field. Let \( R = [x_1, \bar{x}_1] \times \cdots \times [x_n, \bar{x}_n] \), with \( x_k < \bar{x}_k \) for all \( k \in \{1, \ldots, n\} \) be a rectangle of \( \mathbb{R}^n \), delimiting a region of interest for studying the dynamics of (3.1). In this section, we first show how to verify that a given polytope \( P \subseteq R \) is invariant for the dynamical system (3.1). Then, we will show that when we fail to verify that \( P \) is an invariant, sensitivity analysis can help us to modify \( P \) in order to find an invariant polytope for (3.1).

3.1. Polytopic invariant verification. Let us consider a non-empty bounded polytope \( P \subseteq \mathbb{R}^n \) defined by inequality constraints:

\[
P = \{ x \in \mathbb{R}^n | a_k \cdot x \leq b_k, \forall k \in K \}
\]

where \( K = \{1, \ldots, m_K\} \) is a set of indices, \( a_k \in \mathbb{R}^n, b_k \in \mathbb{R} \), for all \( k \in K \). The facets of the polytope are \( \{F_k | k \in K\} \) where

\[
F_k = \{ x \in \mathbb{R}^n | a_k \cdot x = b_k, \text{ and } a_i \cdot x \leq b_i, \forall i \in K \setminus \{k\} \}.
\]

We assume that \( P \subseteq R \) and that all facets \( F_k \) are not empty. As stated in the introduction, \( P \) is an invariant set of the dynamical system (3.1) if and only if

\[\forall k \in K, \min_{x \in F_k} -a_k \cdot f(x) \geq 0.\]

Since for all \( k \in K, F_k \subseteq P \subseteq R \), then this problem is equivalent to showing that the optimal values \( p_k \) of the following optimization problems are non-negative for all \( k \in K \):

\[
\begin{align*}
\text{minimize} & \quad -a_k \cdot f(x) \\
\text{over} & \quad x \in R, \\
\text{subject to} & \quad a_i \cdot x \leq b_i, \quad i \in K \setminus \{k\}, \\
& \quad a_k \cdot x = b_k.
\end{align*}
\]

Since \(-a_k \cdot f\) is a multivariate polynomial, this problem is similar to (2.1). Therefore, by application of Theorem 2.9, we have the following result:

**Proposition 3.1.** For \( k \in K \), let \( d_k^* \) be the optimal value of the linear program:

\[
\begin{align*}
\text{maximize} & \quad t \\
\text{over} & \quad t \in \mathbb{R}, \lambda \in \mathbb{R}^{m_K}, \\
\text{subject to} & \quad \lambda_i \geq 0, \quad i \in K \setminus \{k\}, \\
& \quad t \leq q_k(\tau) + \sum_{i \in K} \lambda_i (a_i' \cdot \tau - b_i), \quad \tau \in \overline{V^i}.
\end{align*}
\]

where \( q_k \) is the blossom of the multivariate polynomial \(-a_k \cdot f\), the vector \( a_i' \) is defined as in Proposition 2.8 for \( i \in K \) and \( \overline{V^i} \) is defined as in Theorem 2.9. If for all \( k \in K \), \( d_k^* \geq 0 \), then \( P \) is an invariant polytope for dynamical system (3.1).

**Proof.** By applying Theorem 2.9 to problem (3.2), we obtain that \( p_k^* \geq d_k^* \), for all \( k \in K \). Then, \( d_k^* \geq 0 \) implies that \( p_k^* \geq 0 \), for all \( k \in K \), and therefore \( P \) is an invariant polytope. \(\square\)
Thus, we have shown that invariance of the polytope $P$ can be verified by solving a set of linear programs (one per facet of $P$). In the case when we fail to verify that the polytope $P$ is invariant, sensitivity analysis may help us in modifying $P$ in order to find an invariant polytope for \((3.1)\).

Remark 3.2. Let us remark that the degrees of the multivariable polynomials $-a_k \cdot f$ may not be all the same. This results in vectors $a_i^*$, and sets $\mathbf{\nabla}^\alpha$ in problem \((3.3)\) that depend on the index $k \in K$. It is possible to avoid this by defining $q_k$ as $-a_k \cdot g$ where $g$ is the blossom of the polynomial vector field $f$ defined as follows. For $i, j \in \{1, \ldots, n\}$, let $\delta_{ij}$ be the degree of $x_i$ in the multivariate polynomial $f_j(x)$ and let $\delta_i = \max_{j \in \{1, \ldots, n\}} \delta_{ij}$. For all $j \in \{1, \ldots, n\}$, it is possible to see $f_j$ as a multivariate polynomial with degrees $\delta_1, \ldots, \delta_n$ possibly with some zero coefficients and define the associated blossom $g_j$ as defined in Definition 2.6. Then, for $j \in \{1, \ldots, n\}$, $g_j$ are the components of the blossom $g : \mathbb{R}^{\delta_1 + \cdots + \delta_n} \rightarrow \mathbb{R}^n$ of the polynomial vector field $f$. Then, for all $k \in K$, $q_k = -a_k \cdot g$ are multi-affine functions defined on $\mathbb{R}^{\delta_1 + \cdots + \delta_n}$ with similar properties to the blossom of $-a_k \cdot f$ that can be used in problem \((3.3)\).

3.2. Polytopic invariant synthesis. Our approach fails in verifying that the polytope $P$ is an invariant for dynamical system \((3.1)\), if $d_k^* < 0$, for some $k \in K$. In that case, we may be interested in modifying $P$ in order to find an invariant polytope. We consider polytopes whose facets are parallel to those of $P$. For $\alpha \in \mathbb{R}^{mk}$, let $P_\alpha$ be the polytope given by

$$P_\alpha = \{ x \in \mathbb{R}^n | a_k \cdot x \leq b_k + \alpha_k, \forall k \in K \}.$$

For $\alpha = 0$, we recover the polytope $P$, we would like to find $\alpha$ such that $P_\alpha$ is an invariant for \((3.1)\). We impose additional constraints on the polytope $P_\alpha$:

- $P_\alpha \subseteq R$: this can be ensured by constraints of the form $b_k + \alpha_k \leq b_k$.
- $P_\alpha$ is not empty: this can be ensured by constraints of the form $b_k \leq b_k + \alpha_k$.
- $P_\alpha$ is relatively close to $P$: this is ensured by imposing $-\varepsilon \leq \alpha_k \leq \varepsilon$ where $\varepsilon$ is a parameter that can be tuned.

Denoting for $k \in K$, $d_k^*(\alpha)$ the optimal values of problems \((3.3)\) for the polytope $P_\alpha$, the sensitivity analysis given in Theorem 2.12 gives us

$$\forall k \in K, \ d_k^*(\alpha) \geq d_k^* + \lambda_k^* \cdot \alpha$$

where $d_k^*$ and $(t_k^*, \lambda_k^*)$ are the optimal values and solutions of problems \((3.3)\) for polytope $P$ and $k \in K$. Then, by Proposition 3.1 for $P_\alpha$ to be an invariant polytope for dynamical system \((3.1)\), it is sufficient that for all $k \in K$, $d_k^* + \lambda_k^* \cdot \alpha \geq 0$. In order to find a suitable $\alpha$, we can solve the following problem:

$$\begin{align*}
\text{maximize} \quad & \min_{k \in K} (d_k^* + \lambda_k^* \cdot \alpha) \\
\text{over} \quad & \alpha \in \mathbb{R}^{mk}, \ \text{subject to} \ \alpha_k \leq \lambda_k^* \cdot \alpha, \quad k \in K
\end{align*}$$

where $\lambda_k^* = \max(-\varepsilon, b_k - b_k)$ and $\alpha_k = \min(\varepsilon, \overline{\alpha_k})$. This problem can be recasted as the following linear program:

$$\begin{align*}
\text{maximize} \quad & t \\
\text{over} \quad & t \in \mathbb{R}, \ \alpha \in \mathbb{R}^{mk}, \ \text{subject to} \ \ t \leq d_k^* - \lambda_k^* \cdot \alpha, \quad k \in K, \ \alpha_k \leq \lambda_k^* \cdot \alpha, \quad k \in K.
\end{align*}$$

Let $(t^*, \alpha^*)$ be a solution of this linear program. If the optimal value of this problem is non-negative then it is sufficient to prove that $P_{\alpha^*}$ is an invariant for the dynamical system \((3.1)\). If the optimal
value is strictly negative, then we compute $d^*_k(\alpha^*)$ by solving problems (3.3) for the polytope $P_{\alpha^*}$ for all $k \in K$. If all $d^*_k(\alpha^*)$ are non-negative, then by Proposition 3.1, $P_{\alpha^*}$ is invariant. If the verification fails as well, then we use sensitivity analysis to modify $P_{\alpha^*}$ in order to find an invariant. This gives an iterative approach for synthesis of polytopic invariants of polynomial dynamical systems.

**Remark 3.3.** Let us remark that the polytope $P_{\alpha^*}$ computed by solving (3.4) may have empty facets. This results, for the empty facet $F_k$, in an unbounded value $d^*_k(\alpha^*) = +\infty$. In order to avoid such situations, it may be useful to replace $\alpha^*$ by $\tilde{\alpha}^*$ such that $P_{\tilde{\alpha}^*}$ has no empty facet and $P_{\alpha^*} = P_{\tilde{\alpha}^*}$ (see Figure 1). Again, this can be done by solving a set of linear programs.

![Figure 1](image)

Figure 1. The polytope $P_{\alpha^*}$ may have empty facets (center polytope), we replace $\alpha^*$ by $\tilde{\alpha}^*$ such that $P_{\tilde{\alpha}^*}$ has no empty facet and $P_{\alpha^*} = P_{\tilde{\alpha}^*}$ (right polytope).

### 3.3. Examples

We implemented our approach in Matlab; in the following, we show for a set of examples borrowed from engineering or biological applications, that our approach is effective in practice. All the reported computations take a few seconds.

#### 3.3.1. Moore-Greitzer jet engine model

We tested our approach on the following polynomial dynamical system, corresponding to a Moore-Greitzer model of a jet engine, with stabilizing feedback [KKK95]:

$$
\begin{align*}
\dot{x}_1 &= -x_2 - \frac{3}{2}x_1^2 - \frac{1}{2}x_1^3, \\
\dot{x}_2 &= 3x_1 - x_2.
\end{align*}
$$

This system has the origin as a stable equilibrium. Using our approach, we tried to find an invariant polytope for the system around the origin. Working in the rectangle $[-0.2, 0.2]^2$, we were able to find an invariant polytope with $m_K = 24$ facets with uniformly distributed orientations (see Figure 2). Starting from the set represented in dashed line, our approach needs 5 iterations to find the invariant polytope depicted in plain line. We can check on the figure that it is effectively an invariant.

At each iteration, we need to solve for invariant verification, $m_K = 24$ linear programs of the form (3.3) with $m_K + 1 = 25$ variables and $m_K - 1 + (\delta_1 + 1) \times (\delta_2 + 1) = 31$ inequality constraints. For invariant synthesis using sensitivity analysis, we need to solve at each iteration 1 linear program with $m_K + 1 = 25$ variables and $2m_K = 48$ inequality constraints.

#### 3.3.2. FitzHugh-Nagumo neuron model

We also applied our approach to the FitzHugh-Nagumo model [Fit61], a polynomial dynamical system modelling the electrical activity of a neuron:

$$
\begin{align*}
\dot{x}_1 &= x_1 - x_1^3/3 - x_2 + I, \\
\dot{x}_2 &= 0.08(x_1 + 0.7 - 0.8x_2),
\end{align*}
$$
where model parameter $I$ is taken equal to $\frac{7}{8}$. This system is known to have a limit cycle. Using our approach, we synthesized an invariant polytope containing the limit cycle. Working in the rectangle $[-2.5, 2.5] \times [-1.5, 3.5]$, we found an invariant polytope with 8 facets with uniformly distributed orientations (see Figure 3). Starting from the set represented in dashed line, our approach needs 15 iterations to find the invariant polytope depicted in plain line. We can check on the figure that it is effectively an invariant. Let us remark that this invariant polytope $\mathcal{P}$ together with the existence of an unstable equilibrium inside $\mathcal{P}$ provides by application of the Poincaré–Bendixon theorem a formal proof of the existence of a limit cycle inside the polytope $\mathcal{P}$.

At each iteration, we need to solve for invariant verification, $m_K = 8$ linear programs of the form (3.3) with $m_K + 1 = 9$ variables and $m_K - 1 + (\delta_1 + 1) \times (\delta_2 + 1) = 15$ inequality constraints. For invariant synthesis using sensitivity analysis, we need to solve at each iteration 1 linear program with $m_K + 1 = 9$ variables and $2m_K = 16$ inequality constraints.

3.3.3. Phytoplankton growth model. The last example we shall consider is model of Phytoplankton growth [BG02]:

$$
\begin{align*}
\dot{x}_1 &= 1 - x_1 - \frac{x_1 x_2}{4}, \\
\dot{x}_2 &= (2x_3 - 1)x_2, \\
\dot{x}_3 &= \frac{x_3}{4} - 2x_3^2.
\end{align*}
$$

This system has a stable equilibrium. Using our approach, we synthesized an invariant polytope containing the equilibrium. Working in the rectangle $[0, 3] \times [-0.1, 2] \times [0, 0.6]$, we were able to find an invariant polytope with $m_K = 18$ facets a regular octagon (see Figure 4). Starting from the polytope represented in left part of the figure, our approach needs 11 iterations to find the invariant polytope depicted in the right part of the figure. We can check on the figure that it is indeed an invariant.
At each iteration, we need to solve for invariant verification, \( m_K = 18 \) linear programs of the form (3.3) with \( m_K + 1 = 19 \) variables and \( m_K - 1 + (\delta_1 + 1) \times (\delta_2 + 1) \times (\delta_3 + 1) = 29 \) inequality constraints. For invariant synthesis using sensitivity analysis, we need to solve at each iteration \( 1 \) linear program with \( m_K + 1 = 19 \) variables and \( 2m_K = 36 \) inequality constraints.

**Figure 3.** Polytopic invariant for the FitzHugh-Nagumo model (represented in plain line) obtained after 15 iterations starting from the dashed polytope. The computed invariant contains the limit cycle.

**Figure 4.** Polytopic invariant for the Phytoplankton growth model (on the right) obtained after 11 iterations starting from the polytope on the left.
4. Conclusion

In this paper, we have presented an approach based on linear programming for verification and synthesis of polytopic invariants for polynomial dynamical systems. It uses the blossoming principle for polynomials, properties of multi-affine functions, Lagrangian duality and sensitivity analysis. Though our approach is conservative (we may fail to verify invariance of a polytope), we have shown on several examples that it can be useful in practical applications. In the future, we plan to use similar ideas for control synthesis for polynomial dynamical systems.

References


Appendix

Proof of Proposition 2.5.

Proof. We first restate problem (2.3) as a minimization problem:

\[
\begin{align*}
\text{minimize} & \quad -t \\
\text{over} & \quad t \in \mathbb{R}, \quad \lambda \in \mathbb{R}^{|I|}, \quad \mu \in \mathbb{R}^{|J|}, \\
\text{subject to} & \quad t - p(v) - \sum_{i \in I} \lambda_i (a_i \cdot v - b_i) - \sum_{j \in J} \mu_j (c_j \cdot v - d_j) \leq 0, \quad v \in V, \\
& \quad -\lambda_i \leq 0, \quad i \in I.
\end{align*}
\]
Then, the Lagrangian of this problem is

\[ L'(t, \lambda, \mu, y, \theta) = -t + \sum_{v \in V} y_v \left( t - p(v) - \sum_{i \in I} \lambda_i (a_i \cdot v - b_i) - \sum_{j \in J} \mu_j (c_j \cdot v - d_j) \right) - \sum_{i \in I} \theta_i \lambda_i \]

where \( y_v \geq 0 \), for all \( v \in V \), and \( \theta_i \geq 0 \), for all \( i \in I \). This can be rewritten as

\[
L'(t, \lambda, \mu, y, \theta) = t \left( \sum_{v \in V} y_v - 1 \right) - \sum_{i \in I} \lambda_i \left( \theta_i + \sum_{v \in V} y_v (a_i \cdot v - b_i) \right) - \sum_{j \in J} \mu_j \left( \sum_{v \in V} y_v (c_j \cdot v - d_j) \right) - \sum_{v \in V} y_v p(v).
\]

Then, minimizing \( L'(t, \lambda, \mu, y, \theta) \) over \( t \in \mathbb{R}, \lambda \in \mathbb{R}^{m_I} \) and \( \mu \in \mathbb{R}^{m_J} \), we obtain

\[
\min_{t \in \mathbb{R}, \lambda \in \mathbb{R}^{m_I}, \mu \in \mathbb{R}^{m_J}} L'(t, \lambda, \mu, y, \theta) = -\sum_{v \in V} y_v p(v), \quad \text{if} \quad \left\{ \begin{array}{l}
\sum_{v \in V} y_v = 1, \\
\theta_i + \sum_{v \in V} y_v (a_i \cdot v - b_i) = 0, \quad i \in I, \\
\sum_{v \in V} y_v (c_j \cdot v - d_j) = 0, \quad j \in J.
\end{array} \right.
\]

or \(-\infty\) otherwise. Now, maximizing over \( y_v \geq 0 \), for all \( v \in V \), and \( \theta_i \geq 0 \), for all \( i \in I \) we obtain the dual of problem (4.1)

\[
\max_{y \in \mathbb{R}^{2n}, \theta \in \mathbb{R}^{m_I}} -\sum_{v \in V} y_v p(v)
\]

subject to

\[
\begin{align*}
y_v & \geq 0, & v & \in V, \\
\theta_i & \geq 0, & i & \in I, \\
\sum_{v \in V} y_v & = 1, \\
\theta_i + \sum_{v \in V} y_v (a_i \cdot v - b_i) & = 0, & i & \in I, \\
\sum_{v \in V} y_v (c_j \cdot v - d_j) & = 0, & j & \in J.
\end{align*}
\]

The variable \( \theta \) can be removed and the problem becomes

\[
\max_{y \in \mathbb{R}^{2n}} -\sum_{v \in V} y_v p(v)
\]

subject to

\[
\begin{align*}
y_v & \geq 0, & v & \in V, \\
\sum_{v \in V} y_v & = 1, \\
\sum_{v \in V} y_v (a_i \cdot v - b_i) & \leq 0, & i & \in I, \\
\sum_{v \in V} y_v (c_j \cdot v - d_j) & = 0, & j & \in J.
\end{align*}
\]

This problem is clearly equivalent to (2.4). Moreover, since (4.1) is a linear program, strong duality holds (see e.g. [BV04]) and the solutions of (4.1) and (4.2) are equal. It follows that the solutions of (2.4) and (2.3) are equal. \( \square \)
Proof of Theorem 2.9

Proof. Let $d^*$ be the optimal value of (2.6) and $\overline{d}^*$ the optimal value of (2.7), we want to show that $d^* = \overline{d}^*$. We start by remarking that (2.6) and (2.7) are linear programs therefore their optimal values are equal to that of their dual problems. Following Proposition 2.5, the dual of (2.6) is

$$\text{minimize} \quad \sum_{v \in V'} y_v q(v)$$

over $y \in \mathbb{R}^{2^\delta_1 + \cdots + \delta_n}$,

subject to $y_v \geq 0$, $v \in V'$,

$$\sum_{v \in V'} y_v = 1,$$

$$a'_i \cdot \sum_{v \in V'} y_v v \leq b_i, \quad i \in I,$$

$$c'_j \cdot \sum_{v \in V'} y_v v = d_j, \quad j \in J,$$

$$e_{k,l} \cdot \sum_{v \in V'} y_v v = 0, \quad k \in \{1, \ldots, n\}, \quad l \in \{1, \ldots, \delta_k - 1\}.$$

Similarly, we can show that the dual of (2.7) is

$$\text{minimize} \quad \sum_{\tau \in \overline{V'}} z_\tau q(\overline{\tau})$$

over $z \in \mathbb{R}^{(\delta_1 + 1) \times \cdots \times (\delta_n + 1)}$,

subject to $z_\tau \geq 0$, $\tau \in \overline{V'}$,

$$\sum_{\tau \in \overline{V'}} z_\tau = 1,$$

$$a'_i \cdot \sum_{\tau \in \overline{V'}} z_\tau \overline{\tau} \leq b_i, \quad i \in I,$$

$$c'_j \cdot \sum_{\tau \in \overline{V'}} z_\tau \overline{\tau} = d_j, \quad j \in J.$$

We first show that $\overline{d}^* \leq d^*$. Let $y \in \mathbb{R}^{2^\delta_1 + \cdots + \delta_n}$ be a feasible point for problem (4.3) such that $d^* = \sum_{v \in V'} y_v q(v)$. For $\overline{\tau} \in \overline{V'}$, let $z_\tau = \sum_{\tau \supseteq \overline{\tau}} y_v$, it is clear that $z_\tau \geq 0$. Further,

$$\sum_{\tau \in \overline{V'}} z_\tau = \sum_{\tau \in \overline{V'}} \sum_{\tau' \supseteq \overline{\tau}} y_v = \sum_{v \in V'} y_v = 1.$$

Since for all $v \equiv \tau$, $a'_i \cdot v = a'_i \cdot \tau$ and $c'_j \cdot v = c'_j \cdot \tau$, we have

$$a'_i \cdot \sum_{\tau \in \overline{V'}} z_\tau \overline{\tau} = \sum_{\tau \in \overline{V'}} \sum_{\tau' \supseteq \tau} y_v (a'_i \cdot \tau) = \sum_{v \in V'} y_v (a'_i \cdot v) \leq b_i,$$

and

$$c'_j \cdot \sum_{\tau \in \overline{V'}} z_\tau \overline{\tau} = \sum_{\tau \in \overline{V'}} \sum_{\tau' \supseteq \tau} y_v (c'_j \cdot \tau) = \sum_{v \in V'} y_v (c'_j \cdot v) = d_j.$$

Therefore, $z$ is feasible for problem (4.4). Finally, since for all $v \equiv \tau$, $q(v) = q(\overline{\tau})$, it follows that

$$\sum_{\tau \in \overline{V'}} z_\tau q(\overline{\tau}) = \sum_{\tau \in \overline{V'}} \sum_{\tau' \supseteq \tau} y_v q(\overline{\tau}) = \sum_{v \in V'} y_v q(v) = d^*.$$
Therefore, \( \overline{d} \leq d^* \). We now show that \( d^* \leq \overline{d} \). Let \( z \in \mathbb{R}^{(\delta_1+1) \times \cdots \times (\delta_n+1)} \) be a feasible point for problem (4.4) such that \( \overline{d} = \sum_{\pi \in \mathcal{V}} z_{\pi q(\pi)} \). Let \( n(\pi) \) denote the number of vertices \( v \in \mathcal{V}' \) such that \( v \equiv \pi \), then for all \( \pi \equiv \pi \), let \( y_\pi = z_{\pi}/n(\pi) \). It is clear \( y_\pi \geq 0 \) and

\[
\sum_{v \in \mathcal{V}'} y_\pi = \sum_{\pi \in \mathcal{V}} \sum_{v \equiv \pi} y_\pi = \sum_{\pi \in \mathcal{V}} z_{\pi} = 1.
\]

We also have that

\[
a'_i \cdot \sum_{v \in \mathcal{V}'} y_\pi v = \sum_{\pi \in \mathcal{V}} \sum_{v \equiv \pi} \sum_{v \equiv \pi} y_\pi (a'_i \cdot v) = \sum_{\pi \in \mathcal{V}} \sum_{v \equiv \pi} \frac{z_{\pi}}{n(\pi)} (a'_i \cdot \pi) = \sum_{\pi \in \mathcal{V}} z_{\pi} (a'_i \cdot \pi) \leq b_i,
\]

and

\[
c'_j \cdot \sum_{v \in \mathcal{V}'} y_\pi v = \sum_{\pi \in \mathcal{V}} \sum_{v \equiv \pi} \sum_{v \equiv \pi} y_\pi (c'_j \cdot v) = \sum_{\pi \in \mathcal{V}} \sum_{v \equiv \pi} \frac{z_{\pi}}{n(\pi)} (c'_j \cdot \pi) = \sum_{\pi \in \mathcal{V}} z_{\pi} (c'_j \cdot \pi) = d_j.
\]

Further,

\[
e_{k,l} \cdot \sum_{v \in \mathcal{V}'} y_\pi v = \sum_{\pi \in \mathcal{V}} \sum_{v \equiv \pi} e_{k,l} \cdot \sum_{v \equiv \pi} \frac{z_{\pi}}{n(\pi)} v = \sum_{\pi \in \mathcal{V}} \frac{z_{\pi}}{n(\pi)} \left( e_{k,l} \cdot \sum_{v \equiv \pi} v \right).
\]

By remarking, that for all \( \pi \in \mathcal{V} \), \( e_{k,l} \cdot \sum_{v \equiv \pi} v = 0 \), it follows that \( e_{k,l} \cdot \sum_{v \in \mathcal{V}'} y_\pi v = 0 \). Therefore, \( y \) is feasible for problem (4.3). Finally,

\[
\sum_{v \in \mathcal{V}'} y_\pi q(v) = \sum_{\pi \in \mathcal{V}} \sum_{v \equiv \pi} z_{\pi} q(\pi) = \sum_{\pi \in \mathcal{V}} z_{\pi} q(\pi) = \overline{d}.
\]

This proves that \( d^* \leq \overline{d} \). The fact that \( d^* \leq p^* \) is a consequence of Propositions 2.3 and 2.8. \( \square \)

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