Fuzzy relation equations and subsystems of fuzzy transition systems

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Abstract

In this paper we study subsystems, reverse subsystems and double subsystems of a fuzzy transition system. We characterize them in terms of fuzzy relation inequalities and equations, as eigen fuzzy sets of the fuzzy quasi-order \( Q \) and the fuzzy equivalence \( E \) generated by fuzzy transition relations, and as linear combinations of aftersets and foresets of \( Q \) and equivalence classes of \( E \). We also show that subsystems, reverse subsystems and double subsystems of a fuzzy transition system \( \mathcal{T} \) form both closure and opening systems in the lattice of fuzzy subsets of \( A \), where \( A \) is the set of states of \( \mathcal{T} \), and we provide efficient procedures for computing related closures and openings of an arbitrary fuzzy subset of \( A \). These procedures boil down to computing the fuzzy quasi-order \( Q \), or the fuzzy equivalence \( E \), which can be efficiently computed using the well-known algorithms for computing the transitive closure of a fuzzy relation.

Keywords: Fuzzy transition systems; Fuzzy automata; Fuzzy relation equations; Fuzzy quasi-orders; Fuzzy equivalences; Eigen fuzzy sets;

1. Introduction

From the very beginning of the theory of fuzzy sets, fuzzy automata and languages are studied as a means for bridging the gap between the precision of computer languages and vagueness and imprecision, which are frequently encountered in the study of natural languages. Study of fuzzy automata and languages was initiated in 1960s by Santos [84–86], Wee [93], Wee and Fu [94], and Lee and Zadeh [54]. From late 1960s until early 2000s mainly fuzzy automata and languages with membership values in the Gödel structure have been considered (cf., e.g., [33, 38, 66]). The idea of studying fuzzy automata with membership values in some structured abstract set comes back to Wechler [92], and in recent years researcher’s attention has been aimed mostly to fuzzy automata with membership values in complete residuated lattices, lattice-ordered monoids, and other kinds of lattices. Fuzzy automata taking membership values in a complete residuated lattice were first studied by Qiu in [74, 75], where some basic concepts were discussed, and later, Qiu and his coworkers have carried out extensive research of these fuzzy automata (cf. [76, 77, 95–99]). From a different point of view, fuzzy automata with membership values in a complete residuated lattice were studied by Ignjatović, Čirić and their coworkers in [20–23, 41, 43, 44, 46, 88, 89].

Fuzzy automata taking membership values in a lattice-ordered monoid were investigated by Li and others [57, 58, 60, 62], fuzzy automata over other types of lattices were the subject of [3, 32, 52, 53, 59, 61, 70–73], and automata which generalize fuzzy automata over any type of lattices, as well as weighted automata over semirings, have been studied recently in [17, 31, 48]. During the decades, fuzzy automata and languages have gained wide field of application, including lexical analysis, description of natural and programming languages, learning systems, control systems, neural networks, knowledge representation, clinical monitoring, pattern recognition, error correction, databases, discrete event systems, and many other areas (cf., e.g., [33, 38, 51, 66, 72]).

On the other hand, fuzzy relation equations and inequalities were first studied by Sanchez, who used them in medical research (cf. [79–82]). Later they found a much wider field of application, and nowadays they are used in fuzzy control, discrete dynamic systems, knowledge engineering, identification of fuzzy systems, prediction of fuzzy systems, decision-making, fuzzy information retrieval, fuzzy pattern recognition, image compression and reconstruction, and in many other areas (cf., e.g., [26, 30, 33, 34, 51, 69, 72]). Recently, fuzzy relation equations and inequalities have been very successfully applied in the theory of fuzzy automata. In [22, 23, 89] they were used in the reduction of the number of states of fuzzy finite automata, and in [20, 21] (see also [18]) they were employed in the study of simulation, bisimulation and equivalence of fuzzy automata. Here, fuzzy relation equations and inequalities are used in the study of subsystems of fuzzy transition systems (by which we mean fuzzy automata...
without fixed fuzzy sets of initial and terminal states).

A common problem that arises in many applications of transition systems and automata is to identify those sets of states that are closed under all transition relations, i.e., sets of states which with each of its states also contain all the states that are accessible from it. Here such sets of states are called subsystems. In the case of deterministic transition systems, if they are regarded as unary algebras, subsystems are precisely subalgebras of these algebras. Also, it is often needed to identify sets of states which are closed under reverse transition relations, i.e., sets of states which with each of its states also contain all the states that are coaccessible to it (c.f., e.g. [12]). We call such sets of states reverse subsystems. Sets of states that are both subsystems and reverse subsystems, called double subsystems, are actually components of decompositions of a transition system into a disjoint union of smaller transition systems with the property that there are no transitions between states from different components. Such decompositions are known as direct sum decompositions and are used when we want to detect and eliminate useless states and transitions of a transition system. In particular, the minimal nonempty double subsystems are the components of the greatest direct sum decomposition of a transition system (cf. [13, 14, 16]).

Subsystems of a fuzzy transition system are defined as fuzzy subsets of its set of states which are closed under all fuzzy transition relations. They were first introduced and studied by Malik, Mordeson and Sen in [63], who described some of their fundamental properties (see also [66]). Later, subsystems were investigated by Das in [24], who showed that they form a topological closure system and identified a number of their topological properties. From a topological point of view, subsystems were also discussed in [87]. All the mentioned papers dealt with fuzzy transition systems over the Gödel structure. Here we study subsystems in a more general context, for fuzzy transition systems over a complete residuated lattice. We also introduce and examine reverse subsystems, which are defined as those fuzzy subsets of the set of states which are closed under reverse transition relations, and double subsystems, which are defined as fuzzy subsets of the set of states which are both subsystems and reverse subsystems. We show that all three types of subsystems can be considered as solutions to some particular systems of fuzzy relation inequalities and equations. Especially important role in our research play the fuzzy quasi-order \( Q \), and fuzzy equivalence \( E \), generated by fuzzy transition relations, which can be efficiently computed using the well-known algorithms for computing the transitive closure of a fuzzy relation. In particular, we characterize subsystems, reverse subsystems and double subsystems respectively as fuzzy fuzzy sets of \( Q \), \( Q^{-1} \) and \( E \) (in the sense of Sanchez [83]), and we also characterize them as linear combinations of aftersets and foresets of \( Q \) and equivalence classes of \( E \) (Theorems 4.3, 4.7 and 4.10). We also show that subsystems, reverse subsystems and double subsystems of a fuzzy transition system \( \mathcal{F} \) form both closure and opening systems in the lattice of fuzzy subsets of \( A \), where \( A \) is the set of states of \( \mathcal{F} \) (Proposition 5.1), and we provide efficient procedures for computing related closures and openings of an arbitrary fuzzy subset of \( A \) (Theorem 5.2). These procedures simply boil down to computing the fuzzy quasi-order \( Q \) or the fuzzy equivalence \( E \).

The results obtained here generalize the results on subsystems of fuzzy transition systems over the Gödel structure from [24, 63, 66, 87], and the results from [13–16] on subsystems, reverse subsystems, double subsystems and direct sum decompositions of ordinary transition systems. Moreover, they are closely related to results from [8, 10] on closure and opening operators defined by fuzzy relations. Compared with these papers, the main advantages of our approach are as follows. We study not only the subsystems and the dual concept of reverse subsystems, but also double subsystems that have been discussed only in the context of ordinary transition systems, in relation to study of direct sum decompositions of transition systems [13–16]. In contrast to the previous articles, where only closures related to subsystems have been studied, here we study both closures and openings related to subsystems, reverse subsystems and double subsystems, and moreover, we deal with a more general structure of truth values. Das in [24] proposed a method for computing closures related to subsystems, but his approach requires computation of all composite fuzzy transition relations, whose number can be exponential in the number of states, and for some structures of truth values this number may even be infinite. This makes his approach computationally inefficient. Our approach does not require computation of composite fuzzy transition relations, instead we have to compute only the union of basic fuzzy transition relations and its reflexive-transitive closure, and then to compute the composition of the resulting fuzzy relation and a fuzzy set. Computationally the most demanding part of this procedure is to compute the transitive closure of a fuzzy relation. When transitivity is defined by a triangular norm on the real unit interval, the transitive closure can be computed using some of the algorithms provided in [27, 28, 55, 68, 90]. Some of them have an impressive complexity of \( O(n^3) \) [68], or even \( O(n^2) \) [28], for the min-transitive closure, where \( n \) is the number of elements of the underlying set, i.e., in our case, the number of states of the considered fuzzy transition system. Hence, our algorithms can achieve the same overall complexity.

Systems of fuzzy relation inequalities that we use to define subsystems, reverse subsystems and double subsystems of a fuzzy transition system are closely related to the so-called weakly linear systems of fuzzy relation inequalities, which have been recently studied in [42, 45, 47], and in [20–23, 89] have been used in the study of fuzzy automata. These systems have similar forms, they consist of inequalities defined by residuated functions, and consequently, closures and openings related to subsystems, reverse subsystems and double subsystems can also be computed using the methodology developed in [42, 45, 47].
However, this methodology is based on an iterative procedure which does not necessarily finishes in a finite number of iterations, depending on certain local properties of the complete residuated lattice that is used as the structure of truth values. In contrast, the methodology developed here provides results for each complete residuated lattice, regardless of its local properties.

The structure of the paper is as follows. In Section 2 we introduce basic notions and notation concerning complete residuated lattices, fuzzy sets, fuzzy relations, fuzzy transition systems and fuzzy automata. In Section 3 we present the basic properties and give the construction of the fuzzy quasi-order and fuzzy equivalence generated by fuzzy transition relations of a fuzzy transition system. Section 4 contains our main results characterizing subsystems, reverse subsystems and double subsystems of a fuzzy transition system. Then in Section 5 we show that subsystems, reverse subsystems and double subsystems form both closure and opening systems and describe the corresponding closure and opening operators.

2. Preliminaries

The terminology and basic notions in this section are according to [4, 5, 29, 33, 34, 36, 37, 51]. For more information on lattices and related concepts we refer to books [6, 7, 78], as well as to books [4, 5, 34, 51], for more information on fuzzy sets and fuzzy relations.

2.1. Complete residuated lattices

We will use complete residuated lattices as the structures of membership (truth) values. Residuated lattices are a very general algebraic structure and generalize many algebras with very important applications (cf. [4, 5, 39, 40]).

A residuated lattice is an algebra \( L = (\mathcal{L}, \wedge, \vee, \otimes, \rightarrow, 0, 1) \) such that

- \((L1)\) \( (L, \wedge, \vee, 0, 1) \) is a lattice with the least element 0 and the greatest element 1,
- \((L2)\) \( (L, \otimes, 1) \) is a commutative monoid with the unit 1,
- \((L3)\) \( \otimes \) and \( \rightarrow \) form an adjoint pair, i.e., they satisfy the adjunction property: for all \( x, y, z \in L \),

\[
x \otimes y \leq z \iff x \leq y \rightarrow z. \tag{1}
\]

If, in addition, \((L, \wedge, \vee, 0, 1)\) is a complete lattice, then \( L \) is called a complete residuated lattice. Emphasizing their monoidal structure, in some sources residuated lattices are called integral, commutative, residuated \( t \)-monoids [40].

The operations \( \otimes \) (called multiplication) and \( \rightarrow \) (called residuum) are intended for modeling the conjunction and implication of the corresponding logical calculus, and supremum (\( \vee \)) and infimum (\( \wedge \)) are intended for modeling of the existential and general quantifier, respectively. An operation \( \leftrightarrow \) defined by

\[
x \leftrightarrow y = (x \rightarrow y) \wedge (y \rightarrow x), \tag{2}
\]
called biresiduum (or biimplication), is used for modeling the equivalence of truth values, and a unary operation \( \neg \) defined by \( \neg x = x \rightarrow 0 \) is called the negation.

It can be easily verified that with respect to \( \leq, \otimes \) is isotone in both arguments, \( \rightarrow \) is isotone in the second and antitone in the first argument, and for any \( x, y, z \in L \) and any \( \{x_i \}_{i \in I}, \{y_i \}_{i \in I} \subseteq L \), the following hold:

\[
\left( \bigvee_{i \in I} x_i \right) \otimes x = \bigvee_{i \in I} (x_i \otimes x), \tag{3}
\]
\[
(x \otimes y) \rightarrow z = x \rightarrow (y \rightarrow z), \tag{4}
\]
\[
\left( \bigvee_{i \in I} x_i \right) \rightarrow y = \bigvee_{i \in I} (x_i \rightarrow y), \tag{5}
\]
\[
x \leq y \Rightarrow \neg y \leq \neg x, \tag{6}
\]
\[
x \leq \neg x. \tag{7}
\]

In general, the opposite inequality in (7) does not hold, and if it holds, i.e., if \( x = \neg x \), for each \( x \in L \), then we say that the complete residuated lattice \( L \) has the double negation property. Note that if \( L \) has the double negation property, then the opposite implication in (6) also hold. For other properties of complete residuated lattices we refer to [4, 5].

The most studied and applied structures of truth values, defined on the real unit interval \([0, 1]\) with \( x \wedge y = \min(x, y) \) and \( x \vee y = \max(x, y) \), are the Lukasiewicz structure (where \( x \otimes y = \max(x + y - 1, 0) \), \( x \rightarrow y = \min(1 - x + y, 1) \)), the Goguen (product) structure \( x \otimes y = x \cdot y, x \rightarrow y = 1 \) if \( x \leq y \), and \( y/x \) otherwise, and the Gödel structure \( x \otimes y = \min(x, y), x \rightarrow y = 1 \) if \( x \leq y \), and \( y \) otherwise. More generally, an algebra \((\{0, 1\}, \wedge, \vee, \otimes, \rightarrow, 0, 1)\) is a complete residuated lattice if and only if \( \otimes \) is a left-continuous \( t \)-norm and the residuum is defined by \( x \rightarrow y = \bigvee \{u \in [0, 1] \mid u \otimes x \leq y\} \) (cf. [5]). Another important set of truth values is the set \( \{a_0, a_1, \ldots, a_n\}, ~0 = a_0 < \cdots < a_n = 1 \), with \( a_i \otimes a_i = a_{\max(i-n, 0)} \) and \( a_i \rightarrow a_i = a_{\min(n-i, n)} \). A special case of the latter algebras is the two-element Boolean algebra of classical logic with the support \([0, 1]\). The only adjoint pair on the two-element Boolean algebra consists of the classical conjunction and implication operations. This structure of truth values we call the Boolean structure. Note that the Łukasiewicz structure has the double negation property, but the Goguen and Gödel structure do not have this property.

2.2. Fuzzy sets

In this paper we deal with fuzzy sets, fuzzy relations, fuzzy transition systems and fuzzy automata with membership values in complete residuated lattices. Therefore, in the rest of the paper, if not noted otherwise, \( L \) will denote a complete residuated lattice.

A fuzzy subset of a set \( A \) over \( L \), or simply a fuzzy subset of \( A \), is any function from \( A \) into \( L \). Ordinary crisp subsets of \( A \) are considered as fuzzy subsets of \( A \) taking membership values in the set \([0, 1] \subseteq L \). The set of all fuzzy subsets of \( A \) over \( L \) will be denoted by \( L^A \). Let \( f \) and \( g \) be
two fuzzy subsets of $A$. The equality of $f$ and $g$ is defined as the usual equality of functions, i.e., $f = g$ if and only if $f(x) = g(x)$, for every $x \in A$. The inclusion $f \leq g$ is also defined pointwise: $f \leq g$ if and only if $f(x) \leq g(x)$, for every $x \in A$. The meet (intersection) $\wedge_{i \in I} f_i$ and the join (union) $\vee_{i \in I} f_i$ of an arbitrary family $\{f_i\}_{i \in I}$ of fuzzy subsets of $A$ are functions from $A$ into $I$ defined by

$$\left(\bigwedge_{i \in I} f_i\right)(x) = \bigwedge_{i \in I} f_i(x),$$

$$\left(\bigvee_{i \in I} f_i\right)(x) = \bigvee_{i \in I} f_i(x).$$

In addition, the product $f \otimes g$ is a fuzzy subset defined by $f \otimes g(x) = f(x) \otimes g(x)$, for every $x \in A$. The quintuple $\mathcal{F}(A) = (L^A, \wedge, \wedge, \emptyset, A)$ is a complete lattice with the least element $\emptyset$ and the greatest element $A$, and it is called the lattice of fuzzy subsets of $A$. The complement $f^-$ of a fuzzy subset $f$ of $A$ is a fuzzy subset defined by $f^-(x) = -f(x)$, for every $x \in A$, and the crisp part of $f$ is the crisp subset $f = \{a \in A | f(a) = 1\}$ of $A$. We will also consider $\neg$ as a function $\neg: A \to L$ defined by $\neg(a) = 1$, if $f(a) = 1$, and $\neg(a) = 0$, if $f(a) < 1$. A subset $\mathcal{C}$ of $L^A$ is called a closure system in $\mathcal{F}(A)$ if $A \in \mathcal{C}$ and $\mathcal{C}$ is closed under arbitrary meets. If $\mathcal{C}$ is a closure system in $\mathcal{F}(A)$ and $\mathcal{X} = \{f_i\}_{i \in I}$ is a family of fuzzy subsets of $A$, then

$$\mathcal{C}_x = \bigwedge\left\{g \in \mathcal{C} \mid f_i \leq g, \text{ for every } i \in I\right\}$$

(8)

is the least element of $\mathcal{C}$ containing all members of $\mathcal{X}$, and it is called the $\mathcal{C}$-closure of the family $\mathcal{X}$. In particular, the $\mathcal{C}$-closure of a family containing only one fuzzy subset $f$ is denoted by $\mathcal{C}_f$, and it is called the $\mathcal{C}$-closure of $f$. If $\mathcal{X} = \{f_i\}_{i \in I}$ is a family of fuzzy subsets of $A$, then it can be easily shown that $\mathcal{C}_x = \mathcal{C}_{f_1}$, where and $f = \bigvee_{i \in I} f_i$. In other words, the $\mathcal{C}$-closure of a family of fuzzy sets coincides with the $\mathcal{C}$-closure of the union of this family.

Analogously, a subset $\mathcal{C}$ of $L^A$ is called an opening (or interior) system in $\mathcal{F}(A)$ if $\emptyset \in \mathcal{C}$ and $\mathcal{C}$ is closed under arbitrary joins. If $\mathcal{C}$ is an opening system in $\mathcal{F}(A)$ and $\mathcal{X} = \{f_i\}_{i \in I}$ is a family of fuzzy subsets of $A$, then

$$\mathcal{C}_x = \bigvee\left\{g \in \mathcal{C} \mid g \leq f_i, \text{ for every } i \in I\right\}$$

(9)

is the least element of $\mathcal{C}$ containing all members of $\mathcal{X}$, and it is called the $\mathcal{C}$-opening of $\mathcal{X}$. In particular, the $\mathcal{C}$-opening of a family containing only one fuzzy subset $f$ is called the $\mathcal{C}$-opening of $f$. It can be also shown that the $\mathcal{C}$-opening of a family of fuzzy sets coincides with the $\mathcal{C}$-opening of the intersection of this family.

If $\mathcal{C}$ is a closure system in $\mathcal{F}(A)$, then for any $a \in A$ we have that the family $\{f \in \mathcal{C} | f(a) = 1\}$ is non-empty, since it contains $A$, so

$$\mathcal{C}_a = \bigwedge\left\{f \in \mathcal{C} \mid f(a) = 1\right\} \in \mathcal{C}.$$  

(10)

In other words, $\mathcal{C}_a$ is the $\mathcal{C}$-closure of the crisp subset $[a]$ of $A$. The fuzzy set $\mathcal{C}_a$ will be called the principal element of $\mathcal{C}$ generated by $a$, and the set $\mathcal{P}(\mathcal{C}) = \{\mathcal{C}_a | x \in A\}$ will be called the principal part of $\mathcal{C}$.

Let $\mathcal{L} = (S, \otimes, \emptyset, 0, 1)$ be a semiring with the zero $0$ and the identity $1$. By a left $\mathcal{L}$-semimodule we mean a commutative monoid $(M, +, 0)$ for which an external multiplication $S \times M \to M$, denoted by $(\lambda, x) \mapsto \lambda x$ and called the left scalar multiplication, is defined and which for all $\lambda, \mu, x \in S$ and $x, x_1, x_2 \in M$ satisfies the following equalities:

$$(\lambda_1 \otimes \lambda_2)x = \lambda_1(\lambda_2x),$$

(11)

$$\lambda(x_1 + x_2) = \lambda x_1 + \lambda x_2,$$

(12)

$$(\lambda_1 \otimes \lambda_2)x = \lambda_1x + \lambda_2x,$$

(13)

$$1x = x,$$

(14)

$$\lambda 0 = 0x = 0.$$  

(15)

The definition of a right $\mathcal{L}$-semimodule is analogous, where the external multiplication is a function $M \times S \to M$, denoted by $(x, \lambda) \mapsto x\lambda$ and called the right scalar multiplication, and conditions dual to (11)–(15) are satisfied.

An $\mathcal{L}$-bimodule is a commutative monoid $(M, +, 0)$ which is both left and right $\mathcal{L}$-semimodule, and for all $\lambda, \mu \in S$ and $x \in M$ the following is true:

$$(\lambda x)\mu = \lambda(x\mu).$$

(16)

If, in addition, left and right scalar multiplications coincide, i.e., if $\lambda x = x\lambda$, for all $\lambda \in S$ and $x \in M$, then we say that $(M, +, 0)$ is a strong $\mathcal{L}$-bimodule (cf. [29, 36, 37]).

If $\mathcal{L} = (L, \wedge, \wedge, \emptyset, 0, 1)$ is a complete residuated lattice, the algebra $\mathcal{L}^\wedge = (L, \vee, \wedge, 0, 1)$ is a semiring and will be called the semiring reduct of $\mathcal{L}$. Let us consider the complete lattice $\mathcal{F}(A) = (L^A, \wedge, \wedge, \emptyset, A)$ of all fuzzy subsets of a set $A$ with membership values in $\mathcal{L}$. For any $\lambda \in L$ and $f \in L^A$ we define the left scalar multiplication $\lambda f$ and the right scalar multiplication $f \lambda$ as follows: $\lambda f(a) = \lambda \otimes f(a)$ and $f \lambda(a) = f(a) \otimes \lambda$, for every $a \in A$. Due to commutativity of the multiplication $\otimes$, we have that the left and right scalar multiplications coincide, i.e., $\lambda f = f \lambda$, for all $a \in A$, and hence, the monoid $(L^A, \vee, 0)$ forms a strong $\mathcal{L}^\wedge$-bimodule. Then, by a scalar multiplication we will mean either the left or the right scalar multiplication, but we will use just the left notation $(\lambda, f) \mapsto \lambda f$ to denote it. The lattice $\mathcal{F}(A)$ equipped with this scalar multiplication will be denoted by $\mathcal{F}_0(A)$ and called the $\mathcal{L}$-lattice of fuzzy subsets of the set $A$. A fuzzy subset $f \in L^A$ is said to be a linear combination of fuzzy subsets $f_i \in L^A (i \in I)$ if there exist scalars $\lambda_i \in L (i \in I)$ such that $f$ can be expressed in the form

$$f = \bigvee_{i \in I} \lambda_i f_i.$$  

(17)

Any subset of $L^A$ which is closed under scalar multiplication and arbitrary meets and joins, and it contains the least and the greatest element of $\mathcal{F}(A)$ will be called a complete $\mathcal{L}$-sublattice of $\mathcal{F}_0(A)$.
2.3. Fuzzy relations

Let A be a non-empty set. A fuzzy relation on a set A is any function from $A \times A$ into $L$, that is, to say, any fuzzy subset of $A \times A$. Therefore, the equality, inclusion (ordering), joins and meets of fuzzy relations are defined as for fuzzy sets, and the quintuple $\mathcal{A}(A) = (L^{A \times A}, \lor, \land, \emptyset, A \times A)$ is a complete lattice, which is called the lattice of fuzzy relations on A. In fact, this lattice is nothing other than the lattice $\mathcal{A}(A \times A)$ of all fuzzy subsets of $A \times A$. The converse (in some sources called inverse or transpose) of a fuzzy relation $R \in L^{A \times A}$ is a fuzzy relation $R^{-1} \in L^{A \times A}$ defined by $R^{-1}(a, b) = R(b,a)$, for all $a, b \in A$. A crisp relation is a fuzzy relation which takes values only in the set $\{0, 1\}$, and if $R$ is a crisp relation on A, then expressions “$R(a,b) = 1$” and “$(a, b) \in R$” will have the same meaning. By $\mathcal{V}_A$ we denote the universal relation on a set A, which is given by $\mathcal{V}_A(a,b) = 1$, for all $a, b \in A$, and by $\Delta_A$ we denote the equality relation on A, which is given by $\Delta_A(a,b) = 1$, if $a = b$, and $\Delta_A(a,b) = 0$, if $a \neq b$, for all $a, b \in A$.

For fuzzy relations $R, S \in L^{A \times A}$, their composition $R \circ S$ is a fuzzy relation on A defined by

$$
(R \circ S)(a,b) = \bigvee_{c \in A} R(a,c) \otimes S(c,b),
$$

(18)

for all $a, c \in A$. If $R$ and $S$ are crisp relations, then $R \circ S = \{(a, b) \in A \times A \mid (3c \in A) (a, c) \in R \& (c, b) \in S\}$, i.e., $R \circ S$ is an ordinary composition of relations, and if $R$ and $S$ are functions, then $R \circ S$ is an ordinary composition of functions, i.e., $(R \circ S)(a) = S(R(a))$, for every $a \in A$. Furthermore, if $f \in L^A$, $R \in L^{A \times A}$ and $g \in L^A$, the compositions $f \circ R$ and $R \circ g$ are fuzzy subsets of A which are defined by

$$
(f \circ R)(a) = \bigvee_{b \in A} f(b) \otimes R(b,a),
$$

(19)

$$(R \circ g)(a) = \bigvee_{b \in A} R(a,b) \otimes g(b),$$

(20)

for every $a \in A$. In particular, for $f, g \in L^A$ we write

$$
f \circ g = \bigvee_{a \in A} f(a) \otimes g(a).
$$

(21)

The value $f \circ g$ can be interpreted as the “degree of overlapping” of $f$ and $g$. In particular, if $f$ and $g$ are crisp sets and $R$ is a crisp relation, then $f \circ R = \{a \in A \mid (3b \in f)(b,a) \in R\}$ and $R \circ g = \{a \in A \mid (3b \in g)(a,b) \in R\}$. For arbitrary $R, S, T \in L^{A \times A}$ we have

$$(R \circ S) \circ T = R \circ (S \circ T),
$$

(22)

(i.e., composition of fuzzy relations is associative), and

$$
R \subseteq S \text{ implies } R^{-1} \subseteq S^{-1},
$$

(23)

$$
R \subseteq S \text{ implies } T \circ R \subseteq T \circ S \text{ and } R \circ T \subseteq S \circ T.
$$

(24)

Further, for any $R, S \in L^{A \times A}$ and $f, g, h \in L^A$ we can easily verify that

$$
(f \circ R) \circ S = f \circ (R \circ S),
$$

(25)

$$
(f \circ R) \circ g = f \circ (R \circ g),
$$

(26)

$$
(R \circ S) \circ h = R \circ (S \circ h)
$$

(27)

and consequently, the parentheses in (23) can be omitted, as well as the parentheses in (21).

For a fuzzy relation $R \in L^{A \times A}$ and $n \in \mathbb{N}$ (where $\mathbb{N}$ denotes the set of natural numbers without zero included and $\mathbb{N}^0 = \mathbb{N} \cup \{0\}$), we define the $n$-th power $R^n$ of $R$ inductively, as follows: $R^0 = \Delta_A$, $R^{n+1} = R^n \circ R$, for each $n \in \mathbb{N}^0$. Clearly, $R^1 = R$.

Finally, for all $R, R_i, S \in L^{A \times A}$ ($i \in I$) we have that

$$
(R \circ S)^{-1} = S^{-1} \circ R^{-1},
$$

(28)

$$
\left(\bigvee_{i \in I} R_i\right)^{-1} = \bigvee_{i \in I} R_i^{-1},
$$

(29)

$$
S \circ \left(\bigvee_{i \in I} R_i\right) = \bigvee_{i \in I} (S \circ R_i),
$$

(30)

$$
\left(\bigvee_{i \in I} R_i\right) \circ S = \bigvee_{i \in I} (R_i \circ S),
$$

(31)

We note that if A is a finite set of cardinality $|A| = n$, then $R, S \in L^{A \times A}$ can be treated as $n \times n$ fuzzy matrices over $\mathcal{Z}$, and $R \circ S$ is the matrix product. Analogously, for $f, g \in L^A$ we can treat $f \circ R$ as the product of an $1 \times n$ matrix $f$ and an $n \times n$ matrix $R$ (vector-matrix product), $R \circ g$ as the product of an $n \times n$ matrix $R$ and an $n \times 1$ matrix $g$, the transpose of $g$ (matrix-vector product), and $f \circ g$ as the scalar product of vectors $f$ and $g$.

A fuzzy relation $R$ on a set A is said to be

- reflexive (or fuzzy reflexive) if $\Delta_A \subseteq R$, i.e., if $(R(a,a) = 1$, for every $a \in A$;
- symmetric (or fuzzy symmetric) if $R^{-1} \subseteq R$, i.e., if $R(a,b) = R(b,a)$, for all $a, b \in A$;
- transitive (or fuzzy transitive) if $R \circ R \subseteq R$, i.e., if for all $a, b, c \in A$ we have $R(a,b) \otimes R(b,c) \subseteq R(a,c)$.

A reflexive and transitive fuzzy relation on A is called a fuzzy quasi-order, and a reflexive and transitive crisp relation on A is called a quasi-order. In some sources quasi-orders and fuzzy quasi-orders are called preorders and fuzzy preorders, but we use the original name introduced by Birkhoff in [6]. Note that a reflexive fuzzy relation $R$ is a fuzzy quasi-order if and only if $R^2 = R$. A reflexive and symmetric fuzzy relation will be called a fuzzy proximity relation. Note that in some sources they are also called fuzzy tolerance, fuzzy similarity and fuzzy compatibility relations.

A reflexive, symmetric and transitive fuzzy relation on A is called a fuzzy equivalence relation (or just a fuzzy equivalence), and a reflexive, symmetric and transitive crisp relation on A is called an equivalence. If $Q$ is a fuzzy quasi-order on A, then $Q \wedge Q^{-1}$ is the greatest fuzzy equivalence contained in $Q$, and it is called the natural fuzzy equivalence of $Q$. For arbitrary fuzzy quasi-orders $Q_1$ and $Q_2$ on A we have that

$$
Q_1 \subseteq Q_2 \iff Q_1 \circ Q_2 = Q_2 \circ Q_1 = Q_2.
$$

(32)

As we have already noted, the lattice $\mathcal{A}(A)$ of fuzzy relations on A is nothing other than the lattice of fuzzy
subsets of $A \times A$, so the concepts of a closure system and an opening system in $\mathcal{O}(A)$, and the $\mathcal{C}$-closure and the $\mathcal{C}$-opening for $\mathcal{C} \subseteq L^{A \times A}$, are defined as for fuzzy sets. The set $\mathcal{O}(A)$ of all fuzzy quasi-orders on a set $A$, and the set $\mathcal{E}(A)$ of all fuzzy equivalences on $A$, form closure systems in $\mathcal{O}(A)$. The $\mathcal{D}(A)$-closure of a family of fuzzy relations or a single fuzzy relation will be called the fuzzy quasi-order closure, or simply the $\mathcal{D}$-closure, and the $\mathcal{E}(A)$-closure will be called the fuzzy equivalence closure, or simply the $\mathcal{E}$-closure. The sets of all reflexive, symmetric and transitive fuzzy relations are also closure systems in $\mathcal{O}(A)$, and correspondingly we define the reflexive closure, symmetric closure and transitive closure of a family of fuzzy relations or a single fuzzy relation. For a fuzzy relation $R$ on $A$, the reflexive closure $R'$ of $R$ can be expressed as $R' = R \cup \Delta_A$, the symmetric closure $R^s$ of $R$ can be expressed as $R^s = (R^T)^T$, and the transitive closure $R^t$ of $R$ can be expressed as $R^t = \bigvee_{n \in \mathbb{N}} R^n$. A quasi-order $\alpha$ is reflexive, symmetric, and transitive if and only if $\alpha = \alpha^s \cap \alpha^t$.

We define the right residual of $\alpha$ by $\alpha_R = \alpha \cap \alpha^s \cap \alpha^t$. The following proposition can be easily proved using the basic properties of residuated lattices [4, 5]. For some related results we refer to [8, 10].

Proposition 2.1. Let $R$ be an arbitrary fuzzy relation on a set $A$. Then the following is true:

(a) Functions $a \mapsto a \circ \alpha$, $a \mapsto \alpha \setminus a$, $a \mapsto \alpha R a$ and $a \mapsto R \setminus a$ are isotope functions of the lattice $\mathcal{O}(A)$ into itself.

(b) If $R$ is reflexive, then functions $a \mapsto a \circ \alpha$ and $a \mapsto \alpha R a$ are extensive, and functions $a \mapsto a / \alpha$ and $a \mapsto R \setminus a$ are intensive.

(c) If $R$ is a fuzzy quasi-order, then all functions $a \mapsto a \circ R$, $a \mapsto a / R$, $\alpha \mapsto \alpha R a$ and $a \mapsto R \setminus a$ are idempotent.

Let $(P, \preceq)$ be a partially ordered set and let $\phi : P \to P$ be a function. If $x \preceq y$ implies $\phi(x) \preceq \phi(y)$, for all $x, y, P$, then $\phi$ is called isotone, if $x \preceq \phi(x)$, for each $x \in P$, then it is called extensive, if $\phi(x) \preceq x$, for each $x \in P$, then it is called intensive, and if $\phi(\phi(x)) = \phi(x)$, for each $x \in P$, then it is called idempotent. An isotone, extensive and idempotent function is called a closure operator on $(P, \preceq)$, and each $x \in P$ for which $\phi(x) = x$ is called a $\phi$-closed element. On the other hand, an isotone, intensive and idempotent function is called an opening operator on $(P, \preceq)$, and each $x \in P$ for which $\phi(x) = x$ is called a $\phi$-open element. In particular, if $\phi$ is a closure operator on the lattice $\mathcal{O}(A)$, then the set of all $\phi$-closed elements form a closure system in $\mathcal{O}(A)$, and conversely, for every closure system $C$ in $\mathcal{O}(A)$ there is a unique closure operator $\phi$ on $\mathcal{O}(A)$ such that $C$ is the set of all $\phi$-closed elements. Analogously, if $\phi$ is an opening operator on $\mathcal{O}(A)$, then the set of all $\phi$-open elements is an opening system in $\mathcal{O}(A)$, and conversely, for every opening system $C$ in $\mathcal{O}(A)$ there is a unique opening operator $\phi$ on $\mathcal{O}(A)$ such that $C$ is the set of all $\phi$-open elements.

According to Proposition 2.1, for any fuzzy quasi-order $Q$ on $A$, functions $a \mapsto a \circ Q$ and $a \mapsto Q \setminus a$ are closure operators on $\mathcal{O}(A)$, and functions $a \mapsto a / Q$ and $a \mapsto Q \setminus a$ are opening operators on $\mathcal{O}(A)$.
2.4. Fuzzy transition systems and fuzzy automata

A fuzzy transition system over $\mathcal{L}$, or simply a fuzzy transition system, is a triple $\mathcal{T} = (X, X, \delta)$, where $A$ and $X$ are non-empty sets, called respectively the set of states and the input alphabet, and $\delta : A \times X \times A \rightarrow L$ is a fuzzy subset of $A \times X \times A$, called the fuzzy transition function. We can interpret $\delta(a, x, b)$ as the degree to which an input letter $x \in X$ causes a transition from a state $a \in A$ into a state $b \in A$. For methodological reasons we sometimes allow the set of states $A$ and the input alphabet $X$ to be infinite. A fuzzy transition system whose set of states and the input alphabet are finite is called a fuzzy finite transition system.

A fuzzy automaton over $\mathcal{L}$, or simply a fuzzy automaton, is a quintuple $\mathcal{A} = (X, X, \delta, \sigma, \tau)$, where $A$, $X$ and $\delta$ are as above, and $\sigma, \tau \in L^A$ are fuzzy subsets of $A$, called respectively the fuzzy set of initial states and the fuzzy set of terminal states. We can interpret $\sigma(a)$ and $\tau(a)$ as the degrees to which $a$ is respectively an input state and a terminal state. A fuzzy automaton whose set of states and input alphabet are finite is called a fuzzy finite automaton.

Let $X^*$ and $X^+$ denote respectively the free semigroup and the free monoid over the alphabet $X$, and let $e \in X^*$ be the empty word. We can extend the fuzzy transition function $\delta : A \times X \times A \rightarrow L$ up to a function $\delta' : A \times X^* \times A \rightarrow L$, as follows: If $a, b \in A$, then
\[
\delta'(a, e, b) = \begin{cases} 1, & \text{if } a = b, \\ 0, & \text{otherwise,} \end{cases} \tag{32}
\]
and if $a, b \in A$, $u \in X^*$ and $x \in X$, then
\[
\delta'(a, ux, b) = \bigvee_{c \in A} \delta'(a, u, c) \otimes \delta(c, x, b). \tag{33}
\]

By (3) and Theorem 3.1 [60] (see also [74, 75, 77]), we have that
\[
\delta'(a, uv, b) = \bigvee_{c \in A} \delta'(a, u, c) \otimes \delta'(c, v, b), \tag{34}
\]
for all $a, b \in A$ and $u, v \in X^*$, i.e., if $w = x_1 \cdots x_n$, for $x_1, \ldots, x_n \in X$, then
\[
\delta'(a, w, b) = \bigvee_{(c_1, \ldots, c_{n-1}) \in A^{n-1}} \delta(a, x_1, c_1) \otimes \cdots \otimes \delta(c_{n-1}, x_n, b). \tag{35}
\]

Intuitively, the product $\delta(a, x_1, c_1) \otimes \cdots \otimes \delta(c_{n-1}, x_n, b)$ represents the degree to which the input word $w$ causes a transition from a state $a$ into a state $b$ through the sequence of intermediate states $c_1, \ldots, c_{n-1} \in A$, and $\delta'(a, w, b)$ represents the supremum of degrees of all possible transitions from $a$ into $b$ caused by $w$. Also, we can visualize a fuzzy finite automaton $\mathcal{A}$ representing it as a labelled directed graph whose nodes are states of $\mathcal{A}$, and an edge from a node $a$ into a node $b$ is labelled by pairs of the form $x/\delta(a, x, b)$, for any $x \in X$.

The reverse fuzzy transition system of a fuzzy transition system $\mathcal{T} = (A, X, \delta)$ is defined as the fuzzy transition system $\mathcal{T} = (A, X, \delta)$ whose fuzzy transition function $\delta : A \times X \times A \rightarrow L$ is defined by $\delta(a, x, b) = \delta(b, x, a)$, for all $a, b \in A$ and $x \in X$. In other words, $\delta x = \delta^{-1} x$, for each $x \in X$. Similarly, the reverse fuzzy automaton of a fuzzy automaton $\mathcal{A} = (X, X, \delta, \sigma, \tau)$ is the fuzzy automaton $\mathcal{A} = (A, X, \delta, \sigma, \tau)$, where the fuzzy transition function $\delta$ is given as above, and the fuzzy sets $\sigma$ and $\tau$ of initial and terminal states are defined by $\delta = \tau$ and $\tau = \sigma$.

If $\mathcal{T} = (A, X, \delta)$ is a fuzzy transition system such that $\delta : A \times X \times A \rightarrow [0, 1]$, i.e., $\delta$ is a crisp subset of $A \times X \times A$, then $\mathcal{T}$ is an ordinary labelled transition system (cf. [1, 64, 65]), or just a transition system, and $\delta$ is a function from $A \times X$ to $A$, then $\mathcal{T}$ is a deterministic transition system. Also, if $\mathcal{A} = (X, X, \delta, \sigma, \tau)$ is a fuzzy automaton such that $\delta$ is a crisp subset of $A \times X \times A$ and $\sigma$ and $\tau$ are crisp subsets of $A$, then $\mathcal{A}$ is an ordinary nondeterministic automaton, and if $\delta$ is a function from $A \times X$ to $A$, then $\mathcal{A}$ is a deterministic automaton. In other words, ordinary transition systems and nondeterministic automata are respectively fuzzy transition systems and fuzzy automata over the Boolean structure. If $\mathcal{A} = (X, X, \delta, \sigma, \tau)$ is a fuzzy automaton such that $\delta$ is a function of $A \times X$ into $A$, $\sigma$ is an one-element crisp subset of $A$, that is, $\sigma = \{a_0\}$, for some $a_0 \in A$, and $\tau$ is a crisp subset of $A$, then $\mathcal{A}$ is called a deterministic fuzzy automaton, and it is denoted by $\mathcal{A} = (A, X, \delta, a_0, \tau)$. In [17, 31] the name crisp-deterministic was used. For more information on deterministic fuzzy automata we refer to [3, 41, 43, 46, 60]. Evidently, if $\delta$ is a crisp subset of $A \times X \times A$, or a function of $A \times X$ into $A$, then $\delta$ is also a crisp subset of $A \times X \times A$, or a function of $A \times X$ into $A$, respectively. A deterministic fuzzy automaton $\mathcal{A} = (A, X, \delta, a_0, \tau)$, where $\tau$ is a crisp subset of $A$, is an ordinary deterministic automaton.

Let $\delta$ be the fuzzy transition function of a fuzzy transition system $\mathcal{T} = (A, X, \delta)$ or a fuzzy automaton $\mathcal{A} = (A, X, \delta, \sigma, \tau)$. For any $u \in X^*$ we define a fuzzy relation $\delta_u$ on $A$ by
\[
\delta_u(a, b) = \delta'(a, u, b), \tag{36}
\]
for all $a, b \in A$. The fuzzy relation $\delta_u$ is called the fuzzy transition relation determined by $u$. Note that now (34) can be written as
\[
\delta_{uv} = \delta_u \circ \delta_v, \tag{37}
\]
for all $u, v \in X^*$. This means that the algebra $M(\mathcal{T}) = \{(\delta_u)_{u \in X^*}, \circ, \delta_0\}$ is a monoid with the identity $\delta_0 = \Delta_A$, and it is called the transition monoid of the fuzzy transition system $\mathcal{T}$. In the case when we deal with a fuzzy automaton $\mathcal{A}$, we talk about the transition monoid of a fuzzy automaton and we denote it by $M(\mathcal{A})$. For a procedure for computing the transition monoid of a fuzzy transition monoid or a fuzzy automaton we refer to [46] (see also [17]).
3. \(\mathcal{D}\)-closures and \(\mathcal{E}\)-closures of fuzzy transition relations

Let \(\delta\) be the fuzzy transition function of a fuzzy transition system \(T = (A, X, \delta)\) or a fuzzy automaton \(\mathcal{A} = (A, X, \delta, \sigma, \tau)\). Define fuzzy relations \(V_\delta\) and \(Q_\delta\) on \(A\) as follows:

\[
V_\delta = \bigvee_{x \in X} \delta_x, \quad Q_\delta = \bigvee_{u \in X} \delta_u. \tag{38}
\]

Then we have the following:

**Theorem 3.1.** Let \(\delta\) be the fuzzy transition function of a fuzzy transition system \(T = (A, X, \delta)\) or a fuzzy automaton \(\mathcal{A} = (A, X, \delta, \sigma, \tau)\). Then \(Q_\delta\) is a fuzzy quasi-order, and it is the \(\mathcal{D}\)-closure of the families \(\{\delta_x\}_{x \in X}\) and \(\{\delta_u\}_{u \in X}\), and the fuzzy relation \(V_\delta\).

The previous theorem gives an efficient procedure for computing the fuzzy quasi-order \(Q_\delta\). Namely, computation of \(Q_\delta\) by means of the second formula in (38) (the definition of \(Q_\delta\)) could lead to problems, because it requires computation of all fuzzy transition relations \(\delta_u, u \in X\). However, the number of these fuzzy transition relations can be exponential in the number of states of a considered fuzzy transition system or a fuzzy automaton, and for some structures of membership values it may even be infinite. This makes such a procedure theoretically inefficient. Nevertheless, \(Q_\delta\) can be efficiently computed as the reflexive-transitive closure of the fuzzy relation \(V_\delta\), using some of the algorithms provided in [25, 27, 28, 55, 90, 91].

The next theorem gives an interesting characterization of the fuzzy quasi-order \(Q_\delta\), in terms of fuzzy relation inequalities.

**Theorem 3.2.** Let \(\delta\) be the fuzzy transition function of a fuzzy transition system \(T = (A, X, \delta)\) or a fuzzy automaton \(\mathcal{A} = (A, X, \delta, \sigma, \tau)\). Then \(Q_\delta\) is the least reflexive solution to any of the following systems of fuzzy relation inequalities and fuzzy relation inequalities:

\[
\begin{align*}
U \circ \delta_x &\leq U \quad (x \in X); \\
\delta_x \circ U &\leq U \quad (x \in X); \\
U \circ \delta_x &\leq U_x, \quad \delta_x \circ U \leq U \quad (x \in X); \\
U \circ V_\delta &\leq U; \\
V_\delta \circ U &\leq U; \\
U \circ V_\delta &\leq U, \quad V_\delta \circ U \leq U,
\end{align*}
\]

where \(U\) is an unknown taking values in \(\mathcal{A}(A)\).

In Section 2.3 we have seen that the \(\mathcal{E}\)-closure \(R^*\) of a fuzzy relation \(R\) can be expressed as \(R^* = ((R^*)^*)^*\). Here we give another characterization of \(R^*\) which will be more convenient for our further work. If \(R\) is a fuzzy relation on a set \(A\), then we define fuzzy relations \(R^a\) and \(R^f\) on \(A\) as follows:

\[
R^a = R \circ R^{-1}, \quad R^f = R^{-1} \circ R. \tag{45}
\]

It is clear that \(R^a\) and \(R^f\) are symmetric fuzzy relations, and if \(R\) is reflexive, then \(R^a\) and \(R^f\) are also reflexive, \(R \subseteq R^a\) and \(R \subseteq R^f\). Moreover, for arbitrary \(a, b \in A\) we have that

\[
\begin{align*}
R^a(a, b) &= \bigvee_{c \in A} (aR(c) \circ (bR)c), \\
R^f(a, b) &= \bigvee_{c \in A} (Ra(c) \circ (Rb)c), \tag{46}
\end{align*}
\]

i.e., \(R^a(a, b)\) can be understood as the degree of intersection of the \(R\)-aftersets of \(Ra\) and \(bR\), and \(R^f(a, b)\) as the degree of intersection of the \(R\)-foresets of \(Ra\) and \(Rb\). That is the reason why we use the notation \(R^a\) and \(R^f\).

Now we prove the following:

**Proposition 3.3.** For an arbitrary fuzzy relation \(R\) on a set \(A\) we have that \(R^{*} = ((R^{*})^{*})^{*} = (R^{f})^{*}\).

Further, for the fuzzy transition function \(\delta\) of a fuzzy transition system \(T = (A, X, \delta)\) or a fuzzy automaton \(\mathcal{A} = (A, X, \delta, \sigma, \tau)\), by \(E_\delta\) we will denote the \(\mathcal{E}\)-closure of the family \(\{\delta_x\}_{x \in X}\), and the fuzzy relations \(V_\delta\) and \(Q_\delta\).

**Theorem 3.4.** Let \(\delta\) be the fuzzy transition function of a fuzzy transition system \(T = (A, X, \delta)\) or a fuzzy automaton \(\mathcal{A} = (A, X, \delta, \sigma, \tau)\). Then \(E_\delta\) is the \(\mathcal{E}\)-closure of the family \(\{\delta_x\}_{x \in X}\), and the fuzzy relations \(V_\delta\) and \(Q_\delta\).

Moreover, \(E_\delta\) is the transitive closure of \(Q_\delta^a\) and \(Q_\delta^f\).

The fuzzy equivalence \(E_\delta\) can be also characterized in terms of fuzzy relation inequalities, as follows.

**Theorem 3.5.** Let \(\delta\) be the fuzzy transition function of a fuzzy transition system \(T = (A, X, \delta)\) or a fuzzy automaton \(\mathcal{A} = (A, X, \delta, \sigma, \tau)\). Then \(E_\delta\) is the least fuzzy equivalence which is a solution to (39), (40), (42) and (43).

Moreover, \(E_\delta\) is the least fuzzy proximity relation which is a solution to (41) and (44).

4. Subsystems, reverse subsystems and double subsystems of fuzzy transition systems

Let \(T = (A, X, \delta)\) be a fuzzy transition system. A fuzzy subset \(A\) of \(A\) will be called a subsystem of \(T\) if it is forward \(\delta_x\)-closed, for each \(x \in X\), i.e., if \(\delta(a) \circ \delta_x(a, b) \leq a(b)\), for all \(a, b \in A\) and \(x \in X\). Equivalently, \(A\) is a subsystem of \(T\) if it is a solution to a system of fuzzy relation inequalities

\[
\chi \circ \delta_x \leq \chi \quad (x \in X), \tag{47}
\]

where \(\chi\) is an unknown taking values in \(L^A\).

**Example 4.1.** Let \(\mathcal{L}\) be the Gödel structure. Consider a fuzzy transition system \(T = (A, X, \delta)\), where \(|A| = 4, X = \{x, y\}\), and the fuzzy transition relations \(\delta_x\) and \(\delta_y\) are given by the following matrices:

\[
\delta_x = \begin{bmatrix}
1 & 0.8 & 0.6 & 0.8 \\
0.8 & 1 & 0.8 & 0.6 \\
0.2 & 0.3 & 0.8 & 0.9 \\
0.2 & 0.3 & 0.8 & 0.9
\end{bmatrix}, \quad \delta_y = \begin{bmatrix}
0.8 & 1 & 0.6 & 0.8 \\
1 & 0.6 & 0.5 & 0.9 \\
0.3 & 0.2 & 0.4 & 0.8 \\
0.5 & 0.3 & 0.3 & 1
\end{bmatrix}.
\]
For $\alpha = [1 1 0.8 0.9]$ and $\beta = [1 1 0.5 0.5]$ we have that

- $\alpha \circ \delta_x = [1 1 0.8 0.9] = \alpha$,
- $\alpha \circ \delta_y = [1 1 0.6 0.9] < \alpha$,
- $\beta \circ \delta_x = [1 1 0.8 0.8] \not\subseteq \beta$,
- $\beta \circ \delta_y = [1 1 0.6 0.9] \not\subseteq \beta$.

Therefore, $\alpha$ is a subsystem, but $\beta$ is not a subsystem of $T$.

The following assertions show the basic properties of subsystems of a fuzzy transition system.

**Proposition 4.2.** The collection $\mathcal{F}(T)$ of all subsystems of a fuzzy transition system $T = (A,X,\delta)$ forms a complete $\mathcal{L}$-sublattice of $F_0(A)$.

**Theorem 4.3.** Let $T = (A,X,\delta)$ be a fuzzy transition system and let $\alpha$ be a fuzzy subset of $A$. Then the following conditions are equivalent:

(i) $\alpha$ is a subsystem of $T$;
(ii) $\alpha$ is forward $V_0$-closed;
(iii) $\alpha$ is forward $\delta_x$-closed, for every $u \in X^*$;
(iv) $\alpha$ is forward $Q_0$-closed;
(v) $\alpha$ is a solution to a fuzzy relation equation

$$\chi \circ Q_0 = \chi,$$

where $\chi$ is an unknown taking values in $F(A)$;
(vi) $\alpha$ can be represented as a linear combination of $Q_0$-after-sets;
(vii) $\alpha$ is a solution to a fuzzy relation equation

$$\chi / Q_0 = \chi,$$

where $\chi$ is an unknown taking values in $F(A)$.

Again, let $T = (A,X,\delta)$ be a fuzzy transition system. A fuzzy subset $\alpha$ of $A$ will be called a reverse subsystem of $T$ if it is backward $V_0$-closed, for each $x \in X$, i.e., if $\delta_x(\alpha, b) \otimes (a, b) \subseteq (a, a)$, for all $a, b \in A$ and $x \in X$. Equivalently, $\alpha$ is a reverse subsystem of $T$ if it is a solution to a system of fuzzy relation inequalities

$$\delta_x \circ \chi \subseteq \chi \quad (x \in X),$$

where $\chi$ is an unknown taking values in $L^A$.

**Example 4.4.** Let $T = (A,X,\delta)$ be the fuzzy transition system from Example 4.1, and let $\alpha = [1 1 0.8 0.9]$ and $\beta = [1 1 0.5 0.5]$ are the same fuzzy subsets that we have already considered in this example. Then

- $\delta_x \circ \alpha = [1 1 0.9 0.9] \not\subseteq \alpha$,
- $\delta_y \circ \alpha = [1 1 0.8 0.9] = \alpha$,
- $\delta_x \circ \beta = [1 1 0.5 0.5] = \beta$,
- $\delta_y \circ \beta = [1 1 0.5 0.5] = \beta$.

Hence, $\alpha$ is not a reverse subsystem whereas $\beta$ is a reverse subsystem of $T$.

We have that the following is true:

- Proposition 4.5. Let $T = (A,X,\delta)$ be a fuzzy transition system. A fuzzy subset $\alpha$ of $A$ is a reverse subsystem of $T$ if and only if it is a subsystem of the reverse fuzzy transition system $T^\prime$.

The next two propositions and a theorem are direct consequences of the previous proposition, Propositions 4.2 and 5.1, and Theorem 4.3.

**Proposition 4.6.** The collection $\mathcal{F}(T)$ of all reverse subsystems of a fuzzy transition system $T = (A,X,\delta)$ is a complete $\mathcal{L}$-sublattice of $F_0(A)$.

**Theorem 4.7.** Let $T = (A,X,\delta)$ be a fuzzy transition system and let $\alpha$ be a fuzzy subset of $A$. Then the following conditions are equivalent:

(i) $\alpha$ is a reverse subsystem of $T$;
(ii) $\alpha$ is backward $V_0$-closed;
(iii) $\alpha$ is backward $\delta_x$-closed, for every $u \in X^*$;
(iv) $\alpha$ is backward $Q_0$-closed;
(v) $\alpha$ is a solution to a fuzzy relation equation

$$Q_0 \circ \chi = \chi,$$

where $\chi$ is an unknown taking values in $F(A)$;
(vi) $\alpha$ can be represented as a linear combination of $Q_0$-foreset;
(vii) $\alpha$ is a solution to a fuzzy relation equation

$$Q_0 / \chi = \chi,$$

where $\chi$ is an unknown taking values in $F(A)$.

Consider again a fuzzy transition system $T = (A,X,\delta)$. A fuzzy subset $\alpha$ of $A$ will be called a double subsystem of $T$ if it is both a subsystem and a reverse subsystem of $T$, i.e., if it is both forward and backward $\delta_x$-closed, for each $x \in X$. In other words, $\alpha$ is a double subsystem of $T$ if it is a solution to a system of fuzzy relation inequalities

$$\delta_x \circ \chi \subseteq \chi, \quad \delta_y \circ \chi \subseteq \chi \quad (x \in X),$$

where $\chi$ is an unknown taking values in $L^A$.

**Example 4.8.** Let $T = (A,X,\delta)$ be the fuzzy transition system from Example 4.1. Clearly, fuzzy subsets $\alpha$ and $\beta$ discussed in Examples 4.1 and 4.4 are not double subsystems of $T$. However, for a fuzzy subset $\gamma = [1 1 0.9 0.9]$ we have that

$$\gamma \circ \delta_x = [1 1 0.8 0.9] < \gamma, \quad \gamma \circ \delta_y = [1 1 0.6 0.9] < \gamma,$$

and consequently, $\gamma$ is a double subsystem of $T$. It can be easily shown that fuzzy subsets $\gamma' = [0.9 0.9 1 0.9]$ and $\gamma'' = [0.9 0.9 0.9 1]$ are also double subsystems.

**Proposition 4.9.** The collection $\mathcal{F}(T)$ of all double subsystems of a fuzzy transition system $T = (A,X,\delta)$ forms a complete $\mathcal{L}$-sublattice of $F_0(A)$.  

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Theorem 4.10. Let $T = (A, X, \delta)$ be a fuzzy transition system and let $\alpha$ be a fuzzy subset of $A$. Then the following conditions are equivalent:

(i) $\alpha$ is a double subsystem of $T$;
(ii) $\alpha$ is forward and backward $V_\delta$-closed;
(iii) $\alpha$ is forward and backward $\delta_\alpha$-closed, for every $\alpha \in X$;
(iv) $\alpha$ is $E_\delta$-closed;
(v) $\alpha$ is a solution to a fuzzy relation equation

$$\chi \circ E_\delta = \chi,$$

where $\chi$ is an unknown taking values in $T(A)$;
(vi) $\alpha$ can be represented as a linear combination of $E_\alpha$-classes;
(vii) $\alpha$ is a solution to a fuzzy relation equation

$$\chi/E_\delta = \chi,$$

where $\chi$ is an unknown taking values in $T(A)$.

Under different names, subsystems, reverse subsystems and double subsystems have already been studied in [13, 16] (see also [14, 15]) in the context of deterministic transition systems. There has been shown that lattices of subsystems and reverse subsystems are conjugated in the Boolean algebra of subsets of the underlying set of states, in the sense that a set of states is a subsystem if and only if its set-theoretical complement is a reverse subsystem. Also, the lattice of double subsystems is self-conjugated, which means that a set of states is a double subsystem if and only if its set-theoretical complement is a double subsystem. In the fuzzy framework situation is different. Namely, in the general case complete residuated lattices do not possess the double negation property, and we have the following.

Theorem 4.11. Let $T = (A, X, \delta)$ be a fuzzy transition system and $\alpha$ a fuzzy subset of $A$. Then the following is true:

(a) if $\alpha$ is a subsystem, then $\alpha^\neg$ is a reverse subsystem;
(b) if $\alpha$ is a reverse subsystem, then $\alpha^\neg$ is a subsystem;
(c) if $\alpha$ is a double subsystem, then $\alpha^\neg$ is also a double subsystem.

In addition, if the underlying complete residuated lattice $L$ has the double negation property, then the opposite implications also hold.

The next example demonstrates that the double negation property is even a necessary condition for the universal validity of the opposite implications in (a), (b) and (c) of Theorem 4.11. In other words, if the underlying complete residuated lattice $L$ does not have the double negation property, we show that there is a fuzzy transition system and a fuzzy subset $\alpha$ of its set of states whose complement $\alpha^\neg$ is a double subsystem, but $\alpha$ is neither a subsystem nor a reverse subsystem.

Example 4.12. Let $L$ be a complete residuated lattice $L = (L, \land, \lor, \cdot, ., \rightarrow, 0, 1)$ which does not have the double negation property. This means that there is $k \in L$ such that $\neg\neg k \not\equiv k$. Consider a fuzzy transition system $T = (A, X, \delta)$, where $|A| = 2$, $X = \{x\}$ and the fuzzy transition relation $\delta_x$ is given by

$$\delta_x = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix},$$

and a fuzzy subset of $A$ given by $a = [\neg\neg k \not\equiv k]$. Then $a^\neg = [\neg k \not\equiv \neg k]$, and we have that $a^\neg \circ \delta_x = \delta_x \circ a^\neg = a^\neg$, whereas $a \circ \delta_x = \delta_x \circ a = [\neg k \not\equiv \neg k] \not\equiv a$. Therefore, $a^\neg$ is a double subsystem, but it is neither a subsystem nor a reverse subsystem.

Let $T = (A_i, X, \delta_i) (i \in I)$ be a family of ordinary (crisp) transition systems with the same input alphabet and pairwise disjoint sets of states, i.e., $A_i \cap A_j = \emptyset$, for $i \neq j$. Then we can define a new transition system $T = (A, X, \delta)$ by letting

$$A = \bigcup_{i \in I} A_i,$$

and $\delta = \bigcup_{i \in I} \delta_i$.

i.e., if $\delta$ is regarded as a function from $A \times X \times A$ to $\{0, 1\}$, then for all $a, b \in A$ and $x \in X$ we have

$$\delta(a, x, b) = \begin{cases} \delta_i(a, x, b) & \text{if } a, b \in A_i, \text{ for some } i \in I, \\ 0 & \text{otherwise}. \end{cases}$$

The transition system $T$ is called the direct sum of transition systems $T_i (i \in I)$, each $T_i$ is called a direct summand of $T$, and the partition of $A$ whose components are different $A_i (i \in I)$ is called the direct sum decomposition of $T$. Clearly, there are no transitions between states from different direct summands, so everything that happens in the transition system takes place within its direct summands. It is therefore very important to find the largest possible direct sum decomposition of a given transition system $T$, i.e., a direct sum decomposition with the smallest possible summands. This problem was discussed in [13, 16]. Among other things, it was proved that every deterministic transition system has the greatest direct sum decomposition and its summands are direct sum indecomposable. The results obtained in [13, 16] for deterministic transition systems are also applicable to transition systems in general, and also follow from more general results concerning direct sum decompositions of quasi-ordered sets given in [14]. Direct summands of a transition system are precisely double subsystems, and the summands in the greatest direct sum decomposition are principal double subsystems, i.e., equivalence classes of the equivalence relation $E_\delta$. Therefore, computing the greatest direct sum decomposition of a transition system $T$ comes down to computing the corresponding equivalence $E_\delta$. 

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In the fuzzy framework, the fuzzy partition corresponding to the fuzzy equivalence $E_3$ can also be regarded as some kind of direct sum decomposition of a fuzzy transition system $T = (A, X, \delta)$. Namely, the crisp parts of the equivalence classes of $E_3$ form an ordinary partition of $A$ such that there are not absolutely certain transitions (transitions of degree 1) between states from different blocks of this partition.

5. Closures and openings corresponding to subsystems, reverse subsystems and double subsystems

Propositions 4.2, 4.6 and 4.9 have shown that the collections of all subsystems, reverse subsystems and double subsystems of a fuzzy transition system $T = (A, X, \delta)$ form complete $\mathcal{L}$-sublattices of the $\mathcal{L}$-lattice $T_0(A)$. In addition, we have the following.

Proposition 5.1. Let $T = (A, X, \delta)$ be a fuzzy transition system. Then

(a) $\mathcal{I}(T)$, $\mathcal{P}(T)$ and $\mathcal{A}(T)$ are both closure and opening systems in $T(A)$.

(b) The principal part of $\mathcal{I}(T)$ consists of all $Q_0$-aftersets.

(c) The principal part of $\mathcal{A}(T)$ consists of all $Q_0$-foresets.

(d) The principal part of $\mathcal{A}(T)$ consists of all $E_0$-classes.

Now, it naturally arises the question of computing closures and openings of a fuzzy subset of $A$ that correspond to the above mentioned closure and opening systems.

Let $T = (A, X, \delta)$ be a fuzzy transition system and let $\alpha$ be a fuzzy subset of $A$. By $\alpha^{sc}$, $\alpha^{rc}$ and $\alpha^{dc}$ we will denote the $\mathcal{I}(T)$-closure, $\mathcal{P}(T)$-closure and $\mathcal{A}(T)$-closure of $\alpha$, respectively, and by $\alpha^{so}$, $\alpha^{ro}$ and $\alpha^{do}$ we will denote the $\mathcal{I}(T)$-opening, $\mathcal{P}(T)$-opening and $\mathcal{A}(T)$-opening of $\alpha$. In other words, $\alpha^{sc}$, $\alpha^{rc}$ and $\alpha^{dc}$ are respectively the least subsystem, reverse subsystem and double subsystem of $T$ containing $\alpha$, i.e., the least solutions to (47), (50) and (53) (or, equivalently, to (48), (51) and (54)) containing $\alpha$. Similarly, $\alpha^{so}$, $\alpha^{ro}$ and $\alpha^{do}$ are respectively the greatest subsystem, reverse subsystem and double subsystem of $T$ contained in $\alpha$, i.e., the greatest solutions to (47), (50) and (53) (or, equivalently, to (48), (51) and (54)) contained in $\alpha$.

The next theorem characterizes the closures $\alpha^{sc}$, $\alpha^{rc}$ and $\alpha^{dc}$ and the openings $\alpha^{so}$, $\alpha^{ro}$ and $\alpha^{do}$ of $\alpha$.

Theorem 5.2. Let $T = (A, X, \delta)$ be a fuzzy transition system and let $\alpha$ be a fuzzy subset of $A$. Then

\[
\begin{align*}
\alpha^{sc} &= \alpha \circ Q_0, \quad \alpha^{rc} = Q_0 \circ \alpha, \quad \alpha^{dc} = \alpha \circ E_0, \\
\alpha^{so} &= \alpha / Q_0, \quad \alpha^{ro} = Q_0 \setminus \alpha, \quad \alpha^{do} = \alpha / E_0.
\end{align*}
\]

(56)

According to the previous theorem, the problem of computing the closures $\alpha^{sc}$, $\alpha^{rc}$ and $\alpha^{dc}$ and the openings $\alpha^{so}$, $\alpha^{ro}$ and $\alpha^{do}$ of $\alpha$ boils down to computing the fuzzy quasi-order $Q_0$ and the fuzzy equivalence $E_3$. As we have seen in Section 3, $Q_0$ can be efficiently computed as the reflexive-transitive closure of the fuzzy relation $V_0$, and $E_3$ as the reflexive-transitive closure of $V_3^\text{r} = V_0 \vee V_0^{-1}$. We again refer to [25, 27, 28, 55, 90, 91] and the sources cited there, where one can find more information about efficient algorithms for computing the transitive closure of a fuzzy relation.

The following example demonstrates the procedure of computing the fuzzy quasi-order $Q_0$, the fuzzy equivalence $E_3$, and related closures and openings.

Example 5.3. Let $T = (A, X, \delta)$ be the fuzzy transition system from Example 4.1. We easily obtain that

\[
V_0 = \begin{bmatrix} 1 & 1 & 0.6 & 0.8 \\ 1 & 1 & 0.8 & 0.9 \\ 0.3 & 0.3 & 0.8 & 0.9 \\ 0.5 & 0.3 & 0.8 & 1 \end{bmatrix}.
\]

\[
Q_0 = (V_0^r)^3 = \begin{bmatrix} 1 & 1 & 0.8 & 0.9 \\ 1 & 1 & 0.8 & 0.9 \\ 0.5 & 0.5 & 0.8 & 1 \end{bmatrix}.
\]

\[
E_3 = (V_3^\text{r})^3 = \begin{bmatrix} 1 & 1 & 0.9 & 0.9 \\ 1 & 1 & 0.9 & 0.9 \\ 0.9 & 0.9 & 0.9 & 1 \end{bmatrix}.
\]

For a fuzzy subset $\alpha$ of $A$ given by $\alpha = \begin{bmatrix} 1 & 0.6 & 0.8 \end{bmatrix}$ we easily obtain that

\[
\alpha^{sc} = \alpha \circ Q_0 = \begin{bmatrix} 1 & 1 & 0.8 \end{bmatrix}, \quad \alpha^{ro} = \alpha / Q_0 = \begin{bmatrix} 0.6 & 0.6 & 0.8 \end{bmatrix}, \quad \alpha^{dc} = \alpha \circ E_3 = \begin{bmatrix} 1 & 1 & 0.9 \end{bmatrix}.
\]

It is clear that $\alpha$ is not closed nor open for any of the considered closure and opening operators.

It is worth noting that Das in [24] defined a function $\alpha \mapsto \alpha^f$, for $\alpha \in L^A$, by

\[
\alpha^f(a) = \bigvee_{b \in A} \left( (\alpha(b) \circ \delta_{\alpha}(b, a)) \right),
\]

(57)

for each $a \in A$, and proved that this function is a topological closure operator on $T(A)$, i.e., a closure operator which preserves joins (see also [87]). It was also shown in [24] that closed fuzzy subsets of $A$ with respect to this closure operator are exactly subsystems of the fuzzy transition system $T = (A, X, \delta)$, and this means that $\alpha^f = \alpha^{rc}$. However, in general, for a given fuzzy subset $\alpha$ of $A$, its closure $\alpha^f$ can not be efficiently computed by means of formula (57). Namely, to compute $\alpha^f(a)$ by means of formula (57) we should know all members of the transition monoid $M(T)$, which may be infinite, and even if it is finite, its number of elements could be exponentially larger than the number of states of $T$.

The next example shows the case when the number of fuzzy transition relations $\delta_u (u \in X^*)$ is infinite.
Example 5.4. Let \( \mathcal{L} \) be the product structure. Consider a fuzzy transition system \( \mathcal{T} = (A, X, \delta) \), where \( |A| = 3 \), \( X = \{x\} \) and the fuzzy transition relation \( \delta \) is given by the following matrix:

\[
\delta_x = \begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2}
\end{bmatrix}.
\]

As shown in Example 11.2 [46], the transition monoid of \( \mathcal{T} \) is infinite, since the transition relations of \( \mathcal{T} \) can be represented by

\[
\delta_{2n} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2}
\end{bmatrix}, \quad \delta_{2n+1} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2}
\end{bmatrix}.
\]

Despite the fact that the number of fuzzy transition relations \( \delta_u (u \in X^*) \) is infinite, we can efficiently compute \( Q_0 \) and \( E_0 \), and obtain that

\[
Q_0 = V_0^* = \delta_x^* = \begin{bmatrix}
1 & 1 & 0 \\
1 & 1 & 0 \\
\frac{1}{2} & \frac{1}{2} & 1
\end{bmatrix},
\]

\[
E_0 = (V_0 \vee V_0^{-1})^* = (\delta_x \vee \delta_x^{-1})^* = \begin{bmatrix}
1 & 1 & \frac{1}{2} \\
1 & 1 & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & 1
\end{bmatrix},
\]

and then efficiently compute closures \( a^{\text{sc}}, a^{\text{rsc}} \) and \( a^{\text{dsc}} \) and openings \( a^{\text{esc}}, a^{\text{esc}} \) and \( a^{\text{edc}} \), for each \( a \in L^3 \).

Closures and openings with respect to fuzzy quasi-orders, represented in the same way as \( a^{\text{sc}} \) and \( a^{\text{edc}} \) in Theorem 5.2, have already been studied by Bodenhofer, De Cock and Kerre [10] and Bodenhofer [8]. In addition, Bodenhofer in [8] found a way to define closure and opening operators starting from an arbitrary fuzzy relation. He proved that the function \( a \mapsto (a \circ R)/R \) is a closure operator, and the function \( a \mapsto (a(R)/R \circ a \) is an opening operator on the lattice \( \mathcal{F}(A) \), for an arbitrary fuzzy relation \( R \) on a set \( A \). However, if we take \( R \) to be the fuzzy relation \( V_0 \), we obtain two operators that are different than the operators \( a \mapsto a^{\text{sc}} \) and \( a \mapsto a^{\text{rsc}} \), and even incomparable with them, as the following example shows.

Example 5.5. Consider again the fuzzy transition system \( \mathcal{T} = (A, X, \delta) \) from Example 5.3, the same fuzzy subset \( a = [1 \ 0.6 \ 0.8 \ 0.9] \), and a fuzzy subset \( \beta \) of \( A \) given by \( \beta = [1 \ 0 \ 0 \ 0] \). Then

\[
a^{\text{sc}} = \alpha \circ Q_0 = [1 \ 1 \ 0.8 \ 0.9] \quad \text{and} \quad (\beta \circ V_0)/V_0 = [1 \ 0.6 \ 0.6 \ 0.6].
\]

which means that \( \gamma \mapsto \gamma^{\text{sc}} \) and \( \gamma \mapsto (\gamma \circ V_0)/V_0 \) are different and mutually incomparable closing operators on the lattice \( \mathcal{F}(A) \). In the same way we show that \( \gamma \mapsto \gamma^{\text{edc}} \) and \( \gamma \mapsto (\gamma/V_0) \circ V_0 \) are different and mutually incomparable opening operators on the lattice \( \mathcal{F}(A) \).

The above considered closure operator \( a \mapsto a \circ Q_0 \) is a direct generalization of the so-called successor operator, which to any set of states of an ordinary transition system assigns the set of all states which are accessible from the states from this set. On the other hand, the closure operator \( a \mapsto Q_0 \circ a \) generalizes the source operator, which to any set of states assigns the set of all states from which the states from this set are accessible. Different generalizations of the successor and source operators, in the fuzzy framework, were studied in [63, 67, 87].

An efficient procedure for computing the principal double subsystems of a deterministic transition system was provided in [13, 16] (see also [14]). It is based on the alternate application of the operators \( a \mapsto a^{\text{sc}} \) and \( a \mapsto a^{\text{rsc}} \), starting from any of them, and stops when the so generated sequence of sets stabilizes. This procedure can be easily derived from the results obtained here. Namely, for an arbitrary fuzzy subset \( a \) of the set of states of a fuzzy transition system \( \mathcal{T} = (A, X, \delta) \) we have that

\[
(a^{\text{esc}})^{n-1} = Q_0 \circ (a \circ Q_0) \circ (a \circ Q_0) \circ Q_0 \circ Q_0 = a \circ (Q_0)^{n-1} = a \circ Q_n,
\]

and applying the operators \( a \mapsto a^{\text{sc}} \) and \( a \mapsto a^{\text{rsc}} \) alternately \( n - 1 \) times, where \( n \) is the number of states of \( \mathcal{T} \), we obtain \( a \circ (Q_0)^{n-1} = a \circ E_0 = a^{\text{dsc}} \). The same happens if we apply the operator \( a \mapsto a^{\text{rsc}} \) first, and then the operator \( a \mapsto a^{\text{esc}} \).

Our final question is how to transform the concepts of subsystems to fuzzy automata. If we make an analogy with deterministic automata, it is natural to define a \textit{subsystem} or a \textit{fuzzy subautomaton} of a fuzzy automaton \( \mathcal{A} = (A, X, \delta, \sigma, \tau) \) to be any fuzzy subset \( \alpha \) of \( A \) which is forward \( \delta \)-closed, for each \( x \in X \), and contains the fuzzy set of initial states, i.e., \( \sigma \leq \alpha \). It can be easily seen that fuzzy subautomata of \( \mathcal{A} \) form a complete lattice with the least element \( \alpha = \sigma \circ Q_0 \), and the same meets and joins as in \( \mathcal{F}(A) \), and the corresponding closure operator is given by \( a \mapsto (a \circ \alpha)^{\text{esc}} = \sigma^{\text{esc}} \circ a^{\text{esc}} \). Similarly, we define \( a \) to be a \textit{reverse fuzzy subautomaton} of \( \mathcal{A} \) if it is backward \( \delta \)-closed, for each \( x \in X \), and contains the fuzzy set of terminal states, i.e., \( \tau \leq \alpha \), and to be a \textit{double fuzzy subautomaton} if it is both a fuzzy subautomaton and a reverse fuzzy subautomaton. Reverse fuzzy subautomata and double fuzzy subautomata also form complete lattices with the least elements \( \tau^{\text{esc}} = Q_0 \circ \tau \) and \( (\sigma \vee \tau)^{\text{dsc}} = (\sigma \circ \tau) \circ E_0 \), and the same meets and joins as in \( \mathcal{F}(A) \). The corresponding closure operators are given by \( a \mapsto (\sigma \circ \alpha)^{\text{esc}} = \sigma^{\text{esc}} \circ a^{\text{esc}} \) and \( a \mapsto (\sigma \circ \tau \circ a)^{\text{dsc}} = \tau^{\text{dsc}} \circ a^{\text{dsc}} \).

6. Concluding remarks

In this paper we discussed subsystems, reverse subsystems and double subsystems of fuzzy transition systems. They were characterized as solutions to certain fuzzy relation inequalities and equations. In particular, we proved that they are respectively eigen fuzzy sets (in the sense of
Sanchez [83]) of \(Q_\delta\), \(Q_\delta^{-1}\) and \(E_\delta\), where \(Q_\delta\) is the fuzzy quasi-order and \(E_\delta\) is the fuzzy equivalence generated by the fuzzy transition relations of the considered fuzzy transition system. They were also characterized as fuzzy sets that can be represented as linear combinations of aftersets of \(Q_\delta\) foresets of \(Q_\delta\) and equivalence classes of \(E_\delta\). Besides, we showed that subsystems, reverse subsystems and double subsystems of a fuzzy transition system \(\mathcal{T}\) form both closure and opening systems in the lattice of fuzzy subsets of \(A\), where \(A\) is the set of states of \(\mathcal{T}\), and we provide efficient procedures for computing related closures and openings of an arbitrary fuzzy subset of \(A\). These procedures comes down to computing the fuzzy quasi-order \(Q_\delta\) or the fuzzy equivalence \(E_\delta\), which can be efficiently computed using the well-known algorithms for computing the transitive closure of a fuzzy relation.

The obtained results generalize the results concerning subsystems of fuzzy transition systems over the G"odel's structure given in [24, 63, 66, 87], as well as the results from [13–16] concerning subsystems, reverse subsystems, double subsystems and direct sum decompositions of crisp transition systems. They are also closely related to results from [8, 10] on closure and opening operators defined by fuzzy relations.

Appendix

**Proof of Theorem 3.1.** As we have noted in Section 2.2, the family \([\delta_x]_{x \in X}\) and its union \(V_\delta\) have the same \(\mathcal{D}\)-closure. For the same reason, the family \([\delta_x]_{x \in X}\) and its union \(Q_\delta\) also have the same \(\mathcal{D}\)-closure, so it remains to show that \(Q_\delta\) is a fuzzy quasi-order and that it is the \(\mathcal{D}\)-closure of the family \([\delta_x]_{x \in X}\).

For arbitrary \(a, b, c \in X\) we have that

\[
Q_\delta(a, b) \otimes Q_\delta(b, c) = \left( \bigvee_{a \in X^+} \delta_a(a, b) \right) \otimes \left( \bigvee_{b \in X^+} \delta_b(b, c) \right) = \bigvee_{a \in X^+, b \in X^+} \left( \bigvee_{a' \in X^+} \delta_{a'}(a, b) \right) \otimes \left( \bigvee_{b' \in X^+} \delta_{b'}(b, c) \right) \leq \bigvee_{a \in X^+, b \in X^+, d \in A} \delta_a(a, d) \otimes \delta_b(b, c) = \bigvee_{a \in X^+, b \in X^+} \delta_{a \cup b}(a, c) \leq \bigvee_{a \in X^+} \delta_a(a, c) = Q_\delta(a, c),
\]

and thus, \(Q_\delta\) is a transitive fuzzy relation. By \(\Delta_X = \delta_x \leq Q_\delta\) we obtain that \(Q_\delta\) is a reflexive fuzzy relation. Therefore, \(Q_\delta\) is a fuzzy quasi-order, which also means that \(Q_\delta\) is the \(\mathcal{D}\)-closure of the family \([\delta_x]_{x \in X}\), and it is clear that \(\delta_x \leq Q_\delta\) for every \(x \in X\).

Let \(Q\) be an arbitrary fuzzy quasi-order on \(A\) such that \(\delta_x \leq Q\), for every \(x \in X\). Then \(\delta_x \leq Q\), and for each \(u \in X^+\) we have that \(u = x_1 \cdots x_n\), for some \(x_1, \ldots, x_n \in X\), \(n \in \mathbb{N}\), so \(\delta_u = \delta_{x_1} \circ \cdots \circ \delta_{x_n} \leq Q^+ = Q\). Now

\[
Q_\delta = \bigvee_{u \in X^+} \delta_u \leq Q.
\]

Hence, we have proved that \(Q_\delta\) is the least fuzzy quasi-order on \(A\) containing fuzzy transition relations \(\delta_x\), for all \(x \in X\), i.e., it is the \(\mathcal{D}\)-closure of the family \([\delta_x]_{x \in X}\).

**Proof of Theorem 3.2.** For an arbitrary \(x \in X\) we have that

\[
Q_\delta \circ \delta_x = \left( \bigvee_{u \in X^+} \delta_u \right) \circ \delta_x = \left( \bigvee_{u \in X^+} \delta_u \circ \delta_x \right) = \left( \bigvee_{u \in X^+} \delta_{u \cup x} \right) = \left( \bigvee_{u \in X^+} \delta_u \right) = Q_\delta.
\]

Thus, \(Q_\delta\) is a solution to (39). Let \(R\) be an arbitrary reflexive fuzzy relation which is a solution to (39). Then by \(R \circ \delta_x \leq R\), for each \(x \in X\), it follows that \(R \circ \delta_x \leq R\), for each \(u \in X^+\), whence

\[
R \circ Q_\delta = R \circ \left( \bigvee_{u \in X^+} \delta_u \right) = \left( R \circ \delta_u \right) \leq R,
\]

and by reflexivity of \(R\) we obtain that \(Q_\delta \leq R \circ Q_\delta \leq R\). Therefore, \(Q_\delta\) is the least reflexive solution to (39).

Next, we will prove that a fuzzy relation \(R\) on \(A\) is a solution to system (39) if and only if it is a solution to inequality (42). Let \(R\) be a solution to (39). Then

\[
R \circ V_\delta = R \circ \left( \bigvee_{x \in X} \delta_x \right) = \left( R \circ \delta_x \right) \leq R,
\]

and hence, \(R\) is a solution to (42). Conversely, let \(R\) be a solution to (42). Then by \(\delta_x \leq V_\delta\), for each \(x \in X\), it follows that \(R \circ \delta_x \leq R \circ V_\delta \leq R\), and we conclude that \(R\) is a solution to (39). Therefore, \(Q_\delta\) is also the least reflexive solution to (42).

Analogously we show that \(Q_\delta\) is the least reflexive solution to (40) and (43). The assertion concerning system (41) follows directly from the assertions concerning systems (39) and (40), whereas the assertion concerning inequality (44) follows directly from the assertions concerning systems (42) and (43).

**Proof of Proposition 3.3.** For the sake of simplicity set \(E = R^\circ\).

Since \((R')^\circ\) and \((R')^\circ\) are reflexive and symmetric fuzzy relations, then \((R')^\circ\) and \((R')^\circ\) are fuzzy equivalences which contain \(R\), so \(E = R^\circ \leq (R')^\circ\) and \(E = R^\circ \leq (R')^\circ\).

On the other hand, by \(R \leq E\) it follows that \(R^\circ \leq E\), so

\[
(R')^\circ = R^\circ \circ (R')^{-1} \leq E \circ E^{-1} = E^2 = E,
\]

and therefore,

\[
((R')^\circ)^\circ \leq E = E' \quad \text{and} \quad ((R')^\circ)^\circ \leq E = E.
\]

This completes the proof of the proposition.

\[\square\]
Proof of Theorem 3.4. By the same arguments used in the proof of the Theorem 3.1 we obtain that $E_b$ is the $\omega$-closure of $V_b$, and that the family $\{\delta_b\}_{b \in X}$ and the fuzzy quasi-order $Q_b$ have the same $\omega$-closure.

Since $Q_b$ is the $\mathcal{Z}$-closure of $V_b$ and $E_b$ is the $\omega$-closure of $V_b$, we conclude that $Q_b \leq E_b$. If $E$ is an arbitrary fuzzy equivalence on $A$ which contains $Q_b$, then $E$ contains $\delta_b$, for every $x \in X$, and consequently, $E_b \leq E$. Therefore, $E_b$ is the least fuzzy equivalence on $A$ containing $Q_b$, i.e., it is the $\omega$-closure of $Q_b$.

Now, by Proposition 3.3 we obtain that

$$E_b = Q_b^\omega = (Q_b^\omega)^\prime = (Q_b^\prime)^\prime,$$

and analogously, $E_b = (Q_b^\prime)^\prime$. $\Box$

Proof of Theorem 3.5. According to (28), from $Q_b \leq E_b$ it follows that $E_b \circ Q_b = Q_b \circ E_b = E_b$. Since $Q_b$ is a solution to (39), for each $x \in X$ we have that

$$E_b \circ \delta_b = E_b \circ Q_b \circ \delta_b \leq E_b \circ Q_b = E_b,$$

and hence, $E_b$ is also a solution to (39).

Let $E$ be an arbitrary fuzzy equivalence on $A$ which is a solution to (39). By Theorem 3.2, $Q_b$ is the least reflexive solution to (39), and consequently, $Q_b \leq E$, which implies that $E \circ Q_b = Q_b \circ E = E$. $Q_b^1 \leq E^{-1} = E$ and $E \circ Q_b^1 = Q_b^1 \circ E = E$. Now we have that

$$E_b \leq E \circ E_b = E \circ \left( \bigvee_{n \in \mathbb{N}} (Q_b \circ Q_b^{1/n}) \right) = \bigvee_{n \in \mathbb{N}} (E \circ (Q_b \circ Q_b^{1/n})^\prime) = E,$$

and therefore, $E_b$ is the least fuzzy equivalence which is a solution to (39). Analogously we prove that $E_b$ is the least fuzzy equivalence on $A$ which is a solution to (40), (42) and (43).

Next, let $R$ be an arbitrary fuzzy proximity relation which is a solution to (41). Then $R \circ \delta_u \leq R$ and $\delta_u \circ R \leq R$, for every $u \in X'$, whence

$$R \circ Q_b = R \circ \left( \bigvee_{u \in X'} \delta_u \right) \leq \bigvee_{u \in X'} R \circ \delta_u \leq R,$$

and similarly, $Q_b \circ R \leq R$. The opposite inequalities follow by reflexivity of $Q_b$, so we obtain that $R \circ Q_b = Q_b \circ R = R$. Moreover, due to the symmetry of $R$, we have that

$$R \circ Q_b^1 = R^{-1} \circ Q_b^1 = (Q_b \circ R)^{-1} = R^{-1} \circ R.$$

Now, due to the reflexivity of $R$, we obtain that

$$E_b \leq R \circ E_b = R \circ \left( \bigvee_{n \in \mathbb{N}} (Q_b \circ Q_b^{1/n})^\prime \right) = \bigvee_{n \in \mathbb{N}} R \circ \left( (Q_b \circ Q_b^{1/n})^\prime \right) \circ R = R.$$

Hence, $E_b$ is the least fuzzy proximity relation which is a solution to (41). Analogously we prove that $E_b$ is the least fuzzy proximity relation which is a solution to (44). $\Box$

Proof of Proposition 4.2. Note again that $\mathcal{S}(\mathcal{J})$ is the set of all solutions to system of fuzzy relation inequalities (47). It is easy to check that the set of all solutions to system (47) is closed under arbitrary meets, joins and scalar multiplications, and $\emptyset$ and $A$ are solutions to (47), i.e., they belong to $\mathcal{S}(\mathcal{J})$. Thus, $\mathcal{S}(\mathcal{J})$ is a complete $\mathcal{Z}$-sublattice of $\mathcal{F}_0$. $\Box$

Proof of Theorem 4.3. (i)$\Rightarrow$(ii). If $a$ is a subsystem of $\mathcal{J}$, i.e., $a \circ \delta_b \leq a$, for every $x \in X$, then

$$a \circ V_b = a \circ \left( \bigvee_{x \in X} \delta_x \right) = \bigvee_{x \in X} (a \circ \delta_x) \leq a,$$

and hence, $a$ is forward $V_b$-closed.

(ii)$\Rightarrow$(i). Let $a$ be forward $V_b$-closed. Then for each $x \in X$, by $\delta_x \leq V_b$ it follows that $a \circ \delta_x \leq a \circ V_b \leq a$, and consequently, $a$ is forward $\delta_x$-closed, for each $x \in X$. In the same way we prove (iv)$\Rightarrow$(i).

(i)$\Rightarrow$(iii). Let $a$ be a subsystem of $\mathcal{J}$, i.e., $a$ is forward $\delta_b$-closed, for every $x \in X$. Then $a \circ \delta_b = a$, so $a$ is forward $\delta_b$-closed, and for an arbitrary $u \in X'$, if $u = x_1 \cdots x_n$, for some $x_1, \ldots, x_n \in X$, $n \in \mathbb{N}$, we have that $a \circ \delta_{x_i} \leq a$, for each $i \in \{1, \ldots, n\}$, and hence,

$$a \circ \delta_u = a \circ \delta_{x_1} \cdots \circ \delta_{x_n} \leq a.$$

Therefore, $a$ is forward $\delta_u$-closed, for every $u \in X'$.

(iii)$\Rightarrow$(iv). If $a$ is forward $\delta_u$-closed, for every $u \in X'$, then by (27) it follows that

$$a \circ Q_b = a \circ \left( \bigvee_{u \in X'} \delta_u \right) = \bigvee_{u \in X'} (a \circ \delta_u) \leq a,$$

and thus, $a$ is forward $Q_b$-closed.

(iv)$\Rightarrow$(v). Let $a$ be forward $Q_b$-closed, i.e., let $a \circ Q_b \leq a$. By reflexivity of $Q_b$, it follows that $a \circ \alpha \leq a \circ Q_b$, and hence, $a \circ Q_b = a$.

(v)$\Rightarrow$(iv). This implication is obvious.

(v)$\Rightarrow$(vi). Let $(v)$ hold. For the sake of simplicity set $Q_b = Q$, and take an arbitrary $a \in \mathcal{S}(\mathcal{J})$. By (iv) we obtain that $a = a \circ Q$, and for any $b \in A$ we have

$$a(b) = (a \circ Q)(b) = \bigvee_{a \in A} (a \otimes Q(a,b)) = \bigvee_{a \in A} (a(a)(aQ))(b),$$

which means that

$$a = \bigvee_{a \in A} (a(aQ)) = \bigvee_{a \in A} \lambda_a(aQ),$$

where $\lambda_a = a(a)$, for each $a \in A$. Therefore, $a$ can be represented as a linear combination of $Q$-aftersets.

(v)$\Rightarrow$(iv). Suppose that $a$ can be represented as a linear combination of $Q$-aftersets, where $Q = Q_b$. Then there are scalars $\lambda_a \in L$, $a \in A$, such that

$$a = \bigvee_{a \in A} \lambda_a(aQ).$$
and for any \(b \in A\) we have that
\[
(\alpha \circ Q)(b) = \bigvee_{c \in A} \alpha(c) \otimes Q(c, b)
\]
\[
= \bigvee_{c \in A} \left( \bigvee_{\lambda \in \mathbb{A}} \lambda(c) \otimes (\alpha Q(c, b)) \right)
\]
\[
= \bigvee_{\lambda \in \mathbb{A}} \left( \bigvee_{c \in A} \lambda(c) \otimes Q(c, b) \right)
\]
\[
= \bigvee_{\lambda \in \mathbb{A}} \lambda(a) \otimes Q(a, b)
\]
\[
= \bigvee_{a \in A} \lambda(a) \otimes Q(a, b)
\]
\[
= (\alpha \circ \lambda)(b) = \alpha(b).
\]

Hence, \(\alpha \circ Q = \alpha\), and we have shown that (iv) holds.

(v)\(\Rightarrow\)(vi). Let \(\alpha\) be a solution to (48). According to (31), we have that \(\alpha \leq a_{Q}\). On the other hand, by Proposition 2.1 it follows that \(a_{Q} \leq \alpha\), and thus, \(\alpha\) is a solution to (49).

Implication (vi)\(\Rightarrow\)(v) can be proved in the same way as (v)\(\Rightarrow\)(vi).

**Proof of Proposition 4.5.** For all \(a, b \in A\) and \(x \in X\) we have \(a(\alpha) \otimes \delta_{0}(a, b) \leq a(\alpha)\) if and only if \(\delta_{0}(b, a) \otimes a(\alpha) \leq a(\beta)\), so \(\alpha\) is a reverse subsystem of \(\mathcal{F}\) if and only if it is a subsystem of \(\mathcal{F}\).

**Proof of Proposition 4.9.** Since \(\mathcal{F}(\mathcal{F}) = \mathcal{F}(\mathcal{F}) \cap \mathcal{F}(\mathcal{F})\), by Propositions 4.2 and 4.6 we obtain that \(\mathcal{F}(\mathcal{F})\) is closed under arbitrary meets, joins and scalar multiplications, and contains \(\emptyset\) and \(A\). Therefore, \(\mathcal{F}(\mathcal{F})\) is a complete \(\mathcal{L}\)-sublattice of \(\mathcal{F}(A)\).

**Proof of Theorem 4.10.** (i)\(\Rightarrow\)(ii) and (i)\(\Rightarrow\)(iii). This follows directly from Theorems 4.3 and 4.7.

(i)\(\Rightarrow\)(v). Let \(\alpha\) be a double subsystem of \(\mathcal{F}\). According to Theorems 4.3 and 4.7, we have that \(\alpha \circ \mathcal{Q}_{0} = \alpha\) and \(\alpha \circ \mathcal{Q}_{0}^{-1} = \mathcal{Q}_{0} \circ \alpha = \alpha\), whence
\[
\alpha \circ \mathcal{E}_{0} = \alpha \circ (\mathcal{Q}_{0}^{-1})' = \alpha \circ \left( \bigvee_{n \in \mathbb{N}} (\mathcal{Q}_{0} \circ \mathcal{Q}_{0}^{-1})' \right)
\]
\[
= \bigvee_{n \in \mathbb{N}} \alpha \circ (\mathcal{Q}_{0} \circ \mathcal{Q}_{0}^{-1})' = \alpha
\]

(iv)\(\Rightarrow\)(v). Let \(\alpha\) be \(\mathcal{E}_{0}\)-closed. Then \(\alpha \circ \mathcal{Q}_{0} \leq \alpha \circ \mathcal{E}_{0} \leq \alpha\) and \(\mathcal{Q}_{0} \circ \alpha \leq \mathcal{E}_{0} \circ \alpha = \alpha \circ \mathcal{E}_{0} \leq \alpha\), and according to Theorems 4.3 and 4.7, \(\alpha\) is a double subsystem of \(\mathcal{F}\).

The implication (v)\(\Rightarrow\)(iv) is obvious, and equivalences (v)\(\Rightarrow\)(vi) and (v)\(\Rightarrow\)(vii) can be proved in the same way as the corresponding equivalences in Theorem 4.3.

**Proof of Theorem 4.11.** For the sake of convenience set \(\mathcal{Q}_{0} = Q\). According to (5) and (4), for any \(a \in A\) we have that
\[
(\alpha \circ Q)(a) = \bigvee_{b \in A} (\alpha(b) \otimes Q(b, a)) \rightarrow 0
\]
\[
= \bigcup_{b \in A} (\alpha(b) \otimes Q(b, a)) \rightarrow 0
\]
\[
= \bigcup_{b \in A} (Q(b, a) \otimes \alpha(b)) \rightarrow 0
\]
\[
= \bigcup_{b \in A} (Q(b, a) \rightarrow (\alpha(b) \rightarrow 0))
\]
\[
= \bigcup_{b \in A} (Q(b, a) \rightarrow \alpha^{-}(b)) = (Q^{-}\alpha)(a),
\]

which means that \((\alpha \circ Q)^{-} = Q^{-}\alpha\). Now, by (6) and (29) we obtain that
\[
\alpha \circ Q \leq \alpha \Rightarrow \alpha^{-} \leq (\alpha \circ Q)^{-} \Rightarrow \alpha^{-} \leq Q^{-}\alpha
\]
\[
\Rightarrow Q \circ \alpha^{-} \leq \alpha^{-},
\]
and hence, if \(\alpha\) is a subsystem, then \(\alpha^{-}\) is a reverse subsystem, i.e., \((a)\) holds. In the same way we prove (b), whereas (c) follows directly from (a) and (b).

In addition, if the complete residuated lattice \(L\) has the double negation property, then the first implication in (58) can be turned into equivalence, which means that \(\alpha\) is a subsystem if and only if \(\alpha^{-}\) is a reverse subsystem. Analogously we show that \(\alpha\) is a reverse subsystem if and only if \(\alpha^{-}\) is a subsystem, and that \(\alpha\) is a double subsystem if and only if \(\alpha^{-}\) is a double subsystem.

**Proof of Proposition 5.1.** (a) This follows directly from Propositions 4.2, 4.6 and 4.9.

(b) To simplify notation set \(\mathcal{Q}_{0} = Q\) and \(\mathcal{F}(\mathcal{F}) = \mathcal{C}\). Consider an arbitrary \(a \in A\). For each \(b \in A\) we have that
\[
(\alpha \circ Q)(b) = \bigvee_{c \in A} (\alpha(c) \otimes Q(c, b)) = \bigvee_{c \in A} Q(a, c) \otimes Q(c, b)
\]
\[
= (Q \circ Q)(a, b) = Q(a, b) = (\alpha \circ \alpha)(b),
\]
and hence, \(\alpha Q\) is a solution to equation (48), i.e., \(a \in \mathcal{C}\). It is clear that \((\alpha Q)(a) = 1\), so \(\mathcal{C}_{a} \leq \alpha Q\).

Take an arbitrary \(\alpha \in \mathcal{C} = \mathcal{F}(\mathcal{F})\) such that \(\alpha(a) = 1\). Then for each \(b \in A\) we have that
\[
(\alpha \circ Q)(b) = Q(a, b) = (\alpha \circ Q)(a, b) \leq \bigvee_{c \in A} a(c) \otimes Q(c, b)
\]
\[
= (\alpha \circ \alpha)(b) = \alpha(b),
\]
so \(\alpha Q \leq \alpha\), whence it follows that
\[
\alpha Q \leq \bigwedge \{ a \in \mathcal{C} | \alpha(a) = 1 \} = \mathcal{C}_{a}.
\]

Therefore, we have proved that \(\mathcal{C}_{a} = a Q\), for each \(a \in A\).

The assertions (c) and (d) can be proved similarly.
Proof of Proposition 5.2. By Proposition 2.1, we have that \((a \circ Q_b) \circ Q_a = a \circ Q_b\), and by Theorem 4.3 we get that \(a \circ Q_b\) is a subsystem of \(T\). Let \(\beta\) be an arbitrary subsystem of \(T\) such that \(a \leq \beta\). Again according to Proposition 2.1 and Theorem 4.3 we obtain \(a \circ Q_b \leq \beta \circ Q_b = \beta\). Thus, \(a \circ Q_b\) is the least subsystem of \(T\) containing \(a\), i.e., \(a \circ Q_b = a^{sc}\).

In the same way we prove the remaining equalities.

References


