Robust Tube-based MPC for Constrained Mobile Robots under Slip Conditions

R. Gonzalez†, M. Fiacchini‡, J.L. Guzman‡, T. Alamo‡

Abstract—This paper focuses on the design of a robust tube-based Model Predictive Control law for the control of constrained mobile robots. A time-varying trajectory tracking error model has been used, where uncertainties are assumed to be bounded and additive. The proposed solution to the control problem is a tube-based MPC ensuring robustness and stability. A comparative simulation example is presented showing the promising behavior of the robust MPC controller.

I. INTRODUCTION

Mobile robotics comprises an important field in which robust control techniques are suitable. Mobile robots must handle both constraints and uncertainties (i.e., physical limitations of actuators, simplified models, noisy measurements, etc.). Furthermore, when a mobile robot operates under off-road conditions, the slip phenomenon is usually presented influencing the control policy [1], [2], [3].

Model Predictive Control (MPC) constitutes a popular control technique to deal with constrained systems [4], [5], [6]. However, in order to tackle uncertainty, robust MPC formulation must be considered [7]. One drawback of the robust MPC techniques is the relatively high online computational effort, which prevents its applicability to many fast systems, as mobile robots. However, some advances have been achieved during the last years, for the case of robot control under slip conditions. For instance, in [2], authors use MPC and adaptive controllers which enable both anticipation of approaching curvature and compensation for lateral slip phenomena for path tracking control of an agricultural vehicle. In the work [8], an MPC strategy is applied to the trajectory tracking problem. In order to avoid vehicle slip, velocity and acceleration are bounded.

An efficient technique for practical implementation of robust MPC is the tube-based MPC [9], [4], [11], [12], although the authors are not aware of its application to mobile robotics.

The basic idea underlying tube-based controllers is to consider all the possible trajectories generated by all the admissible uncertainty realizations. This leads to a sequence of sets which are imposed to be contained in the constraint sets. In [9], authors use a sequence of uncertainty sets for a time-invariant system subject to additive uncertainty in order to obtain a robust MPC controller, whereas [10] and [11] use a single uncertainty set (invariant) for robustify any stabilizing deterministic controller.

In this paper, a robust tube-based MPC for constrained mobile robots is presented. The proposed control is based on ideas presented in [9], [10] which consider a time-invariant system with additive uncertainties. However, in our work additive uncertainties and system dynamics are involved in the trajectory tracking error model of a mobile robot (time-varying system). This model is obtained supposing that the robot operates under slip conditions [1], [3]. State and input constraints represent the physical limitations proper of the mobile robot, such as narrow spaces of operation and saturation on the actuators. Additive uncertainty models the linearization error, the effect of the noise in the slip measurements, and the uncertainty in the robot position estimation. Furthermore, dynamics of the system changes according to reference values, which are assumed to evolve inside a convex bounded set. In order to assure stability of the solution, we use LMI to determine a Lyapunov function constituting the terminal cost for the MPC optimization problem. Finally, a terminal robust positively invariant set is calculated following the ideas detailed in [13], [14].

This paper is organized as follows. Section 2 is concerned with problem statement. In Section 3, we explain the control objective and describe the robust tube-based MPC control law. Section 4 shows a comparative simulation example. Finally, Section 5 presents some conclusions.

Notation: A polyhedron is the (convex) intersection of a finite number of open and/or closed half-spaces and a polytope is the closed and bounded polyhedron. The convex hull of a set of points is defined as the smallest convex set containing the points, and here it is denoted as co. Given two sets $X, Y \subseteq \mathbb{R}^n$, the Minkowski sum is defined by $X \oplus Y \triangleq \{x + y | x \in X, y \in Y\}$, and the Pontryagin set difference is $X \ominus Y \triangleq \{x | x \oplus Y \subseteq X\}$.

II. PROBLEM STATEMENT

In this paper, we consider the continuous-time trajectory tracking error model of a mobile robot under slip conditions [1], [15]. In order to implement the tube-based MPC, this model has been discretized and an additive uncertainty has been included to model the linearization error, the effect of noise in the slip measurements, and the uncertainty in the robot position estimation.

A. Kinematic Model under slip conditions

As addressed in [3], [1], slip can be considered as a penalizing factor of the wheel velocity, that is,

$$v_{r,slip}(t) = v_r(t)(1-i_r(t)),$$

$$v_{l,slip}(t) = v_l(t)(1-i_l(t)),$$ (1)

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$$v_{l,slip}(t) = v_l(t)(1-i_l(t)),$$ (1)
where \( t \) is the continuous time, \( v_{r, \text{slip}} \) and \( v_{l, \text{slip}} \) are the linear velocities under slip of the right and left wheels respectively, \( v_r \) and \( v_l \) are the linear velocities of right and left wheels respectively, and \( i_r \in [v_{r, \text{slip}}, \dot{v}_{r, \text{slip}}] \) and \( i_l \in [v_{l, \text{slip}}, \dot{v}_{l, \text{slip}}] \) are the terms representing the longitudinal slip component. As shown in [1], slip can be estimated in real-time using the appropriate sensors.

Substituting this knowledge in the classical kinematic model of a differential-drive robot [16], we obtain,

\[
\begin{align*}
\dot{x}(t) &= \frac{v_r(t)(1-i_r(t)) + v_l(t)(1-i_l(t))}{2} \cos \theta(t), \\
\dot{y}(t) &= \frac{v_r(t)(1-i_r(t)) + v_l(t)(1-i_l(t))}{2} \sin \theta(t), \\
\dot{\theta}(t) &= \frac{v_r(t)(1-i_r(t)) - v_l(t)(1-i_l(t))}{b},
\end{align*}
\]

where \( x, y, \theta \) represent the location (position and orientation) of the mobile robot, and \( b \) is the distance between the wheels centers.

B. Trajectory tracking error model

Trajectory tracking consists in the problem in which a robot must follow a virtual mobile robot representing the desired positions and velocities. Hence, the objective is to find a feedback control law [15], such that, the error between the desired location and the real location of the mobile robot is close to zero (regulation problem).

This error is expressed with respect to the real robot frame as

\[
\begin{bmatrix}
\dot{e}_x(t) \\
\dot{e}_y(t) \\
\dot{e}_\theta(t)
\end{bmatrix} = \begin{bmatrix}
\cos \theta(t) & \sin \theta(t) & 0 \\
-\sin \theta(t) & \cos \theta(t) & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\dot{x}'(t) - x(t) \\
\dot{y}'(t) - y(t) \\
\dot{\theta}'(t) - \theta(t)
\end{bmatrix},
\]

(3)

where \( e_x \) is the longitudinal error, \( e_y \) is the lateral error, and \( e_\theta \) is the orientation error. The position of the real mobile robot is denoted as \([x, y, \theta]^T\), and the reference position as \([x', y', \theta'^T]\).

As shown in [1], differentiation of equation (3) produces

\[
\begin{align*}
\dot{e}_x(t) &= \alpha(t)e_x(t) + \cos \theta(t) \dot{\eta}'(t) - \eta(t) + \theta(t), \\
\dot{e}_y(t) &= -\alpha(t)e_y(t) + \sin \theta(t) \dot{\eta}'(t), \\
\dot{e}_\theta(t) &= \frac{v_r(t) - v_l(t)}{b} - \alpha(t),
\end{align*}
\]

where \( \alpha = \frac{v_r - \dot{v}_r}{b} - \frac{v_l - \dot{v}_l}{b} \), \( \eta' = \frac{v_r + v_l}{2b}, \eta = \frac{v_r + v_l}{b}, \theta = \frac{v_r - v_l}{b} \), and \( v_r \) and \( v_l \) are the reference linear velocities of right and left wheels respectively.

In order to linearize equation (4) around the reference trajectory, a first-order Taylor expansion has been used. Furthermore, we have defined the following virtual control signals to eliminate some of the nonlinear terms

\[
\begin{align*}
u_1(t) &= \frac{1 + i_r(t)}{2} v_r(t) + \frac{1 - i_l(t)}{2} v_l(t) + \frac{v_r'(t)}{2} + \frac{v_l'(t)}{2} , \\
u_2(t) &= \frac{1 + i_r(t)}{b} v_r(t) + \frac{1 - i_l(t)}{b} v_l(t) + \frac{v_r'(t)}{b} + \frac{v_l'(t)}{b},
\end{align*}
\]

(5)

Afterwards, equation (4) becomes

\[
\begin{bmatrix}
\dot{e}_x(t) \\
\dot{e}_y(t) \\
\dot{e}_\theta(t)
\end{bmatrix} = \begin{bmatrix}
\frac{1}{2} & 0 & 0 \\
0 & \frac{1}{2} & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\frac{1}{2} & 0 & 0 \\
0 & \frac{1}{2} & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\epsilon_e(t) \\
\epsilon_y(t) \\
\epsilon_\theta(t)
\end{bmatrix} + \begin{bmatrix}
\frac{1}{2} & 0 & 0 \\
0 & \frac{1}{2} & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
u_1(t) \\
\epsilon_e(t) \\
\epsilon_y(t) \\
\epsilon_\theta(t)
\end{bmatrix} + \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
u_2(t) \\
\epsilon_e(t) \\
\epsilon_y(t) \\
\epsilon_\theta(t)
\end{bmatrix},
\]

(6)

where \( \epsilon_e = (\dot{v}_r - v_r) - (\dot{v}_l - v_l) \).

Assumption 1: Assume that reference robot wheel velocities are known and bounded, i.e., \( v_r' \in [v_r, \dot{v}_r] \) and \( v_l' \in [v_l, \dot{v}_l] \).

Remark 1: Note that, according to (5), the linear velocities for each wheel are obtained as

\[
\begin{align*}
v_r(t) &= \frac{v_r'(t) - \dot{v}_r(t)}{1 - i_r(t)}, \\
v_l(t) &= \frac{-v_l'(t) + \dot{v}_l(t)}{-1 + i_l(t)},
\end{align*}
\]

(7) \( (8) \)

where \( v_r \in [v_r, v_l'] \) and \( v_l \in [v_l, v_r'] \). This leads to bounds on the space of \( u \), obtained from the constraints on \( v_r, v_l, i_r, i_l, v_r', v_l' \).

Then, the trajectory tracking error model (6) is discretized, obtaining the following linear, time-varying, discrete-time system

\[
e(k + 1) = A_T k e(k) + Bu(k) + w(k),
\]

(9)

where \( k \in \mathbb{Z}^+ \) is the discrete sample, \( e = [e_x, e_y, e_\theta]^T \in \mathbb{R}^3 \) is the current state (error), \( u = [u_1, u_2]^T \in \mathbb{R}^2 \) is the current control input, and \( w \) is a bounded additive uncertainty, satisfying \( w \in W \) where \( W \) is a polytope in the state space \( \mathbb{R}^3 \).

In our case, we suppose that \( W \) includes the error between the nonlinear continuous-time model and the linear discrete-time one, the effects of the noise in the slip measurements, and the uncertainty in the robot position estimation.

Matrices \( A_T \) and \( B \) are defined as

\[
A_T(k) = \begin{bmatrix}
1 & 0 & 0 \\
\epsilon(k) & 1 & \rho(k) \\
0 & 0 & 1
\end{bmatrix}, \quad B = \begin{bmatrix}
T_m & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & T_m
\end{bmatrix},
\]

(10)

where \( e = T_m \left( \frac{1 - i_r}{b} v_r + \frac{1 - i_l}{b} v_l \right) + \rho = T_m \left( \frac{v_r + v_l}{2b} \right) \), being \( T_m \) the sampling time, and \( i_r \) and \( i_l \) the nominal slip of each wheel.

From Assumption 1, \( y(k) = [v_r', v_l']^T \in \mathbb{R}^2 \) is a time-varying vector such that \( y(k) \in \Gamma, \forall k \in \mathbb{Z}^+ \), where \( \Gamma \subseteq \mathbb{R}^2 \) is a polytope. For any admissible realization of parameter \( y \in \Gamma \), a dynamic matrix \( A_T \) is determined. It follows that \( A_T \in \mathcal{A}_\Gamma \), with \( \mathcal{A}_\Gamma \) a polytope in \( \mathbb{R}^{3 \times 3} \) that represents the dynamic state matrix. The model is composed by a family of linear systems (defined by matrix \( A_T \), each of them is controllable provided that \( 0 \leq i_r, i_l < 1 \) and \( v_r', v_l' > 0 \).

States and inputs are subject to the following constraints

\[
e(k) \in E, \quad u(k) \in U,
\]

(11)

where \( E \subseteq \mathbb{R}^3 \) and \( U \subseteq \mathbb{R}^2 \) are polytopes and contain the origin. Recall that state and input constraints represent the physical limitations of the problem, such as narrow spaces of operation and saturation on the actuators.

For sake of notational simplicity, we omit to express the dependence of \( A_T(k) \) on \( k \), employing \( A_T \) to refer to it.
III. ROBUST MPC STRATEGY

In this section, we discuss the strategy followed to implement the robust tube-based MPC adapting the ideas of Chisci et al., in [9] and Mayne and Langson, see [10].

First, we define the “nominal system” as

$$\bar{e}(k+1) = A_e \bar{e}(k) + B g(k),$$  \hspace{1cm} (12)

where $\bar{e} \in \mathbb{R}^3$ is the nominal state, $g \in \mathbb{R}^2$ is the control input for the nominal system.

The control objective is to design a state feedback control law of the form

$$u(k) = K \bar{e}(k) + g(k),$$  \hspace{1cm} (13)

$$\bar{e}(k) = e(k) - \bar{e}(k),$$  \hspace{1cm} (14)

where $K$ is a local controller and its goal is to compensate the additive uncertainty, $\bar{e} = [\bar{e}_x \bar{e}_y \bar{e}_z]^T$ is the difference between the real state and the nominal state, and $g$ is the on-line MPC control input.

Substituting (9) and (12) in (14), we obtain the local uncertain closed-loop system as

$$\bar{e}(k+1) = e(k+1) - \bar{e}(k+1) = (A_e + BK)\bar{e}(k) + w(k).$$  \hspace{1cm} (15)

A. Control objectives

The main objectives of the control policy presented here are:

- Input and state constraints fulfillment: This requirement is guaranteed ensuring constraints satisfaction in the minimization of the MPC control law.
- Robustness: Original constraints (11) have been replaced with more restricted ones which take into account additive uncertainties and time-varying dynamics.
- Stability\footnote{Strictly speaking, asymptotic stability cannot be assured in uncertain systems, since uncertainties may not be vanishing. Some authors use the concept of asymptotically ultimately bounded set [17] which, conceptually, means convergence to a set rather than to a point of the state space.}: It is assured through a quadratic Lyapunov function determined using LMI and a robust positively invariant set for the terminal region of the MPC.
- Real-time implementation: A standard nominal MPC is solved on-line to control the nominal system, since the effect of the uncertainties of the system are already included in the restricted constraints. This fact implies that the robust tube-based MPC strategy fits properly to mobile robotics applications (where high sampling frequencies are employed).

Fig. 1 summarizes the robust tube-based MPC strategy. Off-line, we calculate the reachable sets for the local uncertain closed-loop system (15) (see subsection III-B), the state and input constraints are substituted using the previously calculated reachable sets (subsection III-C), and in order to assure the stability of the MPC, we determine a Lyapunov function and a terminal robustly positively invariant set (discussed in subsection III-E). On-line, we apply a MPC algorithm to the nominal system (detailed in subsection III-D).

B. Local compensation of uncertainty

In this subsection, we design the control law for the local uncertain system (15) and we determine the reachable sets which will be used in the subsection III-C.

First, in order to implement the tube-based MPC, we decompose the state matrix $A_e$ as a time-invariant part, and a time-varying part, whose effects are bounded and partially compensated through local controllers. For that purpose, matrix $A_e$ is expressed as

$$A_e = I_3 + A^r e^r + A^l e^l,$$ \hspace{1cm} (16)

being $I_3 \in \mathbb{R}^{3 \times 3}$ the identity matrix, and

$$A^r = \begin{bmatrix} 0 & \xi & 0 \\ -\xi & 0 & \beta \\ 0 & 0 & 0 \end{bmatrix}, \hspace{0.5cm} A^l = \begin{bmatrix} 0 & -\zeta & 0 \\ \zeta & 0 & \beta \\ 0 & 0 & 0 \end{bmatrix},$$ \hspace{1cm} (17)

where $\xi = T_m \left(\frac{1 - \frac{i}{b}}{\frac{i}{b}}\right)$, $\beta = \frac{b}{2}$, $\zeta = T_m \left(1 - \frac{i}{b}\right)$.

By Assumption 1, we can express

$$e^r = \bar{v}^r + \Delta v^r \Rightarrow \Delta v^r \in [\delta v^m, \delta v^M],$$ \hspace{1cm} (18)

$$e^l = \bar{v}^l + \Delta v^l \Rightarrow \Delta v^l \in [\delta v^m, \delta v^M],$$ \hspace{1cm} (19)

where $\bar{v}^r$ and $\bar{v}^l$ are the nominal reference velocities of the right and left wheels respectively, and $\Delta v^r$ and $\Delta v^l$ can be seen as the new ranges in the reference velocities.

Taking into account previous discussion, equation (16) can be expressed as

$$A_e = A^r + A^l \Delta v^r + A^l \Delta v^l,$$ \hspace{1cm} (20)

where $A^r = I_3 + A^r e^r + A^l e^l$ represents the time-invariant part of matrix $A_e$, and the rest of terms are the time-varying part, which must be bounded.

Now, substituting previous equation into (15), we get

$$\bar{e}(k+1) = [A^r + A^l \Delta v^r + A^l \Delta v^l + BK] \bar{e}(k) + w(k).$$ \hspace{1cm} (21)

In order to compensate the time-varying part, the local control gain $K$ should be defined as

$$K = K^r + K^l \Delta v^r + K^l \Delta v^l,$$ \hspace{1cm} (22)

which implies that equation (21) becomes

$$\bar{e}(k+1) = A^r \bar{e}(k) + A^l \Delta v^r \bar{e}(k) + A^l \Delta v^l \bar{e}(k) + w(k),$$ \hspace{1cm} (23)

where $A^r_{cl} = (A^r + BK^r)$, $A^l_{cl} = (A^l + BK^l)$, and $A^l_{cl} = (A^l + BK^l)$.
Local controller $K^n$ is chosen such that the nominal system is asymptotically stable in closed-loop. For that purpose, we solve an LQR problem using matrices $A^n, B, Q,$ and $R$.

Local controllers $K^r$ and $K^l$ can be chosen such that the effect of the system dynamics in matrices $A^r$ and $A^l$ is compensated in closed-loop. For the case

$$K^r = \begin{bmatrix} 0 & -\frac{\bar{\xi}}{\bar{\tau}_M} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad K^l = \begin{bmatrix} 0 & \frac{\bar{\xi}}{\bar{\tau}_M} & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

we get that

$$A_{ij}^r = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \bar{\xi} \end{bmatrix}, \quad A_{ij}^l = \begin{bmatrix} 0 & 0 & \bar{\xi} \\ 0 & 0 & 0 \end{bmatrix}.$$  

Remark 2: Note that it is possible to completely remove the effects of the uncertainty on the $\bar{\xi} e_r$ and $\bar{\theta} e_l$ components. However, the $\bar{\xi} e_r$-component cannot be compensated directly.

We have to compute the sequence of sets that can be reached by the uncertain system, locally controlled. It can be done iteratively, with $\mathcal{R}_0 = 0$ and the following iteration,

$$\mathcal{R}_{j+1} \triangleq \bigcup_{A_{ij} \in \mathcal{AF}} (A_{ij} + BK) \mathcal{R}_j \oplus W, \quad \forall j = 0, \ldots, N - 1,$$

being $\mathcal{R}_j$ the reachable convex set of the $j$-step in the prediction horizon.

From Remark 2, the reachable sets obtained through (26) have to be solved as

$$\mathcal{R}_{j+1} = A_{ij}^r \mathcal{R}_j \oplus \mathcal{CO}\{A_{ij}^r \Delta v^m \mathcal{R}_j\} \oplus A_{ij}^l \Delta v^M \mathcal{R}_j \} \oplus W,$$

$$\forall j = 0, \ldots, N - 1,$$

since the set of matrices resulting from (23)-(25) leads to a segment in the state space.

Note that, this formulation is slightly different from [9], since now we take into account the parameter dependence of the system dynamics in the reachable sets calculation.

C. Robust Tubed-based MPC

When we deal with uncertain systems, deterministic MPC is limited (although some degree of robustness is achieved due to feedback [17]) because the uncertainties are not explicitly considered in the synthesis of the control law to guarantee robust stability. As explained above, the concept of tube-based MPC consists in replace the original constraints (11) with more restricted ones [9]. Following the ideas of Chisci et al., imposing that $\bar{e}_j \in \bar{E}_j \quad \forall j = 0, \ldots, N$, where $\bar{E}_j$ is defined as

$$\bar{E}_j = E \cap \mathcal{F}_j \quad \forall j = 0, \ldots, N - 1,$$

then constraints satisfaction is ensured and also feasibility is preserved in the presence of uncertainties in the system (9).

Input constraints are also replaced by

$$\bar{U}_j = U \cap [K^n \mathcal{R}_j \oplus K^r \Delta v^m \mathcal{R}_j \oplus K^l \Delta v^M \mathcal{R}_j]$$

$$\forall j = 0, \ldots, N - 1.$$  

D. MPC strategy

MPC also referred as moving or receding horizon can be employed to deal with constrained regulation problems [5], [6], [17]. Assume that a measurement of the state $\bar{e}$ is available at the current time $k$. Then, the optimization problem is stated as follows

$$\min_{\bar{G}(k) \in \{g_k, \ldots, g_{k+N-1}\}} J_N(\bar{G}(k), \bar{e}(k)), \quad \text{s.t.}$$

$$\bar{e}_{k+j|k} \in \bar{E}_j \quad \forall j = 1, \ldots, N,$$

$$g_{k+j|k} \in \bar{U}_j \quad \forall j = 0, \ldots, N - 1,$$

$$\bar{e}_{k+N|k} \in \Omega \ominus \mathcal{R}_N,$$

where the sequence $\bar{G}(k) \triangleq \{g_k, \ldots, g_{k+N-1}\}$ denotes the future sequence of control inputs of the system along the prediction horizon $N$, and $\bar{e}_{k+j|k}$ is the predicted state of the system applying $G$ to system (12). The cost function is given by

$$J_N(\bar{G}(k), \bar{e}(k)) = \sum_{j=0}^{N-1} \bar{e}_{k+j|k}^T Q \bar{e}_{k+j|k} + g_{k+j|k}^T R g_{k+j|k} + Y(\bar{e}_{k+N|k}),$$

where $Q = Q^T \geq 0$ and $R = R^T > 0$.

Notice that the MPC includes state and input constraints (28) and (29), a terminal cost $Y(\cdot)$, and a terminal constraint given by the region $\Omega$, which are calculated in the following subsection.

The MPC control law is based on the following idea: At time $k$, compute the optimal solution $G^*(k) = \{g^*_k, \ldots, g^*_{k+N-1}\}$ to problem (30) and apply

$$g(k) = g^*_k$$

as input to system (12), and repeat the optimization (30) at time $k+1$ based on the new state $\bar{e}(k+1)$, and so on.

E. Terminal constraints for MPC

Given a quadratic function $Y(\bar{e}) = \bar{e}^T P \bar{e}$ the stability of the system is guaranteed if there exists a matrix $P > 0$ such that $Y(\bar{e}(k+1)) - Y(\bar{e}(k)) < 0$, for all $\bar{e}(k) \neq 0$ (Lyapunov function) [18], [19].

Using the LMI formulation [18], [19] to obtain matrix $P$ and optimizing with respect to the $LQR$, we define the following inequality

$$\bar{e}^T (K_{ij}^r)^T P A_{ij}^r \bar{e} - \bar{e}^T P \bar{e} \leq \bar{e}^T \left( -Q - (K^r)^T R K^r \right) \bar{e}.$$  

The closed-loop matrices $A_{ij}^r$ are given by

$$A_{ij}^r = A^r + B K^r,$$  

$$i = 1, \ldots, 4,$$  

Remark 3: It has to be pointed out that the calculation of reachable sets could lead to empty or tiny restricted state and input constraints (sets $\bar{E}$ and $\bar{U}$). This fact constitutes an important issue of tube-based MPC approaches [9], [10], [14]. In the case under analysis, the obtained constraint sets, for the nominal system, have acceptable sizes.
being $\kappa^i$ the gains assuring the stability of the closed-loop system, and
\begin{align}
A_1 &= (A^n + A^r \delta v_m^n + A^l \delta v_M^l), \\
A_2 &= (A^n + A^r \delta v_m^n + A^l \delta v_M^l), \\
A_3 &= (A^n + A^r \delta v_m^n + A^l \delta v_M^l), \\
A_4 &= (A^n + A^r \delta v_m^n + A^l \delta v_M^l).
\end{align}

Now, if we apply the Schur complement to (32), it becomes
\begin{equation}
P - Q - (\kappa^i)^T R \kappa^i (A_i^T)^T P^{-1} \geq 0,
\end{equation}
for the $i$-th vertex. From previous inequality, we get
\begin{align}
P &\begin{bmatrix}
A_i^T \\
Q^i \\
R^i \kappa_i
\end{bmatrix} \\
P^{-1} &\begin{bmatrix}
0 & 0 \\
I_3 & 0 \\
0 & I_3
\end{bmatrix} \\
&\geq 0,
\end{align}
for $i$-th vertex. In order to remove the bilinear terms on $P$, we pre- and post- multiply by
\begin{equation}
\begin{bmatrix}
P^{-1} & 0 & 0 \\
0 & I_3 & 0 \\
0 & 0 & I_3
\end{bmatrix},
\end{equation}
and substituting $S = P^{-1}$, $Y^i = \kappa^i P^{-1}$, and $A_i^T = A^i + B \kappa^i$, we finally obtain
\begin{equation}
\begin{bmatrix}
S & S (A^n)^T + (Y^i)^T B^T & SQ^{1/2} (Y^i)^T R^{1/2} \\
A_i^T \bar{S} + BY^i & S & 0 \\
Q^{1/2} S & 0 & I_3 \\
R^{1/2} Y^i & 0 & I_3
\end{bmatrix} \geq 0.
\end{equation}

The solution to the previous LMI produces four gains ($\kappa^1, \kappa^2, \kappa^3, \kappa^4$) and the matrix $P$.

Finally, in order to calculate the terminal region $\Omega$ as a robust positively invariant set for the system, we have adapted the ideas of Blanchini presented in [13], and Kolmanovsky and Gilbert in [14].

The stability of MPC is ensured through the quadratic Lyapunov function determined by matrix $P$, obtained solving previous LMI, and the robust positively invariant set [4]. Stability properties for analogous control strategies have been analyzed in literature, see for instance [12].

IV. RESULTS AND DISCUSSION

The aim of this section is to check the performance of the robust tube-based MPC control law and to compare it with existing time-varying control techniques. In this case, the linear time-varying controller described in [15] has been implemented. Furthermore, in order to compare our new formulation with a controller that compensates slip effects, the work presented in [1] has been considered. Simulations have been carried out in Matlab® Suite using the LMI toolbox [20] and MPT toolbox [21].

In order to check the robustness of the controllers, nominal slip for each wheel is $20\%$, and we have supposed that slip measurements vary in $\pm 10\%$. The uncertainty set $W = \{w_1, w_2 \in \pm 0.005[m], w_3 \in \pm 0.2[deg]\}$. State constraints are $E = \{e_x, e_y \in \pm 0.3[m], e_\theta \in \pm 20[deg]\}$, reference linear wheel velocities are restricted to $v_r, v_l \in [0.2,1.5][m/s]$, and real linear wheel velocities are restricted to $v_r, v_l \in [-3.3][m/s].$ The rest of parameters are: $b = 0.5[m], T_s = 0.1[s], Q = \text{diag}([1 1 0.01]), R = I_2, N = 5^2$. The parameters of the two selected controllers are $\beta_k = 1$ and $\delta_i = 0.6$ to reach a soft overdamped closed-loop behavior. For more details about these parameters see [1].

Finally, in all cases the initial location of the mobile robot differs from the initial reference, $p = \{x \ y \ \theta\} = [0.1[m] \ 0.1[m] \ 0[deg]]^T$ and $p^r = [0 \ 0 \ 0]^T$.

Fig. 2 shows the trajectories. Although we have tested many trajectories, in this case, we show an 8-shaped reference trajectory which is a typical reference trajectory in mobile robotics. It is possible to observe that although some noise has been added to the simulations, the control laws that compensate the slip effects have a better behavior than the controller proposed in [15].

The errors can be better observed from Fig. 3. As expected the controller presented in this work achieves the smallest error due to the prediction capability of MPC, and to the inclusion in the control law of the additive uncertainty. In this case, the longitudinal error of the controller without slip compensation violates the state constraints, and there are some peaks in the lateral error close to $0.25[m]$. It could be unacceptable for narrow spaces.

As explained, slip decreases the linear velocity, for that reason controllers must increase this component of the velocity to compensate this effect, as shown in Fig. 4.

![Fig. 2. Simulated Trajectory](image)

V. CONCLUSIONS

This paper shows a robust predictive control law for constrained mobile robots under slip conditions. The concept of tubes has been considered to provide robustness to the MPC strategy. The control objective was to regulate the 2Notice that we have employed a short prediction horizon and the size of the matrices involved in the OP problem is relatively small ($3 \times 3$). For that reason, the computational complexity required to obtain the polytopes and geometrical structures is affordable.
The comparative study with previous control laws illustrates the promising behavior of the robust tube-based MPC. Furthermore, this simulation has demonstrated the robustness of the solution.

The main drawback of this solution is a certain degree of conservatism, in future it will be tackled. Furthermore, the proposed strategy will be tested on a real mobile robot.

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