Necessary Condition for Simple Oscillatory Neural Control of Robotic Yoyo

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Abstract—Open-loop unstable rhythmic tasks, like yoyo playing, require proper phase locking to stabilize. Given the phase-locking property of coupled oscillators, and their successful application to open-loop control of rhythmic motion, we investigate their application to closed-loop control of yoyo. In particular, we focus on pulse-coupling, where the yoyo sends a feedback upon reaching the bottom of the string and the onset of the oscillatory cycle is used to trigger the movement. A necessary condition for stable control is derived and applied to disqualify the simply coupled stand-alone leaky-integrate and fire oscillator. In contrast, an alternative circuit level oscillator, known as phase-locked loop, is proposed. The results are verified by numerical simulations and demonstrate that the proposed pulse-coupled oscillatory control provides a model-free control strategy that operates with an easy-to-measure low-rate feedback.

I. INTRODUCTION

For the past two decades, a number of oscillatory neural networks (ONN) have been developed to model the open-loop control of rhythmic movement in biological systems [1]-[6]. The characteristic behavior of these oscillatory networks emerges from the mutual coupling between closely tuned intrinsic oscillators. The coupling evokes phase locking, where all the oscillators oscillate at a single frequency and maintain a constant pattern of relative phases [1], [2]. Furthermore, with a different set of parameters, the same network may oscillate at a different frequency with a different pattern of relative phases. Each pattern of relative phases gives rise to a particular pattern of movement, so one oscillatory neural network may produce multiple patterns of movement [3], [5], [6].

Encouraged by the development of various models of oscillators and their applications in open-loop control of rhythmic motion, we investigate their application to closed loop control. Neural oscillators have also been hypothesized to participate in temporal pattern detection as part of neural phase-locked loops [7]. Phase-locked loops (PLLs) operate as circuit-level oscillators, which may behave differently then single neuron oscillators [8]. Thus, we consider both stand-alone and circuit level oscillators.

Yoyo playing is considered as a representative example of open-loop unstable control problems that involve intermittent dynamical environments. Stable control of yoyo playing relies on a proper phase relationship between the action of the controller and the motion of the yoyo. Given the phase-locked property of coupled oscillators, we investigate the possibility of controlling the yoyo by properly coupling it to an oscillator. Successful control of robotic yoyo playing with a neural oscillator would provide a model-free alternative to the trajectory planning [14] and model-based control of such tasks.

We restrict ourselves to controlling the vertical movement of the yoyo as it rolls up-and-down along the string. Compensation of the pitching, yawing and swinging is not considered here. Thus, the task requires moving the hand up and down at the right phase of the yoyo motion. We focus on pulse-coupling, where the yoyo sends a feedback upon reaching the bottom of the string and the onset of the oscillatory cycle is used to trigger the movement. Pulse-coupling [5], [6] is natural for the task and provides the potential advantage of easy-to-measure low feedback rate control.

The oscillatory control system is described in section II and the open loop responses of the different components are reviewed in section III. The necessary condition for stable control is derived in section IV and used to evaluate the different oscillators. Numerical simulations for verifying the analytic results are presented in section V. We conclude in section VI, and discuss the benefit of the method.

II. CONTROL CONFIGURATION AND MODELING

A. Configuration

Two pulse-coupled oscillators form the basic building block for open-loop movement control [1], [3], [9]. Each oscillator generates an event (spike) at a particular phase of its cycle, which is taken to be the origin. When coupled, the events generated by one oscillators affects the state of the other oscillator and cause its period to prolong (excitatory coupling) or shorten (inhibitory coupling). Excitatory coupling drives the closely tuned coupled oscillators to synchronization so the two oscillators operate together. Inhibitory coupling may drive the closely tuned coupled oscillators to maintain different phase-relationships depending on the strength of the coupling.

Motivated by the oscillatory, albeit damped, nature of the yoyo, we shall investigate the possibility of coupling the yoyo to a neural oscillator in a closed-loop as shown in Fig. 1. The yoyo generates an event upon reaching the bottom of the string, in the form of a peak in the applied force. The event generated by the oscillator triggers the upward acceleration of the robot, while the event generated by the yoyo signals the robot to return to its origin.

B. Oscillatory units

Models of single neuron oscillators are captured by a first or second order nonlinear differential equation with at least one stable periodic solution. Different types of oscillators, including the Van der Pol relaxation oscillator [1], [6], the Stein oscillator [6], and the leaky integrate-and-fire (LIF) oscillator...
[4]-[6], [8], [10], may be used without significantly affecting the overall behavior [6]. As a simple but representative single-neuron model, we consider the LIF oscillator described in [8] with either inhibitory (iLIF) or excitatory (eLIF) coupling.

A PLL includes an intrinsic oscillator, which is indirectly connected to the input via a phase-detector (PD) as shown in Fig. 2b [7], [8]. The PD compares the output of the intrinsic oscillator, whose impulses are referred to as the internal events, with the external input, whose impulses are referred to as the external events, and generates a signal representing their phase relationship. The output from the PD is fed into the intrinsic oscillator and may have either inhibitory (iPLL) or excitatory (ePLL) effect. As a result, the intrinsic oscillator can track the period of the input and respond to its variations. Special properties of PLLs, which distinguish them from single-neuron oscillators, emerge when the PD is sensitive to the correlation, instead of difference, between the two inputs [8]. A particular correlation-based PD, which responds continuously as long as the external and internal events occur within the preceding time window $T_w$, is depicted in Fig. 2c. Specifically, each impulse triggers a unit square wave of fixed duration $T_w$, and the output of the PD is the product of the two square waves. In the sequel, we consider PLLs with LIF intrinsic oscillators and correlation based PDs.

C. Switching control algorithm

The oscillator provides only a timing signal, which triggers the upward movement of the robot. Once initiated, the stereotypical movement continues with a constant acceleration until the yo-yo hits the bottom. To simplify the analysis, we neglect the dynamics of the robot by assuming that the robot can track exactly the following trajectory

$$\ddot{y} = \begin{cases} \kappa g & \text{for } s(t) = 1 \\ -c_1 \dot{y} - c_2 y & \text{for } s(t) = 0 \end{cases}$$

(1)

where the parameter $\kappa > 0$ determines the magnitude of the acceleration, and the parameters $c_1 \geq 0$ and $c_2 > 0$ are the differential and proportional gains of the controller that smoothly moves the robot back to its origin before the yo-yo reaches the next top position. The switching signal $s(t)$ is set by the oscillator and reset by the output from the yo-yo as described below.

D. Switching signal

D.1 Simply generated switching signal

A LIF oscillator generates a train of impulses at $\{T_j\}$. Based on this information, the switching signal is set at the time of occurrence of an impulse and is reset immediately after the yo-yo reaches the bottom at $T_2$. The resulting simply generated switching signal is given by:

$$s(t) = \begin{cases} 1 & \text{for } T_j \leq t < T_{j+1}, j = 1, 2, \cdots \\ 0 & \text{otherwise} \end{cases}$$

(2)

A fixed delay may be added without affecting the subsequent analysis.

D.2 Variably-delayed switching signal

The output of the PLL includes both the impulses from the internal oscillator at $\{T_j\}$ and the pulses of activity evoked by the PD. Hence, the timing of the switching signal may be based not only on the timing of the oscillatory events but also on the duration $\delta_j$ of the PD response. However, in order to use the additional information it is necessary to wait to the PD response. This is natural when the external event (generated by the yo-yo) leads the internal event (from the oscillator) since in this case the internal event occurs too soon to set the switching signal directly, as in (2). In contrast, when the external event lags the internal event, only the timing of the internal event is available before the next occurrence of the yo-yo-event, and (2) should be used.

It is noted that the type of phase relationship between the external and internal events is determined by the nature of the internal coupling in the PLL[8]. Particularly, the external events in an ePLL lag the internal events (lagging input) whereas the external events in an iPLL lead the internal events (leading input).
Thus, the switching signal evoked by the ePLL is simply generated according to (2), based only on the timing of the internal oscillatory events. However, the switching signal evoked by the iPLL should be delayed by an extent that may depend on the duration $\delta_j$ of the PD response. Linear dependence results in the switching signal specified below, which is referred to as variably-delayed switching signal:

$$s(t) = \begin{cases} 
1 & \text{for } T_j + (c+1)\delta_j \leq t < t_{j+1}, j = 1, 2, \ldots \vspace{0.2cm} \\
0 & \text{otherwise} 
\end{cases}$$

where $c > 0$ is a constant coefficient (we shall fix $c = 2$).

E. Yoyo dynamics

The dynamics of the yoyo is captured by the simple one-degree of freedom (DOF) model developed in [11]. As described there, the one DOF model is derived from a more detailed two DOF model by capturing the dynamics at the bottom with an equivalent restitution effect $e_{eq}$. The one-DOF model adequately describes the total energy loss and is therefore sufficient for control analysis and simulation.

III. OPEN LOOP ANALYSIS

Before analyzing the stability of the closed loop system, the open-loop responses of the different oscillators and the yoyo are analyzed.

A. Oscillators

We briefly review the behavior of an oscillator in response to a train of external events with a constant period (see [8] for detailed analysis).

A.1 LIF oscillator

The membrane voltage of a LIF oscillator decays exponentially from its initial value $v_0$ to its equilibrium $v_e$. When the membrane voltage reaches a threshold level it generates a spike and resumes the initial voltage value. Let the decay time constant be $\tau$ and assume the threshold is zero, $v_0 < 0$, and $v_e > 0$, the period of the resulting oscillation is $\tau_{osc} = \tau \ln(\rho)$ where $\rho = 1 - v_0/v_e > 1$.

An excitatory or inhibitory input event increases or decreases the membrane voltage respectively, and thus shortens or lengthens its period. The modified period depends not only on the strength of the input $\alpha$ ($\alpha > 0$ for eLIF, $\alpha < 0$ for iLIF) but also on its timing with respect to the cycle of the oscillator. When the input period is $\tau_{ip}$, the dynamics of the time delay is given by

$$\tau_d(k+1) = \tau_d(k) - \tau_{ip}(k) + \tau_{ip}$$

$$\tau_{ip}(k) = \tau_{osc} + \tau \ln \left(1 - \frac{\alpha e^{\tau(k)/\tau}}{\tau} \right)$$

where $\tau_{ip} \in [0, \tau_{osc}]$ is the time delay with respect to the most recent internal event and $\tau_{ip}$ is the perturbed period.

In the $\tau_d$-plane, the curve of function (4b) is referred to as the working curve of the oscillator. Typical working curves are illustrated in Fig. 3. The range of values of the modified period $\tau_{ip}$ is referred to as the working range. A train of input events at a constant period $\tau_{ip}$ would result in a 1:1 phase locking at the fixed point $\tau_d^*$ such that $\tau_d^*(\tau_d^*) = \tau_{ip}$ if $\tau_{ip}$ is within the working range. The fixed point is stable for eLIF and unstable for iLIF, so the corresponding working curves are plotted as solid and dashed lines, respectively.

![Fig. 3. The working curve of the yoyo and its intersections with those of the oscillators](image)

A.2 PLL oscillator

Consider a PLL with a LIF as the intrinsic oscillator, and the correlation based PD of Fig. 2c. Either ePLL or iPLL may achieve stable 1:1 locking provided that the period of the input train $\tau_{ip}$ is in the corresponding working range. In particular, $\tau_{ip}$ should be less than, or greater than $\tau_{osc}$ to entrain the ePLL or iPLL, respectively. Phase locking in ePLLs is characterized by lagging input and the dynamics of the time difference is given by

$$\tau_d(k+1) = \tau_d(k) - \tau_{ip}(k) + \tau_{ip}$$

$$\tau_{ip}(k) = \tau_{osc} + \tau \ln \left(1 - \beta e^{\tau(k)/\tau} + \beta e^{\tau_{ip}(k)/\tau} \right)$$

In contrast, phase locking in iPLLs is characterized by leading input and the dynamics of the time difference is given by

$$\tau_d(k+1) = \tau_d(k) + \tau_{ip}(k) - \tau_{ip}$$

$$\tau_{ip}(k) = \tau_{osc} + \tau \ln \left(1 + \beta - \beta e^{(\tau(k)-\tau_{ip}(k))/\tau} \right)$$

The parameter $\beta$ defines the strength of the continuous input from the PD to the LIF. Note that $\tau_d \in [0, \tau_x]$ is positive and corresponds to the interval between the two events of a pair. Representative stable working curves for ePLL ($\beta > 0$) and iPLL ($\beta < 0$) are illustrated in Fig. 3.

B. Yoyo working curve

Under the control algorithm (1), the robot is either i) waiting, ii) accelerating upward, or iii) returning smoothly to its origin. The three corresponding intervals are marked $I, II$ and
II respectively (as shown in Fig. 4). Denote the steady state value of these intervals by $I^*_{\mu}, II^*_{\mu}$ and III$^*_{\mu}$ and the steady state period by $T^* = I^*_{\mu} + II^*_{\mu} + III^*_{\mu}$. It can be shown that the ratio between the acceleration interval $II^*_{\mu}$ and the period $T^*$ is constant. In particular,

$$v(\kappa, e_{eq}) \triangleq \frac{II^*_{\mu}}{T^*} = \frac{1-e_{eq}}{2(1+e_{eq})\kappa} > 0 \quad (7)$$

In the $II^*T^*$-plan, this relation defines a straight line passing through the origin with a positive slope $v$. This line is referred to as the working curve of the yoyo.

![Fig. 4. The waves of an iLIF-controlled yoyo](image)

**IV. YOYO CONTROL WITH SIMPLY GENERATED SWITCHING SIGNAL**

**A. The closed-loop return map**

Given the amplitude $\Theta_j$ at the beginning of the $j$-th cycle and the waiting interval $I_j$, the motion in the $j$-th cycle is completely determined (see detailed analysis in Appendix I). To facilitate the formulation, we introduce two auxiliary variables $\Delta_j$ and $\mu_j$, defined such that

$$\Theta_j = \frac{\gamma}{2} \Delta_j^2 \quad (8)$$

where $\gamma = m r/(I + m r^2)$ is the yoyo parameter that depends on its mass $m$, inertia $I$, and the radius of the axle $r$; and

$$I_j = f_1(\mu_j) \Delta_j \quad f_1(\mu) = \sqrt{\mu} \quad (9)$$

In fact, $\Delta_j$ is the free flight interval ($h(t) \equiv 0$) from the top to the bottom. The parameter $\mu_j$ is the square of the ratio of the waiting interval $I_j$ and the free flight interval $\Delta_j$, and is referred to as the waiting ratio. These two relations may be regarded as a coordinate transformation. Thus, we choose $\Delta$ and $\mu$ as two independent state variables.

**Proposition 1:** Under control algorithm (1) and a simply generated switching signal (2), the return map of a closely tuned system, where the period of the yoyo is close to the intrinsic period of the oscillator $\tau_{osc}$, satisfies

$$\Delta_{j+1} = f_4(\mu_j) \Delta_j \quad (10a)$$

$$f_1(\mu_{j+1}) \Delta_{j+1} = \tau_p f_2(\mu_j) \Delta_j - (f_2(\mu_j) + f_3(\mu_j)) \Delta_j \quad (10b)$$

where functions $f_1, f_2$ and $f_3$ are given by (9), (23) and (24), respectively; and

$$f_4(\mu) = \sqrt{f(\mu; \kappa, e_{eq})}$$

in which $f$ is given by (22).

**Proof:** Applying the coordinate transformation (8) to equation (21) immediately yields (10a).

To prove (10b), we use the relation, as shown in Fig. 4, that

$$II_j + III_j + I_{j+1} = \tau_p \tau_d^j \quad (11)$$

We note that under the proposed control algorithm $\tau_d^j$ is identical to $II_j$, and use the relationship in (23), (24) and (9).

**B. The fixed point and its stability**

The system (10) is an autonomous second-order discrete system in implicit form with two parameters $\kappa$ and $e_{eq}$. Although an explicit form may be derived, we shall analyze the fixed point and its local stability directly on the implicit form. This is a key point to simplify the analysis in our system. The next result presents the fixed point of this system.

**Proposition 2:** The discrete system (10) has a fixed point at $(\mu^*, \Delta^*)$ where

$$\mu^* = \frac{((1 + e_{eq})\kappa - (1 - e_{eq})^2)^2}{((1 + e_{eq})\kappa + (1 - e_{eq})^2)^2} \quad (12)$$

and $\Delta^*$ is a solution to

$$(f_1^* + f_2^* + f_3^*) \Delta^* = \tau_p f_2^* \Delta^* \quad (13)$$

in which $f_1^* = f_1(\mu^*), f_2^* = f_2(\mu^*; \kappa)$ and $f_3^* = f_3(\mu^*; \kappa, e_{eq})$.

**Proposition 3:** A necessary condition for stable oscillatory yoyo control with algorithm (1) and a simply generated switching signal (2) is

$$\frac{1}{d \tau_d} \frac{d \tau_d}{d \mu} < \frac{f_2^*}{f_1^* + f_2^* + f_3^*} = \frac{II^*_{\mu}}{T^*} \quad (14)$$

**Proof:** Local stability of the fixed point depends on the eigenvalues of the linearized system around the fixed point. Substituting the following coordinate transformation

$$\mu = \mu^* + \hat{\mu}, \quad \Delta = \Delta^* + \hat{\Delta} \quad (15)$$

in the return map (10), expanding on both sides to the first order, and neglecting higher order terms, yields

$$\begin{bmatrix} 1 & 0 \\ f_1^* & df_1^*/d\mu \end{bmatrix} \begin{bmatrix} x_{j+1} \\ x_j \\ \end{bmatrix} = \begin{bmatrix} 1 \\ -(1 - d\tau_d^j)f_2^* - f_3^* \end{bmatrix} \begin{bmatrix} x_{j+1} \\ x_j \\ \end{bmatrix} \quad (16)$$
where \( x = [\Delta, \Delta^* \mu]^T \) and
\[
df^*_i = -\frac{\partial f_i}{\partial \mu} |_{\mu^*}, \quad i = 1, 2, 3, 4; \quad dr^*_p = -\frac{\partial r_p}{\partial \alpha} |^*
\]
Thus we get a linear system
\[
x_{j+1} = Ax_j
\] (17)
where
\[
A = \begin{bmatrix}
1 & df^*_1 \\
-a_{21} & -a_{22}
\end{bmatrix}
\]
in which
\[
a_{21} = \frac{f_1^* + (1 - dr^*_p) f_2^* + f_3^*}{df^*_1}, \quad a_{22} = \frac{df^*_1 + (1 - dr^*_p) df^*_2 + df^*_3}{df^*_1}
\]
The characteristic equation of this matrix \( a(\lambda) = |\lambda I - A| \) can be easily calculated. According to Jury’s criterion[12], all its roots are located within the unit circle if and only if
\[
a(1) > 0 \quad \text{(18a)}
\]
\[
a(-1) > 0 \quad \text{(18b)}
\]
\[
|a(0)| < 1 \quad \text{(18c)}
\]
Condition (18a) leads to
\[
a(1) = -\frac{f_1^* + (1 - dr^*_p) f_2^* + f_3^*}{df^*_1} > 0
\] (19)
Since
\[
df^*_1 = \frac{1}{2\sqrt{|\mu^*|}} > 0, \quad df^*_4 = \frac{df^*_5}{2\sqrt|f^*|} > 0, \quad \forall \mu^* \in [0, 1]
\] condition (19) is reduced to
\[
f_1^* + (1 - dr^*_p) f_2^* + f_3^* < 0
\] (20)
which is equivalent to the proposed necessary condition.

C. Stable yoyo control

By evaluating the necessary stability condition (14) on the working curves of the different oscillators it can be shown that neither the iLIF, whose working curve is given in (4b) nor the ePLL, whose working curve is given by (5b), may be used to stably operate the yoyo, as is also apparent from Fig. 3. Since the eLIF is unstable under periodic input, this case is not considered. Thus the only network that may stabilize the yoyo is the iPLL, but it requires the variably-delayed switching signal. The resulting closed-loop system may be described by an autonomous third-order discrete system in implicit form. Stability analysis, similar to the one presented here in the context of proposition 3 indicates that the iPLL with the variably-delayed switching signal may indeed stabilize the yoyo [13].

V. SIMULATION

The simulations are conducted by using MATLAB Simulink toolbox. We illustrate that the iLIF-based controller is unstable while the iPLL-based controller is stable with an iLIF oscillator (\( \alpha = -0.1703, \tau_{osc} = 1.67 \)) starting from two close initial periods. In the former the period is 1.9689 s, which corresponds to an amplitude of 20 cm in the yoyo; whereas in the latter the period is 1.9787 s, which corresponds to an amplitude of 20.2 cm (1% increase) in the yoyo. Both cases show that the response gradually deviate from the desired equilibrium condition, with decreasing or increasing amplitude.

![Figure 5a and Fig. 5b show the time response with an iLIF oscillator (\( \alpha = -0.1703, \tau_{osc} = 1.67 \)) starting from two close initial periods. In the former the period is 1.9689 s, which corresponds to an amplitude of 20 cm in the yoyo; whereas in the latter the period is 1.9787 s, which corresponds to an amplitude of 20.2 cm (1% increase) in the yoyo. Both cases show that the response gradually deviate from the desired equilibrium condition, with decreasing or increasing amplitude.](image)

In contrast, Fig. 6 demonstrates that properly designed iPLL oscillator can stably control the yoyo. The iPLL oscillator is designed such that \( \beta = -1, \tau_{osc} = 1.67 \) s. More simulation
results on the fixed point and its stability are presented in [13].

Fig. 6. Control with iPLL (Yoyo-1, $e_{eq} = 0.9183$, $\gamma = 0.0817$): Last 20 s of the response (for $\kappa = 0.06$, $\tau_{osc} = 1.67$ s)

VI. DISCUSSION

We have studied the application of simple oscillatory neural networks for closed-loop control of yoyo, with either stand-alone LIF oscillator or the circuit level PLL oscillator. Using a stand-alone oscillator, the switching signal that activates the yoyo has to be simply generated based on the timing of the oscillatory events, since only this information is available. The switching signal should be also simply generated when operating with the ePLL since the oscillatory events lead the control with the simply generated switching control and yoyo. Detailed analysis along the lines of the analysis presented here holds directly when the oscillator-based approach provides a model-free low feedback rate control of open-loop unstable systems.

We thus conclude that only the iPLL may stably operate the yoyo. Detailed analysis along the lines of the analysis presented here verifies that the iPLL may stably operate the yoyo [13] as demonstrated here.

The work presented here may be extended to other periodic systems for which the working curve may be determined. In particular the results presented here holds directly when the resulting working curve is linear as in the case of the yoyo. The oscillator-based approach provides a model-free low feedback rate control of open-loop unstable systems.

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APPENDIX: YOYO ANALYSIS

**Proposition 4:** Given the $j$-th amplitude $\Theta_j$ and the waiting time ratio $\mu_j$, the consecutive amplitude $\Theta_{j+1}$ of the simplified yoyo under control algorithm (1), satisfies

$$\Theta_{j+1} = f(\mu_j; \kappa; e_{eq})\Theta_j$$  \hspace{1cm} (21)

where

$$f(\mu_j; \kappa; e_{eq}) = \kappa \varphi^2(\mu_j; \kappa) + \left( e_{eq} \sqrt{1 + (1 - \mu_j) \kappa} + \kappa \varphi(\mu_j; \kappa) \right)^2$$  \hspace{1cm} (22a)

in which

$$\varphi(\mu_j; \kappa) = \frac{-\sqrt{\mu_j + \sqrt{1 + (1 - \mu_j) \kappa}}}{1 + \kappa}$$  \hspace{1cm} (22b)

**Proposition 5:** Given the $j$-th amplitude $\Theta_j$ and the waiting time ratio $\mu_j$, the duration of each interval $\overline{II}$ and $\overline{III}$ is given, respectively, by

$$\overline{II}_j = f_2(\mu_j; \kappa) \Delta_j$$

$$\overline{III}_j = f_3(\mu_j; \kappa, e_{eq}) \Delta_j$$ \hspace{1cm} (23)

$$f_3(\mu_j; \kappa, e_{eq}) = e_{eq} \sqrt{1 + (1 - \mu_j) \kappa} + \kappa \varphi(\mu_j; \kappa)$$ \hspace{1cm} (24)

where $\varphi(\mu, \kappa)$ is defined by (22b).

REFERENCES


