MINIMAX INEQUALITIES IN G-CONVEX SPACES

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In this paper we establish two minimax theorems of Sion-type in G-convex spaces. As applications we obtain generalisations of some theorems concerning compatibility of systems of inequalities.

1. INTRODUCTION AND PRELIMINARIES

Motivated by Nash equilibrium and the theory of non-cooperative games, Fan [4] generalised Sion’s minimax theorem obtaining the following two-function minimax inequality:

THEOREM 1. Let $X$ and $Y$ be compact convex subsets of topological vector spaces and $f, g : X \times Y \to \mathbb{R}$. Suppose that $f$ is lower semicontinuous on $Y$ and quasiconcave on $X$, $g$ is upper semicontinuous on $X$ and quasiconvex on $Y$, and $f \leq g$ on $X \times Y$. Then $\min_{y \in Y} \sup_{x \in X} f(x, y) \leq \max_{x \in X} \inf_{y \in Y} g(x, y)$.

Granas and Liu [6, 7] obtained generalisations and versions of Theorem 1 involving three real functions $f, g, h$. On the other hand Park [14] extended Ky Fan’s result to G-convex spaces. In this paper we obtain a unified generalisation of all these results. Also we give a version of our main result for the case when $X$ is a convex subset of a topological vector space. As applications we obtain generalisations of some theorems of Granas and Liu [6, 7] and Liu [11] concerning compatibility of some systems of inequalities.

Let us recall some notions necessary in our paper.

A generalised convex space or a G-convex space $(X, D; \Gamma)$ consists of a topological space $X$ and a nonempty set $D$ such that for each $A \in \langle D \rangle$ with the cardinality $|A| = n + 1$, there exist a subset $\Gamma(A)$ of $X$ and a continuous function $\Phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\Phi_A(\Delta_J) \subset \Gamma(J)$.

Here $\langle D \rangle$ denotes the set of all nonempty finite subsets of $D$, $\Delta_n$ any $n$-simplex with vertices $\{e_i\}_{i=0}^n$ and $\Delta_J$ the face of $\Delta_n$ corresponding to $J \in \langle A \rangle$; that is, if $A = \{u_0, u_1, \ldots, u_n\}$ and $J = \{u_{i_0}, u_{i_1}, \ldots, u_{i_k}\} \subset A$, then $\Delta_J = \text{co}\{e_{i_0}, e_{i_1}, \ldots, e_{i_k}\}$. In case $D \subset X$ then $(X, D; \Gamma)$ will be denoted by $(X \supset D; \Gamma)$. For $(X \supset D; \Gamma)$, a subset $C$ of $X$ is said to be G-convex if $\Gamma(A) \subset C$ whenever $A \in \langle C \cap D \rangle$.
The main example of $G$-convex space corresponds to the case when $X = D$ is a convex subset of a Hausdorff topological vector space and for each $A \in \langle X \rangle$, $\Gamma(A)$ is the convex hull of $A$. For other major examples of $G$-convex spaces see [15, 16].

Let $(X, D; \Gamma)$ be a $G$-convex space. A function $f : X \to \mathbb{R} = \mathbb{R} \cup \{\pm \infty\}$ is said to be $G$-quasiconcave (respectively, $G$-quasiconvex) if for any finite set $\{u_1, \ldots, u_n\} \subset D$ and for each $x \in \Gamma(\{u_1, \ldots, u_n\})$ we have $f(x) \geq \min_{1 \leq i \leq n} f(u_i)$ (respectively, $f(x) \leq \max_{1 \leq i \leq n} f(u_i)$). We note that $f$ is $G$-quasiconcave (respectively, $G$-quasiconvex) if and only if, for each $\lambda \in \mathbb{R}$ the set $\{x \in X : f(x) > \lambda\}$ (respectively, $\{x \in X : f(x) < \lambda\}$) is $G$-convex. A function $f : X \times Y \to \mathbb{R}$ ($Y$ nonempty set) is said to be $G$-quasiconcave (respectively, $G$-quasiconvex) on $X$ if for each $y \in Y$ the function $x \to f(x, y)$ is $G$-quasiconcave (respectively, $G$-quasiconvex). Inspired by [1] and [9] we shall introduce two more general concepts.

Let $(X, D; \Gamma)$ be a $G$-convex space, $Y$ be a nonempty set and $f : D \times Y \to \mathbb{R}$, $g : X \times Y \to \mathbb{R}$. We say that $g$ is $G$-$f$-quasiconcave on $X$ if for any finite set $\{u_1, \ldots, u_n\} \subset D$ and for each $y \in Y$ we have

$$g(x, y) \geq \min_{1 \leq i \leq n} f(u_i, y)$$

for all $x \in \Gamma(\{u_1, \ldots, u_n\})$.

Note that the notion introduced above coincides with the corresponding notion in [9, Definition 2] only when $D = X$.

When $X$ is a convex subset of a topological vector space the concept of $G$-$f$-quasiconcavity reduces to that of $f$-quasiconcavity introduced by Chang and Yen in [1]. More precisely, in this case, if $f, g : X \times Y \to \mathbb{R}$ we say that $g$ is $f$-quasiconcave on $X$ if for any $\{x_1, \ldots, x_n\} \in \langle X \rangle$ and each $y \in Y$ we have

$$g(x, y) \geq \min_{1 \leq i \leq n} f(x_i, y)$$

for all $x \in \co(\{x_1, \ldots, x_n\})$.

Similarly, if $X$ is a nonempty set, $(Y, D; \Gamma)$ a $G$-convex space and

$$g : X \times Y \to \mathbb{R},$$

$$h : X \times D \to \mathbb{R}$$

two functions, we say that $g$ is $G$-$h$-quasiconvex on $Y$ if for any $\{v_1, \ldots, v_n\} \in \langle D \rangle$ and each $x \in X$ we have

$$g(x, y) \leq \max_{1 \leq i \leq n} h(x, v_i)$$

for all $y \in \Gamma(\{v_1, \ldots, v_n\})$.

**Remark 1.** It is easy to see that if $D \subset Y$, $g$ is $G$-$h$-quasiconvex on $Y$ whenever there exists a function $k : X \times Y \to \mathbb{R}$ such that:

(i) $g \leq k$ on $X \times Y$;

(ii) $k \leq h$ on $X \times D$;
(iii) \( k \) is \( G \)-quasiconvex on \( Y \).

Let \( X \) be a nonempty set, \( (Y,D;\Gamma) \) be a \( G \)-convex space and \( G : Y \rightarrow X, H : D \rightarrow X \) be two mappings (that is, set-valued functions). We say that \( H \) is a \emph{generalised \( G \)-KKM mapping with respect to \( G \)} if for each \( A \in D \), \( G(\Gamma(A)) \subset H(A) \). If \( X \) is a topological space, \( G : Y \rightarrow X \) is said to have the \( G \)-KKM property if for any mapping \( H : D \rightarrow X \) generalised \( G \)-KKM with respect to \( G \), the family \( \{\overline{H(v)} : v \in D\} \) has the finite intersection property (where \( \overline{H(v)} \) denotes the closure of \( H(v) \)).

Let \( X \) be a topological space and \( Y \) be a nonempty set. A function \( f : X \times Y \rightarrow \mathbb{R} \) is said to be \( \lambda \)-transfer upper semicontinuous (respectively \( \lambda \)-transfer lower semicontinuous) on \( X \) for some \( \lambda \in \mathbb{R} \) if for all \( x \in X, y \in Y \) with \( f(x,y) < \lambda \) (respectively \( f(x,y) > \lambda \)) there exist a neighbourhood \( V(x) \) of \( x \) and a point \( y' \in Y \) such that \( f(z,y') < \lambda \) (respectively \( f(z,y') > \lambda \)) for all \( z \in V(x) \). If \( f \) is \( \lambda \)-transfer upper (respectively lower) semicontinuous on \( X \) for any \( \lambda \in \mathbb{R} \), we say that \( f \) is transfer upper (respectively lower) semicontinuous on \( X \).

It is clear that every function upper semicontinuous (respectively, lower semicontinuous) on \( X \) is \( \lambda \)-transfer upper semicontinuous (respectively, \( \lambda \)-transfer lower semicontinuous) on \( X \) for any real \( \lambda \), but the converse is not true (see [2]).

### 2. Main Results

First we state three results from the literature which will be used in this section. The following continuous selection theorem is well-known (see [10, 13, 17]).

**Lemma 2.** Let \( (X,D;\Gamma) \) be a \( G \)-convex space and \( Y \) be a compact topological space. Let \( F : Y \rightarrow D, G : Y \rightarrow X \) be two mappings satisfying the following conditions:

(a) for each \( y \in Y, A \in \langle F(y) \rangle \) implies \( \Gamma(A) \subset G(y) \);

(b) \( Y = \bigcup \{\text{int} F^{-1}(u) : u \in D\} \).

Then \( G \) has a continuous selection; that is, there exists a continuous function \( p : Y \rightarrow X \) such that \( p(y) \in G(y) \) for each \( y \in Y \).

The next result is a particular case of Corollary in [12].

**Lemma 3.** Let \( X \) be a topological space and \( (Y,D;\Gamma) \) be a \( G \)-convex space, then any continuous function \( p : Y \rightarrow X \) has the \( G \)-KKM property.

Combining assertions (ii) and (iii) in Lemma 3 and assertion (ii) in Lemma 4 in [8] one obtains

**Lemma 4.** Let \( X \) be a topological space and \( D \) a nonempty set. If \( h : X \times D \rightarrow \mathbb{R} \) is \( \lambda \)-transfer upper semicontinuous, then \( \bigcap_{v \in D} H(v) = \bigcap_{v \in D} \overline{H(v)} \), where

\[
H(v) = \{x \in X : h(x,v) \geq \lambda\}.
\]

The main result of the paper is as shown in the following theorem.
THEOREM 5. Let \((X,D;\Gamma_1)\) and \((Y,D;\Gamma_2)\) be two compact \(G\)-convex spaces and let \(f : D_1 \times Y \rightarrow \mathbb{R}\), \(g : X \times Y \rightarrow \mathbb{R}\), \(h : X \times D_2 \rightarrow \mathbb{R}\) be three functions such that:

(i) \(g\) is \(G\)-\(f\)-quasiconcave on \(X\);
(ii) \(g\) is \(G\)-\(h\)-quasiconvex on \(Y\);
(iii) \(f\) is transfer lower semicontinuous on \(Y\);
(iv) \(h\) is transfer upper semicontinuous on \(X\);

Then \(\inf_{y \in Y} \sup_{u \in D_1} f(u,y) \leq \sup_{z \in X} \inf_{v \in D_2} h(x,v)\).

PROOF: We may suppose that \(\inf_{y \in Y} \sup_{u \in D_1} f(u,y) > -\infty\). It suffices to prove that for any real \(\lambda < \inf_{y \in Y} \sup_{u \in D_1} f(u,y)\) we have \(\lambda \leq \sup_{z \in X} \inf_{v \in D_2} h(x,v)\). Fix such a \(\lambda\) and define the mappings \(F : Y \rightarrow D_1\), \(G : Y \rightarrow X\), \(H : D_2 \rightarrow X\) by

\[
F(y) = \{u \in D_1 : f(u,y) \geq \lambda\}, \quad G(y) = \{x \in X : g(x,y) \geq \lambda\} \quad \text{and} \quad H(v) = \{x \in X : h(x,v) \geq \lambda\}.
\]

First we show that \(G\) and \(F\) satisfy the conditions of Lemma 2. Let \(y \in Y\), \(\{u_1,\ldots,u_n\} \subset F(y)\) and \(x \in \Gamma_1(\{u_1,\ldots,u_n\})\). Since \(g\) is \(f\)-quasiconcave on \(X\), \(g(x,y) \geq \min_{1 \leq i \leq n} f(u_i,y) \geq \lambda\), hence \(x \in G(y)\). Thus \(\Gamma_1(\{u_1,\ldots,u_n\}) \subset G(y)\).

For each \(y \in Y\) there exists \(u \in D_1\) such that \(f(u,y) > \lambda\) (as consequence of \(\lambda < \inf_{y \in Y} \sup_{u \in D_1} f(u,y)\)). By (iii) there exist \(u' \in D_1\) and a neighbourhood \(V(y)\) of \(y\) such that

\[
\bigcap_{z \in V(y)} \{u \in D_1 : f(u,z) > \lambda\} \subset \bigcap_{z \in V(y)} F(z),
\]

hence \(y \in \text{int} F^{-1}(u')\). Thus condition (b) in Lemma 2 is satisfied. By Lemma 2, there exists a continuous function \(p : Y \rightarrow X\) such that \(p(y) \in G(y)\) for every \(y \in Y\).

Next we prove that \(H\) is a generalised \(G\)-KKM mapping with respect to \(G\). Suppose that there exist a nonempty finite set \(\{v_1,\ldots,v_n\} \subset D_2\) and a point \(x \in X\) such that

\[
x \in G\left(\Gamma_2(\{v_1,\ldots,v_n\})\right) \setminus \bigcup_{i=1}^n H(v_i).\]

Since \(x \in G\left(\Gamma_2(\{v_1,\ldots,v_n\})\right)\), there exists \(y \in \Gamma_2(\{v_1,\ldots,v_n\})\) such that \(g(x,y) \geq \lambda\).

By \(x \notin \bigcup_{i=1}^n H(v_i)\) we get \(h(x,v_i) < \lambda\) for each \(i \in \{1,\ldots,n\}\). Taking into account (ii) we obtain the following contradiction

\[
\lambda \leq g(x,y) \leq \max_{1 \leq i \leq n} h(x,v_i) < \lambda.
\]

Thus \(H\) is a generalised \(G\)-KKM mappings with respect to \(G\), and consequently it is generalised \(G\)-KKM mapping with respect to \(p\), too. By Lemma 3, the family of sets
\{\overline{H}(v) : v \in D_2\} has the finite intersection property. Since for each \(v \in D_2\), \(\overline{H}(v)\) is a closed subset of compact space \(Y\), by Lemma 4 we infer that \(\bigcap_{v \in D_2} H(v) \neq \emptyset\), that is, \(\sup_{x \in X} \inf_{v \in D_2} h(x, v) \geq \lambda\).

**Remark 2.** Following the proof of Theorem 5 it seems that if \(\inf_{y \in Y} \sup_{u \in D_1} f(u, y) > -\infty\), instead of conditions (iii) and (iv) it would be sufficient to put the following conditions:

(iii') \(f\) is \(\lambda\)-transfer lower semicontinuous on \(Y\) for any \(\lambda < \inf_{y \in Y} \sup_{u \in D_1} f(u, y)\);

(iv') \(h\) is \(\lambda\)-transfer upper semicontinuous on \(X\) for any \(\lambda < \inf_{y \in Y} \sup_{u \in D_1} f(u, y)\).

But this clearly less demanding conditions make really no difference. In fact, assume

\[ a = \inf_{y \in Y} \sup_{u \in D_1} f(u, y) > -\infty \]

and define the functions

\[ f'(u, y) = \min(f(u, y), a) \]
\[ g'(x, y) = \min(g(x, y), a) \]
\[ h'(x, v) = \min(h(x, v), a) \]

We observe that:

(a) if conditions (i), (ii) in Theorem 5 hold for \(f, g, h\), then they hold also for \(f', g', h'\);

(b) if \(f\) is \(\lambda\)-transfer lower semicontinuous on \(Y\) (respectively, \(h\) is \(\lambda\)-transfer upper semicontinuous on \(X\)) whenever \(\lambda < a\), then \(f'\) is transfer lower semicontinuous on \(Y\) (respectively, \(h'\) is transfer upper semicontinuous on \(X\));

(c) \(\inf_{y \in Y} \sup_{u \in D_1} f'(u, y) \leq \sup_{x \in X} \inf_{v \in D_2} h'(x, v)\) implies \(\inf_{y \in Y} \sup_{u \in D_1} f(u, y) \leq \sup_{x \in X} \inf_{v \in D_2} h(x, v)\).

A mapping \(F : Y \to X\) (\(X\) nonempty set, \(Y\) topological space) is said to have the **local intersection property** (see [18]) if for each \(y \in Y\) with \(F(y) \neq \emptyset\), there exists an open neighbourhood \(V(y)\) of \(y\) such that \(\bigcap_{z \in V(y)} F(z) \neq \emptyset\).

The following continuous selection theorem is [18, Theorem 1].

**Lemma 6.** Let \(X\) be a nonempty subset of a topological vector space and \(Y\) be a paracompact topological space. Suppose that \(F, G : Y \to X\) are two mappings satisfying the following conditions:

(a) for each \(y \in Y\), \(F(y)\) is nonempty and \(\text{co} F(y) \subset G(y)\);

(b) \(F\) has local intersection property.

Then \(G\) has a continuous selection.
It can be easily prove that if \( D = X \) and \( F \) is a mapping with nonempty values, then conditions (b) in Lemmas 2 and 6 are equivalent (see [8, Proposition 1]).

The following version of Theorem 5 shows that in the case when \( X \) is a convex subset of a topological vector space the conclusion holds if the \( G \)-convex space \((Y, D; \Gamma)\) is only paracompact. The proof is similar to that of Theorem 5 using as argument Lemma 6 instead of Lemma 2.

**Theorem 7.** Let \( X \) be a compact convex subset of a topological vector space and \((Y, D; \Gamma)\) be a paracompact \( G \)-convex space. Let \( f, g : X \times Y \to \mathbb{R} \) and \( h : X \times D \to \mathbb{R} \) be three functions such that:

- (i) \( g \) is \( f \)-quasiconcave on \( X \);
- (ii) \( g \) is \( G \)-\( h \)-quasiconvex on \( Y \);
- (iii) \( f \) is transfer lower semicontinuous on \( Y \);
- (iv) \( h \) is transfer upper semicontinuous on \( X \).

Then
\[
\inf_{y \in Y} \sup_{x \in X} f(x, y) \leq \sup_{x \in X} \inf_{v \in D} h(x, v).
\]

Let \( Y \) be an arbitrary set and \( D \) a nonempty subset of \( Y \). Given two families of functions \( G = \{g : Y \to \mathbb{R}\} \) and \( H = \{h : D \to \mathbb{R}\} \) we write \( G \subseteq H \) on \( D \) if for every \( g \in G \) there is \( h \in H \) such that \( g(v) \leq h(v) \) for all \( v \in D \). Following Ky Fan [3] a family of functions \( H = \{h : D \to \mathbb{R}\} \) is said to be concave provided given any \( h_1, \ldots, h_n \in H \) and \( x_1, \ldots, x_n \in \mathbb{R} \) such that \( x_i \geq 0 \) and \( \sum_{i=1}^{n} x_i = 1 \) there is an \( h \in H \) satisfying \( h(v) \geq \sum_{i=1}^{n} x_i h_i(v) \) for all \( v \in D \).

In what follows we denote by \( \Delta_{n-1} \) the standard \((n - 1)\)-simplex; that is
\[
\Delta_{n-1} = \left\{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_i \geq 0, \sum_{i=1}^{n} x_i = 1 \right\}.
\]

The next result generalises under many aspects in [7, Theorem 9.2].

**Theorem 8.** Let \((Y \supseteq D; \Gamma)\) be a compact \( G \)-convex space and let
\[
\mathcal{F} = \{f : Y \to (-\infty, +\infty]\},
\]
\[
\mathcal{G} = \{g : Y \to (-\infty, +\infty]\},
\]
\[
\mathcal{H} = \{h : D \to (-\infty, +\infty]\}
\]
be three families of functions such that:

- (i) \( \mathcal{F} \subseteq \mathcal{G} \) on \( Y \) and \( \mathcal{G} \subseteq \mathcal{H} \) on \( D \);
- (ii) for any finite subfamily \( \{g_1, \ldots, g_n\} \) of \( \mathcal{G} \) and for each \( (x_1, \ldots, x_n) \in \Delta_{n-1} \) the function \( y \to \sum_{i=1}^{n} x_i g_i(y) \) is \( G \)-quasiconvex on \( Y \);
- (iii) each \( f \in \mathcal{F} \) is lower semicontinuous on \( Y \);
(iv) \textit{the family }\mathcal{H} \textit{ is concave.}

Then \( \inf_{y \in Y} \sup_{f \in \mathcal{F}} f(x) \leq \sup_{h \in \mathcal{H}} \inf_{v \in D} h(v) \).

\textbf{Proof:} Let \( \beta = \sup_{h \in \mathcal{H}} \inf_{v \in D} h(v) \). We may suppose that \( \beta \) is finite. For each \( f \in \mathcal{F} \) let

\[ S(f) = \{ y \in Y : f(y) \leq \beta \}. \]

We have to show that \( \bigcap_{f \in \mathcal{F}} S(f) \neq \emptyset \). Since \( Y \) is compact and the sets \( S(f) \) are closed it suffices to prove that the family \( \{ S(f) : f \in \mathcal{F} \} \) has the finite intersection property.

Let \( f_1, \ldots, f_n \in \mathcal{F} \); choose \( g_1, \ldots, g_n \in \mathcal{G} \) and \( h_1, \ldots, h_n \in \mathcal{H} \) such that

\[ f_i \leq g_i \text{ on } Y \text{ and } g_i \leq h_i \text{ on } D. \]

Define the functions \( f, g, h : \Delta_{n-1} \times Y \to (-\infty, +\infty] \), \( h : \Delta_{n-1} \times D \to (-\infty, +\infty] \) by

\[ f(x, y) = \sum_{i=1}^{n} x_i f_i(y), \quad g(x, y) = \sum_{i=1}^{n} x_i g_i(y) \quad \text{and} \]

\[ h(x, v) = \sum_{i=1}^{n} x_i h_i(v) \text{ for } x = (x_1, \ldots, x_n) \in \Delta_{n-1}, y \in Y, v \in D. \]

One readily verifies that \( f, g, h \) satisfy assertions (i), (iii), (iv) in Theorem 7, for \( X = \Delta_{n-1} \). Assertion (ii) of the same theorem is also proved taking into account condition (ii) in present theorem and Remark 1.

Since \( \Delta_{n-1} \) and \( Y \) are compact and \( f \) is continuous on \( \Delta_{n-1} \) and lower semicontinuous on \( Y \) the conclusion of Theorem 7 becomes

\[ \min_{x \in Y} \max_{x \in \Delta_{n-1}} f(x, y) \leq \sup_{x \in \Delta_{n-1}} \inf_{v \in D} \sum_{i=1}^{n} x_i h_i(v). \]

On the other hand by (iv) we have

\[ \sup_{x \in \Delta_{n-1}} \inf_{v \in D} \sum_{i=1}^{n} x_i h_i(v) \leq \sup_{h \in \mathcal{H}} \inf_{v \in D} h(v) = \beta. \]

Consequently, there exists \( y_0 \in Y \) such that for each \( x \in \Delta_{n-1} \)

\[ \sum_{i=1}^{n} x_i f_i(y_0) = f(x, y_0) \leq \beta, \]

thus we have necessarily \( f_i(y_0) \leq \beta \) for each \( i \in \{1, \ldots, n\} \), that is, \( y_0 \in \bigcap_{i=1}^{n} S(f_i) \). \( \square \)

Theorem 8 can be stated for convenience in the form of an alternative, obtaining in this way generalisations of [5, Theorem 1] and of [7, Theorem 9.1].
THEOREM 9. Assume that \( Y, \mathcal{F}, \mathcal{G}, \mathcal{H} \) satisfy conditions of Theorem 8. Then given any \( \lambda \in \mathbb{R} \) one of the following properties holds:

(a) there is a \( h \in \mathcal{H} \) such that \( h(y) > \lambda \) for all \( y \in Y \);

(b) there is a \( y_0 \in Y \) such that \( f(y_0) \leq \lambda \) for all \( f \in \mathcal{F} \).

The following theorem generalises under many aspects a result of Liu [11, Theorem 3] which in turn improves a well-known theorem of Ky Fan concerning compatibility of some systems of inequalities.

THEOREM 10. Let \((Y \supset D; \Gamma)\) be a compact \( G \)-convex space and let

\[
\{ f_i : Y \to (-\infty, +\infty) \}_{i \in I}, \quad \{ g_i : Y \to (-\infty, +\infty) \}_{i \in I}
\]

be two families of functions such that:

(i) \( f_i \leq g_i \) for each \( i \in I \);

(ii) for each \( i \in I \) \( f_i \) is lower semicontinuous on \( Y \);

(iii) for each \( n \geq 1 \), \( \{i_1, \ldots, i_n\} \subset I \) and \((x_1, \ldots, x_n) \in \Delta_{n-1}\) the function \( y \to \sum_{i=1}^{n} x_ig_i(y) \) is G-quasiconvex on \( Y \);

(iv) for each \( n \geq 1 \), \( \{i_1, \ldots, i_n\} \subset I \) and \((x_1, \ldots, x_n) \in \Delta_{n-1}\) there is a \( v \in D \) such that \( \sum_{i=1}^{n} x_ig_i(v) \leq 0 \).

Then there exists \( y_0 \in Y \) such that \( f_i(y_0) \leq 0 \).

PROOF: Apply Theorem 8 when

\[
\mathcal{F} = \{ f_i \}_{i \in I},
\]
\[
\mathcal{G} = \{ g_i \}_{i \in I},
\]
\[
\mathcal{H} = \left\{ \sum_{i=1}^{n} x_ig_i : n \geq 1, g_i \in \mathcal{G}, (x_1, \ldots, x_n) \in \Delta_{n-1} \right\}.
\]

Our last result generalises [7, Theorem 9.3].

THEOREM 11. Let \((Y \supset D; \Gamma)\) be a compact \( G \)-convex space, \( X \) an arbitrary set and let \( f, g : X \times Y \to (-\infty, +\infty), h : X \times D \to (-\infty, +\infty] \) be three functions such that

(i) \( f(x,y) \leq g(x,y) \) for each \((x,y) \in X \times Y\) and \( g(x,y) \leq h(x,y) \) for all \((x,y) \in X \times D\);

(ii) for any \( x_1, \ldots, x_n \in X \) and for each \( (\alpha_1, \ldots, \alpha_n) \in \Delta_{n-1}\) the function \( y \to \sum_{i=1}^{n} \alpha_ig_i(x_i, y) \) is G-quasiconvex on \( Y \);

(iii) \( f \) is lower semicontinuous on \( Y \);

(iv) for any \( x_1, \ldots, x_n \in X \) and for each \( (\alpha_1, \ldots, \alpha_n) \in \Delta_{n-1}\) there is a \( x \in X \) such that \( h(x,y) \geq \sum_{i=1}^{n} \alpha_i h(x_i, y) \) for all \( y \in Y \).
Then
\[
\inf_{y \in Y} \sup_{x \in X} f(x, y) \leq \sup_{x \in X} \inf_{y \in Y} h(x, y).
\]

**PROOF:** Apply Theorem 8 when
\[
\mathcal{F} = \{ f(x, \cdot) \}_{x \in X}, \\
\mathcal{G} = \{ g(x, \cdot) \}_{x \in X}, \\
\mathcal{H} = \{ h(x, \cdot) \}_{x \in X}.
\]

REFERENCES


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