Two minimax inequalities in $G$-convex spaces

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Abstract

In this work we obtain two minimax inequalities in $G$-convex spaces which extend and improve a large number of generalizations of the Ky Fan minimax inequality and of the von Neumann–Sion minimax principle.

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1. Introduction

Motivated by the well-known works of Horvath [1,2], there have appeared many generalizations of the concept of convex subset of a topological vector space. The most general one seems to be that of the generalized convex space or $G$-convex space introduced by Park and Kim [3] which extends many generalized convex structures on topological vector spaces. This will be the framework in which we obtain in this work two very general minimax inequalities. The first of them originates from the Ky Fan minimax inequality [4]. The origin of the second one goes back to a well known two-function minimax inequality due also to Ky Fan [5] which in turn generalizes the von Neumann–Sion minimax principle [6]. Our results improve and generalize a large number of generalizations of the above-mentioned results.

Let us recall the terminology needed in the following. For a set $D$ let $\langle D \rangle$ denote the class of all nonempty finite subsets of $D$.

A generalized convex space or a $G$-convex space (see [7,8]) consists of a topological space $X$ and a nonempty set $D$ such that for each $A \in \langle D \rangle$ with the cardinality $|A| = n + 1$ there exists a subset $\Gamma(A)$ of $X$ and a continuous function $\Phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\Phi_A(\Delta_J) \subseteq \Gamma(J)$.
Here $\Delta_n$ denotes any $n$-simplex with vertices $\{e_i\}_{i=0}^n$ and $\Delta_J$ the face of $\Delta_n$ corresponding to $J$, that is, if $A = \{z_0, z_1, \ldots, z_n\}$ and $J = \{z_{i_0}, z_{i_1}, \ldots, z_{i_k}\} \subset A$, then $\Delta_J = \text{co}\{e_{i_0}, e_{i_1}, \ldots, e_{i_k}\}$.

Note that in the original definition of a $G$-convex space [3], $D$ is a subset of $X$ and the mapping $\Gamma$ satisfies the condition $A, B \in \langle D \rangle, A \subset B \Rightarrow \Gamma(A) \subset \Gamma(B)$.

The main example of a $G$-convex space corresponds to the case where $X = D$ is a convex subset of a Hausdorff topological vector space, and for each $A \in \langle X \rangle$, $\Gamma(A)$ is the convex hull of $A$. For other major examples of $G$-convex spaces see [9,10].

Let $(X, D; \Gamma)$ be a $G$-convex space, $Y$ a nonempty set and $f : D \times Y \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm \infty\}$, $g : X \times Y \to \overline{\mathbb{R}}$. We say that:

(i) $f$ is $G - g$-quasiconvex in the first variable if for any $\{z_1, z_2, \ldots, z_n\} \in \langle D \rangle$ and for each $y \in Y$ we have

$$f(u, y) \leq \max_{1 \leq i \leq n} g(z_i, y), \text{ for all } u \in \Gamma([z_1, z_2, \ldots, z_n]);$$

(ii) $f$ is $G - g$-quasiconcave in the first variable if for any $\{z_1, z_2, \ldots, z_n\} \in \langle D \rangle$ and for each $y \in Y$ we have

$$f(u, y) \geq \min_{1 \leq i \leq n} g(z_i, y), \text{ for all } u \in \Gamma([z_1, z_2, \ldots, z_n]).$$

It is easily seen that $f$ is $G - g$-quasiconvex if and only if $-f$ is $G - (-g)$-quasiconvex.

Note that the notions introduced above coincide with the corresponding notions in Definition 2 in [11] only when $D = X$. The origin of all these concepts goes back to the notion of $g$-quasiconcavity introduced by Chang and Yen [12].

**Remark 1.** It is easy to see that, when $D \subset X$, $f$ is $g$-quasiconcave whenever there is a function $\varphi : X \times Y \to \overline{\mathbb{R}}$ such that:

(i) $\varphi \leq f$ on $X \times Y$;

(ii) $g \leq \varphi$ on $D \times Y$;

(iii) $\varphi$ is $G$-quasiconcave, i.e., for any $\{z_1, \ldots, z_n\} \in \langle D \rangle$ and for each $y \in Y$,

$$\varphi(x, y) \geq \min_{1 \leq i \leq n} \varphi(z_i, y), \text{ for all } x \in \Gamma([z_1, \ldots, z_n]).$$

For a set-valued mapping (simply a map) $F : X \to Y$ the lower inverse of $F$ is the map $F^{-} : Y \to X$ defined by $F^{-}(y) = \{x \in X : y \in F(x)\}$.

Let $X$ be a nonempty space and $Y$ be a topological set. A map $F : X \to Y$ is said to be transfer open valued [13] if for any $x \in X$ and $y \in F(x)$ there exists an $x' \in X$ such that $y \in \text{int} F(x')$.

Let $X$ and $Y$ be two topological spaces. A function $f : X \times Y \to \overline{\mathbb{R}}$ is said to be transfer upper semicontinuous (resp. transfer lower semicontinuous) in the first variable [14] if for each $\lambda \in \mathbb{R}$ and all $x \in X$, $y \in Y$ with $f(x, y) < \lambda$ (resp. $f(x, y) > \lambda$), there exists a $y' \in Y$ and a neighborhood $V(x)$ of $x$ such that $f(u, y') < \lambda$ (resp. $f(u, y') > \lambda$) for all $u \in V(x)$.

**Remark 2.** (a) It is clear that if for each $y \in Y$, $x \to f(x, y)$ is upper (lower) semicontinuous, then $f$ is transfer upper (lower) semicontinuous in the first variable.

(b) It is easy to prove (see [14, Lemma 2.2]) that $f$ is transfer upper (lower) semicontinuous in the first variable if and only if the map $F : Y \to X$, $F(y) = \{x \in X : f(x, y) < \lambda\}$ (resp. $F(y) = \{x \in X : f(x, y) > \lambda\}$) is transfer open valued.

From now on we assume for simplicity that all topological spaces are Hausdorff.
2. Main results

The following extension to $G$-convex spaces of the Fan–Browder fixed point theorem is well known. For instance, it is a particular case of Theorem 3 in [15].

**Lemma 1.** Let $(X, D; \Gamma)$ be a compact $G$-convex space and $T : X \to X, S : X \to D$ be two maps satisfying the following conditions:

(i) for each $x \in X$, $A \in \langle S(x) \rangle$ implies $\Gamma(A) \subset T(x)$;
(ii) $X = \cup \{\text{Int } S^{-1}(z) : z \in D\}$.

Then $T$ has a fixed point.

From Lemma 1 we derive the following:

**Theorem 2.** Let $(X, D; \Gamma)$ be a compact $G$-convex space, $Y$ be a nonempty set. Suppose that $T : X \to X, S : X \to D$ and $F : X \to Y$ are three maps satisfying the following conditions:

(i) for each $x \in X$, $A \in \langle S(x) \rangle$ implies $\Gamma(A) \subset T(x)$;
(ii) $F$ has nonempty values;
(iii) $F^-$ is transfer open valued;
(iv) for each $y \in Y$, there exists $z \in D$ such that $F^-(y) \subset S^-(z)$.

Then $T$ has a fixed point.

**Proof.** It suffices to show that condition (ii) in Lemma 1 is verified. Let $x \in X$. Since $F(x) \neq \emptyset$ there exists $y \in Y$ such that $x \in F^-(y)$. By (iii) and (iv) there are $y' \in Y$ and $z \in D$ such that $x \in \text{int } F^-(y') \subset \text{int } S^-(z)$. \qed

**Theorem 3.** Let $(X, D; \Gamma)$ be a compact $G$-convex space and $Y$ be a nonempty set. Let $t : X \times X \to \mathbb{R}$, $s : X \times D \to \mathbb{R}$ and $f : X \times X \to \mathbb{R}$ be three functions such that:

(i) $t$ is $G - s$-quasiconvex in the second variable;
(ii) $f$ is transfer upper semicontinuous in the first variable;
(iii) for each $y \in Y$ there exists $z \in D$ such that $s(\cdot, z) \leq f(\cdot, y)$.

Then $\inf_{x \in X} t(x, x) \leq \sup_{z \in X} \inf_{y \in Y} f(x, y)$.

**Proof.** We may assume that $\sup_{x \in X} \inf_{y \in Y} f(x, y) < \infty$. Let $\lambda > \sup_{x \in X} \inf_{y \in Y} f(x, y)$ be arbitrarily fixed and define the maps $T : X \to X, S : X \to D, F : X \to Y$ by

$$T(x) = \{u \in X : g(x, u) < \lambda\}, \quad S(x) = \{z \in D : g(x, z) < \lambda\},$$

$$F(x) = \{y \in Y : f(x, y) < \lambda\}.$$

From $\lambda > \sup_{x \in X} \inf_{y \in Y} f(x, y)$ it follows that $F$ has nonempty values, and by (ii), via Remark 2(b), $F^-$ is transfer open valued.

By (iii), for each $y \in Y$ there exists $z \in D$ such that $F^-(y) \subset S^-(z)$.

We prove that condition (i) in Theorem 2 holds. Let $x \in X, \{z_1, z_2, \ldots, z_n\} \in \langle S(x) \rangle$ and $u \in \Gamma(\{z_1, z_2, \ldots, z_n\})$. Since $z_i \in S(x)$ and $t$ is $G - s$-quasiconvex in the second variable we have

$$t(x, u) \leq \max_{1 \leq i \leq n} s(x, z_i) < \lambda,$$

and hence $u \in T(x)$.\n
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From Lemma 1 we derive the following:
By Theorem 2, $T$ has a fixed point $x_0$. It follows that
\[
\inf_{x \in X} t(x, x) \leq t(x_0, x_0) < \lambda,
\]
and the proof is complete. \qed

In our opinion it would be of some interest to compare Theorem 3 with other extensions to $G$-convex spaces of the Ky Fan minimax inequality established by Park in [3], [16] and [17].

Theorem 4. Let $(X_1, D_1; \Gamma_1)$, $(X_2, D_2; \Gamma_2)$ be compact $G$-convex spaces and $Y_1$, $Y_2$ be nonempty sets. Let $t_1, t_2 : X_1 \times X_2 \to \mathbb{R}$, $f_1 : Y_1 \times X_2 \to \mathbb{R}$, $f_2 : X_1 \times Y_2 \to \mathbb{R}$, $s_1 : D_1 \times X_2 \to \mathbb{R}$, $s_2 : X_1 \times D_2 \to \mathbb{R}$ be six functions satisfying the following conditions:

(i) $t_1(x_1, x_2) \leq t_2(x_1, x_2)$ for all $(x_1, x_2) \in X_1 \times X_2$;

(ii) $t_1$ is $G - s_1$-quasiconcave in the first variable;

(iii) $t_2$ is $G - s_2$-quasiconvex in the second variable;

(iv) $f_1$ is transfer lower semicontinuous in the second variable;

(v) $f_2$ is transfer upper semicontinuous in the first variable;

(vi) for each $y_1 \in Y_1$ there exists $z_1 \in D_1$ such that $s_1(z_1, \cdot) \geq f_1(y_1, \cdot)$ on $X_2$;

(vii) for each $y_2 \in Y_2$ there exists $z_2 \in D_2$ such that $s_2(\cdot, z_2) \leq f_2(\cdot, y_2)$ on $X_1$.

Then $\inf_{x_2 \in X_2} \sup_{y_1 \in Y_1} f_1(y_1, x_2) \leq \sup_{x_1 \in X_1} \inf_{y_2 \in Y_2} f_2(x_1, y_2)$.

Proof. The triplet $(X_1 \times X_2, D_1 \times D_2, \Gamma')$ is a $G$-convex space (see [15, Lemma 4]) if for each $(z_1^n, z_2^n, \ldots, z_1^n, z_2^n) \in (D_1 \times D_2)$ we put \[ \Gamma'((z_1^n, z_2^n), (z_1^n, z_2^n), \ldots, (z_1^n, z_2^n)) = \Gamma_1((z_1^n, z_2^n), \ldots, (z_1^n, z_2^n)) \times \Gamma_2((z_1^n, z_2^n), \ldots, (z_1^n, z_2^n)). \]

By contradiction, suppose that there exists a real $\lambda$ such that
\[
\sup_{x_1 \in X_1} \inf_{y_2 \in Y_2} f_2(x_1, y_2) < \lambda < \inf_{x_2 \in X_2} \sup_{y_1 \in Y_1} f_1(y_1, x_2). \tag{(*)}
\]

Define the maps $T : X_1 \times X_2 \to X_1 \times X_2$, $S : X_1 \times X_2 \to D_1 \times D_2$ and $F : X_1 \times X_2 \to Y_1 \times Y_2$ by

\[
T(x_1, x_2) = \{u_1 \in X_1 : t_1(u_1, x_2) > \lambda\} \times \{u_2 \in X_2 : t_2(x_1, u_2) < \lambda\},
\]

\[
S(x_1, x_2) = \{z_1 \in D_1 : s_1(z_1, x_2) > \lambda\} \times \{z_2 \in D_2 : s_2(x_1, z_2) < \lambda\}
\]

and

\[
F(x_1, x_2) = \{y_1 \in Y_1 : f_1(y_1, x_2) < \lambda\} \times \{y_2 \in Y_2 : f_2(x_1, y_2) > \lambda\}.
\]

By (\(\ast\)), $F$ has nonempty values. By (iv) and (v), for each $(y_1, y_2) \in Y_1 \times Y_2$ there exists $(z_1, z_2) \in D_1 \times D_2$ such that $F^-(y_1, y_2) \subset S^-(z_1, z_2)$.

Let $(y_1, y_2) \in Y_1 \times Y_2$ and $(x_1, x_2) \in F^-(y_1, y_2)$. It follows that $f_1(y_1, x_2) > \lambda$ and $f_2(x_1, y_2) < \lambda$. By (iii) and (iv), there exist $(y_1', y_2') \in Y_1 \times Y_2$ and a neighborhood (in $X_1 \times X_2$) $V_1(x_1) \times V_2(x_2)$ of $(x_1, x_2)$ such that
\[
f_2(u_1, y_2') < \lambda < f_1(y_1', u_2), \text{ for all } (u_1, u_2) \in V_1(x_1) \times V_2(x_2).
\]

Thus $(x_1, x_2) \in \int F^-(y_1', y_2')$; hence $F^-$ is transfer open valued.

We prove now that $S$ and $T$ satisfy condition (i) in Theorem 2. Let $(x_1, x_2) \in X_1 \times X_2$, \((z_1^n, z_2^n, \ldots, z_1^n, z_2^n) \in (S(x_1, x_2))\) and $(u_1, u_2) \in \Gamma'((z_1^n, z_2^n), (z_1^n, z_2^n), \ldots, (z_1^n, z_2^n)) = \Gamma_1((z_1^n, z_2^n), \ldots, (z_1^n, z_2^n)) \times \Gamma_2((z_1^n, z_2^n), \ldots, (z_1^n, z_2^n))$. Since $(z_1^n, z_2^n) \in S(x_1, x_2)$, $s_1(z_1^n, x_2) > \lambda > s_2(x_1, z_2^n)$ and
by (ii₁) and (ii₂) we have
\[ t₁(u₁, x₂) ≥ \min_{1 ≤ i ≤ n} s₁(z₁^i, x₂) > \lambda, \]
\[ t₂(x₁, u₂) ≤ \max_{1 ≤ i ≤ n} s₂(x₁, z₂^i) < \lambda, \]
and hence \((u₁, u₂) ∈ T(x₁, x₂)\).

Applying Theorem 2 we get a point \((x₁, x₂) ∈ X₁ × X₂\) such that \((x₁, x₂) ∈ T(x₁, x₂)\). Taking into account condition (i) we obtain the following contradiction:
\[ λ < t₁(x₁, x₂) ≤ t₂(x₁, x₂) < \lambda, \]
and the proof is complete. □

Theorem 4 extends to \(G\)-convex spaces and improves in many aspects Theorem 4 in [18], Theorem 1 in [19] and Theorems 5.5 and 5.6 in [20].

References