The authors are with the Departament de Matemàtica Aplicada I Técnica, Universitat Politècnica de Catalunya, Barcelona, Spain.
E-mail: matcp@mat.upc.es.

Manuscript received July 18, 1994; revised Nov. 16, 1995.
For information on obtaining reprints of this article, please send e-mail to: transcom@computer.org, and reference IEEECS Log Number C90026.

Comments

Comments on

"Line Digraph Iterations and Connectivity Analysis of de Bruijn and Kautz Graphs"

C. Padró, P. Morillo, and M.A. Fiol

Abstract—The aim of this note is to present some counterexamples to the results in the paper by Du, Lyuu, and Hsu published in this Transactions [1].

Index Terms—Connectivity, diameter-vulnerability, fault-tolerance, line digraph iteration, spread.

We present in this paper, for any $d \geq 2$, a digraph $G \in L^1(d, 4) \cap L^2(d - 1, 4)$ such that $LG \in L^1(d - 1, 5)$ and $L^2G \in L^1(d, 6)$. Therefore, this is a counterexample to Lemma 3.2 and Theorem 3.3 of [1].

The digraphs we consider here are members of a family of bipartite digraphs, called BD(d, n), constructed by Fiol and Yebra [2]. See this paper for the definition and some properties of these digraphs. The digraph BD(d, n) is $d$-regular and bipartite with partite sets $V_0 = \{0\} \times Z_d$ and $V_1 = \{1\} \times Z_d$. One important fact about this family is that the line digraph LBBD(d, n) is isomorphic to BD(d, nd). For any vertex $(a, i)$ of BD(d, $d^2 + 1$), we have that $V_1(a, i) = V_0(a, i)$. Then, there is an unique path of length 2 between any pair of different vertices in $V_A$ and there are no cycles of length 2 in BD(d, $d^2 + 1$). That is, all the vertices which appear in the tree formed by all paths of length at most 2 from $x$ are different. If $x$ and $y$ are two different vertices of BD(d, $d^2 + 1$) in the same partite set, $(x, y)$ will denote the unique path of length 2 from $x$ to $y$.

PROPOSITION 1. For all $d \geq 2$, BD(d, $d^2 + 1$) $\in L^1(d, 4) \cap L^2(d - 1, 4)$.

PROOF. Let $x, y, y_1, ..., y_{d-1}$ be vertices (not necessarily different) of BD(d, $d^2 + 1$). We are going to prove that there exist $d$ disjoint paths (one for each $y_i$) of positive length at most 4 from $x$ to the vertices $y$. Since BD(d, $d^2 + 1$) is vertex-symmetric, we can suppose that $x \in V_0$. We consider first all possible disjoint paths of length 1 from $x$ to the vertices $y$. Rearranging the subindices, we can suppose that these paths are $P_i = x, y_1, 1 \leq i \leq p$, where $0 \leq p \leq d$. After that, we take all possible disjoint paths of length 2 from $x$ to the vertices $y$, that is, the paths that are obtained with the paths constructed before. We will denote these paths $P_i = (x, z_1, ..., z_q, y_{i-q}, y_{i-q-1}, ..., y_1)$, where $p + 1 \leq i \leq q$ and $p \leq q \leq d$. If $1 \leq i \leq p$, put $z_i = y_i$, and put $z_i = b(x, y_i)$ if $p + 1 \leq i \leq q$. Consider $G^1 = (x, z_1, ..., z_q, y_{i-q}, y_{i-q-1}, ..., y_1)$, which is isomorphic to BD(d, $d^2 + 1$). We rearrange now the vertices $y_i$ in such a way that $y_{i-q}, y_{i-q-1}, ..., y_1 \in V_0$, $q \leq r \leq d$, and $y_{r-q}, y_{r-q-1}, ..., y_{r-1} \in V_1$. Consider then the paths $P_i = x, y_1, ..., y_{d-1}$ and $y_{d-1}, b(x, y_{d-1}) = z_k$ for some $k = 1, ..., q$, and, of course, $z_k$ is not adjacent to $y_{d-1}$ if $k \neq i$. For $i = r + 1, ..., d$, consider $P_i = x, (z_k, w), y_{i-1}$, where $w$ is a vertex adjacent to $y_{i-1}$ such that $w \neq b(x, y_{i-1}) = z_k, z_k \neq y_{i-1}$ if $q + 1 \leq j \leq r$ and $w \neq y_{i-1}$ if $r + 1 \leq j \leq d$. It is not difficult to check that the paths $P_i, i = 1, ..., d$, constructed in this way are disjoint. Therefore, we have proved that BD(d, $d^2 + 1$) $\in L^1(d, 4)$. We will denote by $(x, y_1, ..., y_{d-1})$ a set of $d$ disjoint paths of length at most 4 from $x$ to the vertices $y_1$, constructed as before.

We have to prove now that BD(d, $d^2 + 1$) $\in L^2(d - 1, 4)$. That is, for any vertices $x, y_1, ..., y_{d-1}$, and for any pair of arcs in the form $e_1 = (u, v)$ and $e_2 = (v, w)$, we have to prove the existence of $d - 1$ disjoint paths of positive length at most 4 from $x$ to the vertices $y_1$, avoiding the arcs $e_1$ and $e_2$. If $v \neq x$, $y_1, 1 \leq i \leq d - 1$, we will find these paths in the set $(x, y_1, ..., y_{d-1}, u, v)$. If $x \neq u, v$ and $u = y_1$, for some $j = 1, ..., d - 1$, the paths we are looking for are in the set $(x, y_1, ..., y_{d-1}, u)$. If $x = u, v$ and $y_1 = y_1, 1 \leq j \leq d - 1$, we construct, with the algorithm described above, the set of paths $(x, y_1, ..., y_{d-1}, u)$, but being aware of taking $P_i = u, v, w$. Then, the paths from $x$ to the vertices $y_i$ are constructed as before. If $x = u, v$, we construct the paths $(x, y_1, ..., y_{d-1}, w)$ beginning with $P_1 = x, u, v, w$. If $x = u, v, y_1 = y_1, 1 \leq j \leq d - 1$, we consider the path $P_i = x, z_k, y_{i-1}$, taking into account that $w$ has to be different from $u$.

PROOF. Let $x$ be a vertex of BD(d, $d^2 + 1$). Then $x = xy$, where $(x, y)$ is an arc of BD(d, $d^2 + 1$). We put $G^1 = (x, y_1, ..., y_{d-1})$. Since $y$ is not adjacent to $x$, there are exactly $d$ paths of length 3 from $y$ to $x$: the paths $P_i = y, y_{i-1}, ..., y_1.$ We take in BD(d, $d^2 + 1$) the arcs $e_1 = (y, x, y_1)$ and $e_2 = (x, y_1, y_2)$. Then, there are, in BD(d, $d^2 + 1$), exactly $d$ cycles of length at most 5 containing the vertex $x$, which are obtained from the paths of length 3 from $y$ to $x$ in BD(d, $d^2 + 1$). The cycle $C_1 = y, x, y_1, y_2, x$ contains the arc $e_1$, and the cycle $C_2 = x, y_1, x, y_2, y_1$ contains the arc $e_2$. Since $C_1 \neq C_2$, there are exactly $d$-2 cycles of length at most 5 which contain the vertex $x$ and avoid the arcs $e_1$ and $e_2$. Therefore, BD(d, $d^2 + 1$) $\notin L^2(d - 1, 5)$.

We consider now in BD(d, $d^2 + 1$) the vertices $x_1 = x, y_1$ and $x_2 = x, y_2$ that is, the vertices which represent the arcs $e_1$ and $e_2$ of BD(d, $d^2 + 1$). There are exactly $d + 1$ paths of length at most 6 from $x_1$ to $x_2$: the arc $(x_1, x_2)$ and the paths obtained from the $d$ paths of length 3 from $x$ to $x$ in BD(d, $d^2 + 1$). These of these paths contain the arc $(x_1, x_2)$. Three of these paths contain the arc $(x_1, x_2)$: $P_1 = x, y_1, x$, which is the arc itself, the path $P_2 = x, y_1, x, y_2$, and the path $P_3 = x, y_2, x, y_1$. Then, there are only $d$-1 disjoint paths of length at most 6 from $x_1$ to $x_2$. Therefore, BD(d, $d^2 + 1$) $\notin L^2(d, 6)$.

REFERENCES