Spectral and Geometric Properties of $k$-Walk-Regular Graphs

M.A. Fiol$^{1,2}$ E. Garriga$^{1,3}$

Dep. Matemàtica Aplicada IV
Universitat Politècnica de Catalunya
Barcelona, Catalonia (Spain)

Abstract

Let us consider a connected graph $G$ with diameter $D$. For a given integer $k$ between 0 and $D$, we say that $G$ is $k$-walk-regular if the number of walks of length $\ell$ between vertices $u, v$ only depends on the distance between $u$ and $v$, provided that such a distance does not exceed $k$. Thus, in particular, a 0-walk-regular graph is the same as a walk-regular graph, where the number of cycles of length $\ell$ rooted at a given vertex is a constant through all the graph. In the other extreme, the distance-regular graphs correspond to the case $k = D$. In this talk we discuss some algebraic characterizations of $k$-walk-regularity, in terms of the local spectrum and pre-distance-polynomials of $G$. Moreover, some results on the relationship between the diameter and the spectrum, as well as some geometric properties, of walk-regular graphs are presented.

Keywords: Walk-regular graph, Pre-distance-polynomials, Spectrum.

1 Research supported by the Spanish Research Council under project MTM2005-08990-C02-01 and by the Catalan Research Council under project 2005SGR00256.
2 Email: fio1@ma4.upc.edu
3 Email: egarriga@ma4.upc.edu
1 Introduction

Throughout this paper, \( G = (V, E) \) denotes a simple, connected graph, with order \( n = |V| \) and adjacency matrix \( A \). The spectrum of \( G \) (\( A \)) is denoted by \( \text{sp} G = \{ \lambda_0^m, \lambda_1^m, \ldots, \lambda_d^m \} \), where \( \lambda_0 > \lambda_1 > \cdots > \lambda_d \) and the superindexes stand for the multiplicities \( m_i = m(\lambda_i) \). Let \( Z = \prod_{i=0}^d (x - \lambda_i) \) be the minimal polynomial of \( A \). The vector space \( \mathbb{R}_d[x] \) of real polynomials of degree at most \( d \) is isomorphic to \( \mathbb{R}[x]/(Z) \), and each polynomial \( p \in \mathbb{R}_d[x] \) operates on the vector \( w \in \mathbb{R}^n \) by \( pw = p(A)w \). For every \( 0 \leq k \leq d \), the orthogonal projection of \( \mathbb{R}^n \) onto \( \text{Ker}(x - \lambda_k) \) is given for the polynomial of degree \( d P_k = \frac{1}{\phi_k} \prod_{i=0, i \neq k}^d (x - \lambda_i) = \frac{(-1)^k}{\pi_k} \prod_{i=0, i \neq k}^d (x - \lambda_i) \), where \( \phi_k = \prod_{i=0, i \neq k}^d (\lambda_k - \lambda_i) \) and \( \pi_k = |\phi_k| \). The matrices \( E_k = P_k(A) \) are called the (principal) idempotents of \( A \). Then, the orthogonal decomposition of the unitary vector \( e_u \), representing vertex \( u \), is:

\[
e_u = z_u^0 + z_u^1 + \ldots + z_u^d, \text{ where } z_u^k = P_k e_u = E_k e_u.
\]

2 Spectral regularity and walk-regularity

From the decomposition (1), we define the \( u \)-local multiplicity of eigenvalue \( \lambda_k \) as \( m_u(\lambda_k) = (E_k)_{uu} = \|z_u^k\|^2 \), satisfying \( \sum_{k=0}^d m_u(\lambda_k) = 1 \) and \( \sum_{u \in V} m_u(\lambda_k) = m_k, 0 \leq k \leq d \). (see [2]). We say that \( G \) is spectrally regular when, for any \( k = 0, 1, \ldots, d \), the \( u \)-local multiplicity of \( \lambda_k \) does not depend on \( u \in V \). Then, the above equations imply that \( m_u(\lambda_k) = m_k/n \). In particular, since \( m_u(\lambda_0) = \|z_u^0\|^2 = v_u^2/\|v\|^2 \), the spectral regularity implies the regularity of the graph.

Let \( C_u^{(r)} \) denote the number of closed walks of length \( r \) rooted at vertex \( u \). Then, as it is well known, \( C_u^{(r)} = (A^r)_{uu} \), the \( u \)-th element in the diagonal of \( A^r \). When the number \( C_u^{(r)} \) only depends on \( r \), the graph \( G \) is called walk-regular (see, for instance, [3]). Notice that, as \( C_u^{(2)} = \delta(u) \), the degree of vertex \( u \), every walk-regular graph is also regular. In our context, we also have the following result:

**Proposition 2.1** A connected graph \( G \) is spectrally regular if and only if it is walk-regular.

Let us see one of the implications. If \( G \) is spectrally regular, we have, for any \( u \in V \) and \( r \geq 0 \), \( C_u^{(r)} = (A^r)_{uu} = \langle x^r e_u, e_u \rangle = \langle x^r \sum_{k=0}^d z_u^k, \sum_{k=0}^d z_u^k \rangle = \sum_{k=0}^d \|z_u^k\|^2 \lambda_k^r = \frac{1}{n} m_k \lambda_k^r \), so that \( C_u^{(r)} \) is independent of \( u \) and \( G \) is walk-regular.
From the spectrum of a given graph \( G = \{ \lambda_0^{m_0}, \lambda_1^{m_1}, \ldots, \lambda_d^{m_d} \} \), we consider the following scalar product in \( \mathbb{R}_d[x] \):

\[
\langle p, q \rangle = \frac{1}{n} \operatorname{tr}(p(A)q(A)) = \frac{1}{n} \sum_{k=0}^{d} m_k p(\lambda_k)q(\lambda_k).
\]

Then, by using the Gram-Schmidt method and normalizing appropriately, it is immediate to prove the existence and uniqueness of an orthogonal system of polynomials \( \{ p_k \}_{0 \leq k \leq d} \) called the pre-distance polynomials which, for any \( 0 \leq h, k \leq d \), satisfy:

\[
dgr(p_k) = k, \quad \langle p_h, p_k \rangle = 0 \quad (h \neq k), \quad \| p_k \|^2 = p_k(\lambda_0).
\]

The pre-distance polynomials can be thought of as a generalization of the so-called “distance polynomials.” Recall that, in a distance-regular graph, such polynomials satisfy \( p_k(A) = A_k \), \( 0 \leq k \leq d \), where \( A_k \) stands for the adjacency matrix of the \( k \)-distance graph \( G_k \), usually called the \( k \)-th distance matrix of \( G \) (see, for instance, [1]). More generally, the following result links the spectral regularity to the pre-distance polynomials:

**Proposition 2.2** Let \( G \) be a connected graph with adjacency matrix \( A \) having \( d + 1 \) distinct eigenvalues, and with pre-distance polynomials \( \{ p_0, p_1, \ldots, p_d \} \). Then, the two following statements are equivalent:

(a) \( G \) is spectrally regular.

(b) The matrices \( p_k(A) \), \( 1 \leq k \leq d \), have null diagonals.

Note that, from the above comments, property (b) is also satisfied in the case of distance regularity, as \( p_k(A) = A_k \).

The above result can be generalized if we consider the following new definition. Let \( G \) be a connected graph with diameter \( D \). For a given integer \( k \), \( 0 \leq k \leq D \), we say that \( G \) is \( k \)-walk-regular if the number of walks of length \( \ell \) between vertices \( u, v \) only depends on the distance between \( u \) and \( v \), provided that \( \operatorname{dist}(u, v) \leq k \). Thus, a 0-walk-regular graph is the same thing as a walk-regular graph whereas, at the other extreme, the distance-regular graphs correspond to the case of \( D \)-walk-regular graphs. (In the following result “\( \circ \)” stands for the Schur or Hadamard—componentwise—product of matrices.)

**Theorem 2.3** For a graph \( G \) as above and a given integer \( k \), \( 0 \leq k \leq D \), the two following statements are equivalent:

(a) \( G \) is \( k \)-walk-regular.
(b) \( A_k = A_k \circ p_k(A) \) for any \( 1 \leq k \leq d \).

3 Spectrum and Diameter

Consider the sets \( T_k = \{ z_u^k = E_k e_u : u \in V \} \) of vectors in the \( m_k \)-dimensional space \( \text{Ker}(x - \lambda_k) \). (These sets are usually called eutactic stars [4].) Then the spectral regularity of the graph is equivalent to state that, for every \( k = 0, 1, \ldots, d \), such vectors define \( n \) (not necessarily different) points on the sphere with radius \( \sqrt{m_k/n} \). Moreover, for any \( k = 1, 2, \ldots, d \), the “center of mass” of the set \( T_k \) is \( \sum_{u \in V} z_u^k = E_k \sum_{u \in V} e_u = E_k j = 0 \). Let \( \gamma_{u,v} = \gamma(z_u^k, z_v^k) \) denote the angle between the two vectors \( z_u^k, z_v^k \). Note that, since \( z_0^u = (1/n) j \), we always have \( \gamma_{u,v}^0 = 0 \).

Proposition 3.1 Let \( G = (V, E) \) be a spectrally regular graph with \( d + 1 \) eigenvalues. Then, two vertices \( u, v \in V \) are at (spectrally maximum) distance \( d \) if and only if
\[
\cos \gamma_{u,v}^k = \frac{(-1)^k \pi_0}{m_k} \frac{\pi_k}{\pi_0} \quad (0 \leq k \leq d).
\]

The above cosines were already considered by Godsil [3] when \( G \) is a distance-regular graph. As a direct consequence of the above proposition, we have the following result.

Corollary 3.2 The eigenvalue multiplicities of a (connected) spectrally regular graph with spectrally maximum diameter satisfy \( m_k \geq \frac{\pi_0}{\pi_k} \), \( 0 \leq k \leq d \).

Let \( \alpha \equiv \alpha(G) \equiv \alpha_{d-1}(G) \) be the \((d - 1)\)-independence number of \( G \); that is, the maximum number of vertices which are at distance \( d \) from each other. Note that, for a graph \( G \), the property of having spectrally maximum diameter is equivalent to have \( \alpha \geq 2 \).

Proposition 3.3 The \((d - 1)\)-independence number of a spectrally regular graph \( G \) satisfies the bound
\[
\alpha \leq 1 + \min_{\begin{subarray}{c}1 \leq k \leq d \\ d \text{ odd} \end{subarray}} \left\{ m_k \frac{\pi_k}{\pi_0} \right\}.
\]

Then, from the above results we get:

Corollary 3.4 Let \( G \) be a (connected) spectrally regular graph with spectrum \( \text{sp}G = \{\lambda_0^{m_0}, \lambda_1^{m_1}, \ldots, \lambda_d^{m_d}\} \), spectrally maximum diameter \( d \), and \((d - 1)\)-independence number \( \alpha \). Then, the eigenvalue multiplicities satisfy the bounds \( m_k \geq \frac{\pi_0}{\pi_k} \) if \( k \) is even, and \( m_k \geq (\alpha - 1) \frac{\pi_0}{\pi_k} \) if \( k \) is odd.
4 The Geometry of $d$-Cliques

As before, we are assuming that the (connected) graph $G = (V, E)$ is spectrally regular and has spectrally maximum diameter. Let $C_r = \{u_1, u_2, \ldots, u_r\}$ be a set of $r(\leq \alpha)$ vertices which are at distance $d$ from each other. For each $k = 0, 1, \ldots, d$, we are here interested in studying the geometry of the projection set $\{z^k_u = E_k e_u : u \in C_r\}$ onto the eigenspace $\text{Ker}(x - \lambda_k) \subset \mathbb{R}^n$ with dimension $m_k$. According to the parity of $k$ and whether or not the equality in Corollary 3.4 applies, there are four cases to be considered. Here we present the results only for the two cases of even $k$.

Assume first that $m_k = \frac{\pi_0}{\pi_k}$. Then, by Proposition 3.1 we have that $\cos \gamma^k_{u_i, u_j} = \frac{1}{m_k} \frac{\pi_0}{\pi_k} = 1$. Because of the spectral regularity, the vector $z^k_{u_i}$ is independent of $u_i \in C_r$, and it has norm $\sqrt{m_k/n}$. Hence, the projection of $C_r$ is a single point.

Now, suppose that $m_k > \frac{\pi_0}{\pi_k}$, and let $C_r^0 = \frac{1}{r} \sum_{i=1}^{r} z^k_{u_i} = \frac{1}{r} w^k$ be the baricenter of the projections onto $\text{Ker}(x - \lambda_k)$.

**Proposition 4.1** Let $G$ be a graph as above, and suppose that, for some even $k$, $m_k > \frac{\pi_0}{\pi_k}$. Then the projected points $\{P_k e_u : u \in C_r\}$ are the vertices of a regular tetrahedron with center $C_r^0$, radius $\sqrt{\frac{r-1}{rn} (m_k - \frac{\pi_0}{\pi_k})}$ and edge length $\sqrt{\frac{2}{n} (m_k - \frac{\pi_0}{\pi_k})}$. Moreover, the angle $\beta$ formed by the vectors going from the center to each vertex $z^k_{u_i}$ satisfies $\cos \beta = -1/(r-1)$.

Note that the above value of $\cos \beta$ is a known result for regular tetrahedrons. In particular, when $r = 2$, we have $\beta = \pi$ and the tetrahedron collapses into a segment.

**References**


