On the Local Spectra of the Subconstituents of a Vertex Set and Completely Pseudo-Regular Codes

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Abstract

The local spectrum of a vertex set in a graph has been proven to be very useful to study some of its metric properties. It also has applications in the area of pseudo-distance-regularity around a set and can be used to obtain quasi-spectral characterizations of completely (pseudo-)regular codes. In this paper we study the relation between the local spectrum of a vertex set and the local spectrum of each of its subconstituents. Moreover, we obtain a new characterization for completely pseudo-regular codes, and consequently for completely regular codes, in terms of the relation between the local spectrum of an extremal set of vertices and the local spectrum of its antipodal set. We also present a new proof of the version of the Spectral Excess Theorem for extremal sets of vertices.

Keywords: Pseudo-distance-regularity; Local spectrum; Subconstituents; Predistance polynomials; Completely regular code.

1. Introduction

The notion of local spectrum was first introduced in [8] for a single vertex of a graph. In that paper, such a concept was used to obtain several quasi-spectral characterizations of local (pseudo)-distance-regularity. In the study of pseudo-distance-regularity around a set of vertices [7], which particularizes to that of completely regular codes when the graph is regular, the local spectrum is generalized to a set of vertices. As commented in the same paper, when we study the graph from a “base” vertex subset, its local spectrum plays a role

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similar to the one played by the (standard) spectrum for studying the whole graph.

In this work we are interested in the study of the relation between the local spectrum of a vertex set and the local spectra of the elements of the distance partition associated to it (also known as its subconstituents). Thus, Section 2 is devoted to define the local spectrum of a subset of vertices in a graph. In Section 3 completely pseudo-regular codes are introduced and we discuss some known results of special interest. Our main results can be found in Sections 4 and 5, where we give sufficient conditions implying a tight relation between the local spectrum of a set of vertices and that of each of its subconstituents. As a consequence, we obtain a new characterization of completely (pseudo-)regular codes. In the way we also obtain some information about the structure of the local spectrum of the subconstituents associated to a completely pseudo-regular code and we give a new proof of a result from [7], which can be seen as the Spectral Excess Theorem [5] for sets of vertices.

Before going into our study, let us first give some notation. In this paper \( \Gamma = (V, E) \) stands for a simple connected graph with vertex set \( V = \{1, 2, \ldots, n\} \). Each vertex \( i \in V \) is identified with the \( i \)-th unit coordinate (column) vector \( e_i \) and \( V \cong \mathbb{R}^n \) denotes the vector space of formal linear combinations of its vertices. The adjacencies in \( \Gamma \), \( \{i, j\} \in E \), are denoted by \( i \sim j \) and \( \partial(\cdot, \cdot) \) stands for the distance function in \( \Gamma \). Given a set of vertices \( C \subset V \), the distance from a vertex \( i \) to \( C \) is given by the expression \( \partial(i, C) = \min \{ \partial(i, j) \mid j \in C \} \).

We denote by \( \varepsilon_C = \max_{i \in V} \partial(i, C) \) the eccentricity of \( C \). Notice that, in the context of coding theory, the parameter \( \varepsilon_C \) corresponds to the covering radius of the code \( C \).

As usual, \( A \) stands for the adjacency matrix of \( \Gamma \), with set of different eigenvalues \( \text{ev} \Gamma = \{\lambda_0, \lambda_1, \ldots, \lambda_d\} \), where \( \lambda_0 > \lambda_1 > \cdots > \lambda_d \). The spectrum of \( \Gamma \) is

\[
\text{sp} \Gamma := \text{sp} A = \{\lambda_0^{m(\lambda_0)}, \lambda_1^{m(\lambda_1)}, \ldots, \lambda_d^{m(\lambda_d)}\},
\]

where \( m(\lambda_i) \) is the multiplicity of the eigenvalue \( \lambda_i \). We denote by \( \mathcal{E}_l = \ker(A - \lambda_l I) \) the eigenspace of \( A \) corresponding to \( \lambda_l \). Recall that, since \( \Gamma \) is connected, \( \mathcal{E}_0 \) is one-dimensional, and all its elements are eigenvectors having all its components either positive or negative (see e.g. [1, 4]). Denote by \( \mathbf{v} = (\nu_1, \nu_2, \ldots, \nu_n) \in \mathcal{E}_0 \) the unique positive eigenvector of \( \Gamma \) with minimum component equal to 1.

Note that \( V \) is a module over the quotient ring \( \mathbb{R}[x]/(Z) \), where \( (Z) \) is the ideal generated by the minimal polynomial of \( A \), \( Z = \prod_{l=0}^d(x - \lambda_l) \), with product defined by

\[
pu := p(A)u \quad \text{for every } p \in \mathbb{R}[x]/(Z) \text{ and } u \in V.
\]

With this notation, let us remark that the orthogonal projection \( E_l \) of \( V \) onto the eigenspace \( \mathcal{E}_l \) corresponds to

\[
E_l u = Z_l u, \quad u \in V,
\]
where $Z_l$, $l = 0, 1, \ldots, d$, is the Lagrange interpolating polynomial satisfying $Z_l(\lambda_h) = \delta_{lh}$, that is:

$$Z_l = \frac{(-1)^l}{\pi_l} \prod_{0 \leq h \leq d, h \neq l} (x - \lambda_h),$$

with $\pi_l$ being the moment-like parameter given by $\pi_l := \prod_{0 \leq h \leq d, h \neq l} |\lambda_l - \lambda_h|$.

2. C-Local spectrum

Given a set of vertices $C$ of $\Gamma$, define the map $\rho : \mathcal{P}(V) \to V$ by $\rho \emptyset := \emptyset$ and $\rho C := \sum_{i \in C} \nu_i e_i$ for $C \neq \emptyset$. Consider the spectral decomposition of the unit vector $e_C = \rho C/\|\rho C\| = z_c(\lambda_0) + z_c(\lambda_1) + \cdots + z_c(\lambda_d)$, that is $z_c(\lambda_i) = E_i e_C \in \mathcal{E}_i$, $0 \leq l \leq d$. The C-multiplicity of the eigenvalue $\lambda_l$ is defined by

$$m_c(\lambda_l) := (E_l e_C, e_C) = \|z_c(\lambda_l)\|^2.$$

If $\mu_0 > \mu_1 > \cdots > \mu_{dc}$ are the eigenvalues of $\Gamma$ with nonzero C-multiplicity, the C-local spectrum of $\Gamma$ is defined by

$$\text{sp}_C \Gamma := \{m_0^{m_c(\mu_0)}, m_1^{m_c(\mu_1)}, \ldots, m_{dc}^{m_c(\mu_{dc})}\},$$

and we denote by $\text{ev}_C \Gamma := \{\mu_0, \mu_1, \ldots, \mu_{dc}\}$, $\mu_0 > \mu_1 > \cdots > \mu_{dc}$, the set of different eigenvalues in the C-local spectrum. Let us remark that, as $E_0 e_C = \frac{e_C}{\|e_C\|} \cdot \nu = \frac{\|\rho C\|^2}{\|\rho C\|^2} \nu$, we have $m_c(\lambda_0) = \frac{\|\rho C\|^2}{\|\rho C\|^2} \neq 0$, and hence $\mu_0 = \lambda_0$. The parameter $d_c$ is called the dual degree of $C$ and it provides an upper bound for the eccentricity of the vertex set, $\varepsilon_c \leq d_c$ (see [7]). When the equality is attained we say that $C$ is extremal.

Consider the idempotents $E_l^C$, $0 \leq l \leq d_c$, corresponding to the members of the C-spectrum, that is $E_l^C$ is the projection of $\mathcal{V}_C = \bigoplus_{\lambda_{lh} \in \text{ev}_C \Gamma} \mathcal{E}_h$ onto the eigenspace corresponding to $\mu_l$. As we have done for the standard spectrum of the graph, we define for each $\mu_l \in \text{ev}_C \Gamma$ the moment-like parameter $\pi_l(\Gamma) := \prod_{0 \leq h \leq d_c, h \neq l} |\mu_l - \mu_h|$ and consider the polynomial

$$Z_l^C := \frac{(-1)^l}{\pi_l(\Gamma)} \prod_{0 \leq h \leq d_c, h \neq l} (x - \mu_h)$$

which gives $Z_l^C(A) = E_l^C$.

3. Completely pseudo-regular codes

In this section we review some known results on completely pseudo-regular codes. These results are formulated in terms of C-local pseudo-distance-regularity,
which extends the notion of local distance-regularity from single vertices to subsets of vertices and from regular to non-necessarily regular graphs. Completely pseudo-regular codes where introduced in [7] with the aim of generalizing both local distance-regularity and completely regular codes [9]. Let us consider the distance partition of a vertex set $C \subset V$ of $\Gamma$, given by the sets $C_k = \{i \in V \mid \partial(i,C) = k\}$, $k = 0, 1, \ldots, \varepsilon_C$, which are known as the sub-constituents associated to $C$. Consider also the functions $a, b, c : V \rightarrow [0, \lambda_0]$ acting on a vertex $i \in C_k$ as follows:

$$c(i) = \begin{cases} 0 & (k = 0); \\ \frac{1}{\nu_i} \sum_{j \in \Gamma(i) \cap C_{k-1}} \nu_j & (1 \leq k \leq \varepsilon_C). \end{cases}$$

$$a(i) = \frac{1}{\nu_i} \sum_{j \in \Gamma(i) \cap C_k} \nu_j \quad (0 \leq k \leq \varepsilon_C).$$

$$b(i) = \begin{cases} \frac{1}{\nu_i} \sum_{j \in \Gamma(i) \cap C_{k+1}} \nu_j & (0 \leq k \leq \varepsilon_C - 1); \\ 0 & (k = \varepsilon_C). \end{cases}$$

We say that $C$ is a completely pseudo-regular code (or that $\Gamma$ is $C$-local pseudo-distance-regular) when the values of $c$, $a$ and $b$ do not depend on the chosen vertex $i \in C_k$ but only on $k$; that is, when the distance partition with respect to $C$ is pseudo-regular (see [6]). If this is the case, we denote by $c_k, a_k$ and $b_k$ the values of these three functions on the vertices of $C_k$, $k = 0, 1, \ldots, \varepsilon_C$, and refer to them as the $C$-local pseudo-intersection numbers.

Remark that, since $\nu \in \mathcal{E}_0$, the sum of these three functions is constant over all the vertices of $\Gamma$:

$$c(i) + a(i) + b(i) = \frac{1}{\nu_i} \sum_{j \in \Gamma(i)} \nu_j = \frac{1}{\nu_i} (A\nu)_i = \lambda_0 \quad \text{for all } i \in V.$$

Thus, in a sense, we can say that the weight $\nu_i$, given to every vertex $i$, regularizes the graph. Note also that, when $\Gamma$ is a regular graph we have that $\nu = j$, the all-1 vector, and the definition of a completely pseudo-regular code particularizes to that of a completely regular code.

3.1. Some characterizations of $C$-local pseudo-distance-regularity

As established in [3], one can consider two approaches to completely pseudo-regular codes. One is based on the combinatorial definition given in the previous section and the other relies upon the study of the Terwilliger algebra [11] associated to a vertex set. Although the results presented here can be obtained from both points of view, we will focus the attention on the combinatorial approach, which makes more evident the key role of the local spectrum.
From now on, all the polynomials must be considered in the quotient ring $\mathbb{R}[x]/(Z(x))$, where $Z(x) = \prod_{\mu \in eV} \Gamma(x - \mu)$. We begin by defining the $C$-local scalar product as follows:

$$\langle p, q \rangle_C := \langle pe_C, qe_C \rangle = \sum_{l=0}^{d_C} m_C(\mu_l)p(\mu_l)\rho(\mu_l).$$

Then, we consider the family of $C$-local predistance polynomials, $(p_k)_{0 \leq k \leq d_C}$, which is the unique orthogonal system with respect to the $C$-local scalar product, such that $\deg p_k = k$ and $\langle p_k, p_h \rangle_C = \delta_{kh}p_k(\mu_0), 0 \leq k \leq d_C$; see [2]. In [7] one can find some quasi-spectral characterizations of $C$-local pseudo-distance-regularity, which are based on these polynomials. In particular, the following two theorems are of special interest in our work.

**Theorem 1.** Let $\Gamma = (V, E)$ be a graph and let $C \subset V$ have eccentricity $\varepsilon_C$. Then, $\Gamma$ is $C$-local pseudo-distance-regular if and only if the $C$-local predistance polynomials satisfy $p_Ck = p_{\varepsilon_C}$, $k = 0, 1, \ldots, \varepsilon_C$.

The following result establishes that, if we assume that $C$ is extremal, it is enough to check the condition of Theorem 1 for the set of vertices at maximum distance $C_{\varepsilon_C}$. We refer to this set as the antipodal set of $C$ and, if there is no possible confusion, we write $D = C_{\varepsilon_C}$.

**Theorem 2.** Let $\Gamma = (V, E)$ be a graph and let $C$ be an extremal set with eccentricity $\varepsilon_C = d_C$. Then, $\Gamma$ is $C$-local pseudo-distance-regular if and only if there exists a polynomial $p \in \mathbb{R}_{d_C}[x]$ such that $ppC = \rho D$, in which case $p$ is the $C$-local predistance polynomial $p_{d_C}$.

This last result points out the relevance of the antipodal set in the study of completely pseudo-regular codes. In fact, it is known from [7] that a set of vertices is a completely pseudo-regular code if and only if its antipodal set is. This symmetry is illustrated in Fig. 1 where the $p_k$’s are the $D$-local predistance polynomials.

### 4. Spectra of the subconstituents

As showed in [3], for an extremal set of vertices $C$ with antipodal set $D$, the relation between the local spectra of $C$ and $D$ is tight and can be expressed in terms of the following proposition. We include the proof for the sake of completeness.

**Proposition 1.** Let $C$ be an extremal set and let $D$ be its antipodal set. Then, $eV \subset eV \Gamma$ and the $C$-multiplicities and $D$-multiplicities satisfy

$$m_C(\mu_l)m_D(\mu_l) \geq \frac{\pi^2(\mu_l)}{\pi^2(\mu_l)} \frac{\|pC\|^2\|\rho D\|^2}{\|\pi^4\}}$$

for all $\mu_l \in eV \Gamma$. (2)
Moreover:

(i) Equality holds in Eq. (2) for an eigenvalue $\mu_l$ if and only if the vectors $z_C(\mu_l)$ and $z_D(\mu_l)$ are linear dependent.

(ii) Equality in Eq. (2) for every $\mu_l \in \text{ev}_C \Gamma$ is equivalent to the existence of a polynomial $p \in \mathbb{R}_{\varepsilon_c}[x]$ such that

$$p_D = p p_C + z,$$

where $z \in \bigoplus_{\lambda_l \in \text{ev}_D \Gamma \setminus \text{ev}_C \Gamma} \mathcal{E}_l$.

**Proof.** Consider the $C$-local Hoffman polynomial (see [3, 10])

$$H_C = \frac{||\nu||^2}{||p_C||^2} Z_0^C = \frac{||\nu||^2}{\pi_0(C) ||p_C||^2} \prod_{l=1}^{d_c} (x - \mu_l),$$

which satisfy $H_C p_C = \nu$. Since both $Z_l^C$, defined in Eq. (1), and $H_C$ have degree $d_C$ and their leading coefficients are, respectively, $\frac{(-1)^l}{\pi_l(C)}$ and $\frac{||\nu||^2}{\pi_0(C) ||p_C||^2}$, the polynomial

$$T = \pi_0(C) \frac{||p_C||^2}{||\nu||^2} H_C - (-1)^l \pi_l(C) Z_l^C$$

has degree less than $d_C = \varepsilon$. Then, the vectors $Te_C$ and $e_D$ are orthogonal, giving:

$$\langle Te_C, e_D \rangle = \pi_0(C) \frac{||p_C||^2}{||\nu||^2} \langle H_C e_C, e_D \rangle - (-1)^l \pi_l(C) \langle Z_l^C e_C, e_D \rangle$$

$$= \pi_0(C) \frac{||p_C|| \cdot ||p_D||}{||\nu||^2} \langle H_C p_C, p_D \rangle - (-1)^l \pi_l(C) \langle z_C(\mu_l), e_D \rangle$$

$$= \pi_0(C) \frac{||p_C|| \cdot ||p_D||}{||\nu||^2} - (-1)^l \pi_l(C) \langle z_C(\mu_l), z_D(\mu_l) \rangle = 0, \quad (3)$$
since \( \langle H_c, \rho C, \rho D \rangle = \langle \nu, \rho D \rangle = \| \rho D \|^2 \). Therefore, by the Cauchy-Schwarz inequality,

\[
\frac{\pi^2(C) \| \rho C \|^2 \| \rho D \|^2}{\pi^1(C) \| \nu \|^4} = \langle z_c(\mu_l), z_D(\mu_l) \rangle^2 \leq m_c(\mu_l)m_D(\mu_l),
\]

where equality occurs if and only if \( z_c(\mu_l) \) and \( z_D(\mu_l) \) are colinear, and (i) is also proved. Moreover, as all the terms involved are positive, \( m_D(\mu_l) > 0 \) and hence \( \mu_l \in \text{ev}_\alpha \Gamma \).

In order to prove (ii), suppose now that the equality holds for every eigenvalue of the \( C \)-local spectrum. Then, given \( \mu_l \in \text{ev}_c \Gamma \), the vectors \( z_D(\mu_l), z_c(\mu_l) \) are proportional. More precisely, by Eq. (3), there exist \( \xi_l > 0 \) such that \( z_D(\mu_l) = (-1)^l \xi_l z_c(\mu_l) \). Let \( p \) be the unique polynomial in \( \mathbb{R} \xi_c[x] \) such that \( p(\mu_l) = (-1)^l \frac{\| \rho D \|}{\| \rho C \|} \xi_l \) for all \( \mu_l \in \text{ev}_c \Gamma \). We have

\[
E_l \rho D = \| \rho D \| z_D(\mu_l) = (-1)^l \| \rho D \| \| \rho C \| z_c(\mu_l) = (-1)^l \| \rho D \| \| \rho C \| \xi_l E_l \rho C = p(\mu_l) E_l \rho C = E_l p \rho C.
\]

Thus \( z = \rho D - p \rho C \in \bigoplus_{\lambda_l \in \text{ev}_D \Gamma \setminus \text{ev}_C \Gamma} E_l \). Conversely, assuming the existence of \( p \in \mathbb{R} \xi_c[x] \) satisfying \( \rho D = p \rho C + z \), with \( z \in \bigoplus_{\lambda_l \in \text{ev}_D \Gamma \setminus \text{ev}_C \Gamma} E_l \), by projecting onto the eigenspace of \( \mu_l \) (\( \mu_l \in \text{ev}_c \Gamma \)) we obtain

\[
\| \rho D \| z_D(\mu_l) = p(\mu_l) \| \rho C \| z_c(\mu_l)
\]

and, by (i), equality in Eq. (2) holds for every \( \mu_l \in \text{ev}_c \Gamma \).

Although the last result cannot be directly extended to the other subconstituents, it suggests that the existence of the distance polynomials guarantees a tight relation between the local spectra of the subconstituents. The following proposition supports this claim.

**Proposition 2.** Let \( C \) be a completely pseudo-regular code in a graph \( \Gamma \). Denote by \( C_k, k = 0, 1, \ldots, \varepsilon (= d_c) \) the subconstituents associated to \( C \). Then \( \text{ev}_c \Gamma \subset \text{ev}_c \Gamma \). Moreover, for each \( l = 0, 1, \ldots, d_c \), the projections of \( \rho C_k \) and \( \rho C \) onto the eigenspace \( E_l \) are linearly dependent.

**Proof.** Let \( (p_k)_{0 \leq k \leq \varepsilon} \) be the \( C \)-local predistance polynomials. From Theorem 1 we have that \( \rho C_k = p_k \rho C \), \( k = 0, 1, \ldots, \varepsilon \). By projecting onto the eigenspace \( E_l \) we obtain

\[
E_l \rho C_k = E_l p_k \rho C = p_k(\lambda_l) E_l \rho C, \text{ so that } E_l \rho C_k \text{ and } E_l \rho C \text{ are colinear. Moreover, since } e_{C_k} = \frac{\rho C_k}{\| \rho C_k \|} \text{ and } e_C = \frac{\rho C}{\| \rho C \|} \text{ we obtain:}
\]

\[
m_{C_k}(\lambda_l) = \| E_l e_{C_k} \|^2 = \frac{\| E_l \rho C_k \|^2}{\| \rho C_k \|^2} = \frac{\| p_k(\lambda_l) E_l \rho C \|^2}{\| \rho C_k \|^2} = \frac{\| \rho C \|^2}{\| \rho C_k \|^2} (p_k(\lambda_l))^2 \| E_l e_{C} \|^2 = \frac{\| \rho C \|^2}{\| \rho C_k \|^2} (p_k(\lambda_l))^2 m_c(\lambda_l).
\]
Thus, $\lambda_l \in \text{ev}_{c_k} \Gamma$ and $\text{ev}_{c_k} \Gamma \subset \text{ev}_c \Gamma$. 

Note that, since for an extremal set we have $\text{ev}_c \Gamma \subset \text{ev}_D \Gamma$, the last result shows that, in particular, for a completely pseudo-regular code we have $\text{ev}_c \Gamma = \text{ev}_D \Gamma$.

As a by-product, from Eq. (4), case $k = d_c$, and Proposition 1(ii) we have that, for a completely pseudo-regular code,

$$m_c(\mu_l) = \frac{\pi_0(C) \|\rho_D\|^2}{\pi_l(C) \|\upsilon\|^2} \frac{1}{p_{dc}(\mu_l)};$$  \hspace{1cm} (5) 

$$m_D(\mu_l) = \frac{\pi_0(C) \|\rho_C\|^2}{\pi_l(C) \|\upsilon\|^2} \frac{1}{p_{dc}(\mu_l)};$$  \hspace{1cm} (6) 

for all $\mu_l \in \text{ev}_c \Gamma$. As commented, recall that in this case $D$ is also a completely pseudo-regular code and $C$ is its antipodal set. If we denote by $\bar{p}$ the $D$-local predistance polynomial with maximum degree and use the existing symmetry (see Fig. 1) Eq. (5) gives:

$$m_D(\mu_l) = \frac{\pi_0(C) \|\rho_C\|^2}{\pi_l(C) \|\upsilon\|^2} \frac{1}{\bar{p}(\mu_l)} \quad \text{for all } \mu_l \in \text{ev}_c \Gamma.$$ 

This jointly with Eq. (6) leads us to the existing relation between the $C$-local and $D$-local predistance polynomials of maximum degree:

$$\bar{p}(\mu_l) = \frac{1}{p_{dc}(\mu_l)} \quad \text{for all } \mu_l \in \text{ev}_c \Gamma.$$ 

Recall that the $C$-local predistance polynomials satisfy a three term recurrence (see [2]) and, in particular, for a completely pseudo-regular code with intersection numbers $a_k, b_k$ and $c_k$, $k = 0, 1, \ldots, \varepsilon_c (= d_c)$, we have:

$$xp_k = b_{k-1}p_{k-1} + a_k p_k + c_{k+1}p_{k+1} \quad (0 \leq k \leq d_c),$$

where $b_{-1} = c_{d_c+1} = 0$ and $b_{k+1} > 0$ (see [2]). Thus, for an eigenvalue $\mu_l \in \text{ev}_c \Gamma$

$$p_{k+1}(\mu_l) = \frac{(\mu_l - a_k)p_k(\mu_l) - b_{k-1}p_{k-1}(\mu_l)}{c_{k+1}},$$

and, since $\text{ev}_c \Gamma = \text{ev}_D \Gamma$, Eq. (4) ensures us that in a completely pseudo-regular code there cannot exist $k$ such that $\mu_l \notin \text{ev}_{c_k} \Gamma \cup \text{ev}_{c_{k+1}} \Gamma$. That is, for every $k = 0, 1, \ldots, \varepsilon_c - 1$, $\text{ev}_c \Gamma = \text{ev}_{c_k} \Gamma \cup \text{ev}_{c_{k+1}} \Gamma$. Remark also that in this case we have that $\varepsilon_{c_k} \geq d_c - 1$ and, in general, $\varepsilon_{c_k} \geq \max\{k, d_c - k\}$. Thus, since $d_{c_k} \geq \varepsilon_{c_k}$, the dual degree of the $k$-th subconstituent satisfies the bound

$$d_{c_k} \geq \max\{k, d_c - k\}.$$ 

Consequently, in a completely pseudo-regular code, the $C$-local predistance polynomial $p_k$ vanishes at most at $\min\{k, d_c - k\}$ different eigenvalues of the $C$-local spectrum.
5. Characterizations of completely pseudo-regular codes

Some of the results given in the previous section can be used to obtain characterizations for completely pseudo-regular codes. In particular, we get an alternative proof for extremal sets of a result in [7], which can be seen as the version of the Spectral Excess Theorem [5] for sets of vertices.

**Theorem 3.** Let $C$ be an extremal set and $D$ its antipodal set. Then

$$
\frac{\|\rho D\|^2}{\|\rho C\|^2} \leq \frac{1}{m_C(\lambda_0)^2} \sum_{l=0}^{d_C} 1/m_C(\mu_l)\pi_l^2(C),
$$

and the equality holds if and only if $C$ is a completely pseudo-regular code.

**Proof.** By Proposition 1 we have:

$$
m_D(\mu_l) \geq \frac{\pi_0^2(C)}{\pi_l^2(C)} \frac{\|\rho D\|^2}{\|\rho C\|^2} \frac{1}{m_C(\mu_l)} \text{ for all } \mu_l \in \text{ev}_C \Gamma.
$$

By adding up for all $\mu_l \in \text{ev}_C \Gamma$ and using that we obtain $m_C(\lambda_0) = \|e_D\|^2 = 1$, we have:

$$
\sum_{l=0}^{d_C} \frac{\pi_0^2(C)}{\pi_l^2(C)} \|\rho D\|^2 m_C(\lambda_0) \leq \sum_{l=0}^{d_C} m_D(\mu_l) \leq \sum_{l=0}^{d_C} m_D(\lambda_l) = \|e_D\|^2 = 1,
$$

(8)

which gives Eq. (7). In case of equality in Eq. (8) we obtain $m_D(\mu_l) = 0$ if $\lambda_l \notin \text{ev}_C \Gamma$, so that $\text{ev}_C \Gamma = \text{ev}_D \Gamma$. Moreover, in this case Proposition 1(ii) applies and there exist a polynomial $p \in \mathbb{R}_{\text{ev}_C} [x]$ such that $p \rho C = \rho D$, or, equivalently, $C$ is a completely pseudo-regular code. $\Box$

Next theorem gives a new approach to completely pseudo-regular codes and shows that, when the considered set of vertices is extremal, the converse of Proposition 2 also holds.

**Theorem 4.** Let $C$ be an extremal set with eccentricity $\varepsilon = d_C$ and antipodal set $D = C_\varepsilon$. Then $C$ is a completely pseudo-regular code if and only if the orthogonal projections of $\rho C$ and $\rho D$ onto each eigenspace of $\Gamma$ are colinear.

**Proof.** Proposition 2 guarantees the necessity. Assume now that $E_l \rho D$ and $E_l \rho C$ are colinear for each $l = 0, 1, \ldots, d$. That is, there exist constants $\alpha_l$ satisfying $E_l \rho D = \alpha_l E_l \rho C$. Let $p_\varepsilon$ be the unique polynomial of degree $\varepsilon$ such that $p_\varepsilon(\lambda_l) = \alpha_l$. Then,

$$
\rho D = \sum_{l=0}^{d} E_l \rho D = \sum_{l=0}^{d} \alpha_l E_l \rho C = \sum_{l=0}^{d} p_\varepsilon(\lambda_l) E_l \rho C = p_\varepsilon \rho C,
$$

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and the result follows from Theorem 2.

In particular, if the underlying graph \( \Gamma \) is regular, the theorem gives a new characterization of completely regular codes.

References


