The Optimization of Spring Stiffness for Passive Dynamic Walker

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Abstract—On a passive dynamic walker, we find that an appropriate spring placed on the mass center of two legs can greatly improve the walker’s walking performance, which includes its walking speed, disturbance rejection ability and step length. However, the extent to which spring influences these properties is unknown and how to choose an appropriate spring stiffness to achieve optimal walking performance still needs to be discussed. In this paper, we present a study on the effect of spring on these properties by constructing a synthesis index $P$ to assess the walking performance. Through numerical simulation on three models, we find the walker with spring has a better walking performance over a pure passive dynamic walker and the extension spring on the mass center of two legs can improve a walker’s overall walking performance much better.

I. INTRODUCTION

In 1990, McGeer first proposed the concept of passive dynamic walking, showing that a fully unactuated and uncontrolled two-legged mechanism can perform a stable walking motion down a slight slope [1]. This theory has been greatly explored by many researchers and it has been applied in many real robots [2][3]. Generally, there are alternative ways to obtain a stable walking on a slope. Hass et al. optimized the mass distributions to obtain the maximum speed of a straight leg point-foot model [4]. Wisse et al. proposed the use of ankle springs instead of arc-shaped feet to improve the walking disturbance handling ability [5]. But all these means depend on modifying some parameters to achieve only one specific property. It is hard to say that the achieved walking is really good. Those properties are contradictory in many cases. When we construct a walker with high energy efficiency, the walker’s disturbance rejection ability would not be satisfying. It is necessary to find a way to construct a walker that can walk well with a balance of multiple good properties.

Recently, on a passive dynamic walker-THUDA PDWbot, which is modified from handon robot in Dynamic Walking 2009 (see Fig. 1), we found a spring placed on the mass center of two legs greatly improve the robot walking. With the spring, the walker can walk over 48 steps easily\(^1\) , which indicates the walker has proper step length and good disturbance rejection ability. Mizuno et al. also showed that utilizing restoring force of spring or rubber will make the passive dynamic walking robot walk easier [6]. But his research only found initial-value spaces of the passive dynamic walking robot with elastic materials is greater, which is not enough to explain this phenomenon. Kuo studied that the use of a hip torque on the swing leg can tune walking frequency [7]. But hip torque can do much more than just tuning walking frequency. This hip spring is just like human’s thigh tension or muscle, which can adjust its walking gait to achieve much better walking performance.

Though the hypothesized properties of spring working in the robot have been known, the extent of those properties influenced by the spring is still unknown. Besides, which spring stiffness works the best still needs to be discussed.

In this paper, we will first define an index $P$ to measure the overall performance of the passive dynamic walker. The index $P$ is very crucial, which includes velocity, step length and disturbance rejection ability. The optimal spring stiffness is to make the index $P$ maximum. The goal of this paper is to provide an idea to construct the index $P$ and to get the optimal spring stiffness by simulation.

This paper is organized as follows. Section II describes the models used in this study. In Section III, the overall walking performance measurement index is introduced. In Section IV, we use the index to analysis the models and to get the optimal spring stiffness. In Section V, we compare the properties and analyze the energy efficient respectively among the models. Section VI is about the conclusions.

II. MODELS

This Section describes the composition of three models and the equations of walking dynamics.
A. The Description of Models

To get the comparison result of elastic work, this study is based on three models. Fig. 2 shows a passive walking model. It consists of two equal legs with arc foot, which was introduced by McGeer [1]. Due to the symmetric construction of the prototype, it has quasi-2D dynamics. To focus more on the elastic and make the conclusions more generalizable, we fix and normalize our model parameters. The method to choose these parameters is similar to McGeer’s research [1], which are given in Table I. The leg length is \( L \). Foot radius is \( R \). \( B \) and \( C \) show the relative position from center of mass to hip. The radius of leg equivalent inertia is represented by \( r \). We call this model PASSIVE for short. As shown in Fig. 3, the second model is added with torsion spring in the hip, which is called HIP for short. The third model is added with extension spring on the mass center of mass to hip. The radius of leg equivalent inertia is \( r \). In the following sections, we discuss the relationship between walking performance and the spring stiffness.

The PASSIVE model is used as a reference to illustrate the improvement of other models. Due to the absence of knees, we cover the walking slope with some tiles to avoid stubbing its toes when the swing leg passes the stance leg in the real experiments. This is why we consider the desired step length as an important part of overall walking performance index. We assume a perfectly inelastic impact, no slip, no rolling resistance. The energy loss only happens at the impact. We allow the swing foot pass through the floor during moving in the simulation.

B. Governing Equations

With the assumption illustrated above, the entire moving equation can be expressed with stride function [1]. The stride function composes of motion equation and impact equation.

1) Equation of Motion: As shown in Fig. 2, the robot configuration can be described by \((\theta_{st}, \theta_{sw})\), where \( \theta_{st} \) and \( \theta_{sw} \) are the stance leg angle and swing leg angle with slope vertical direction (counter clockwise positive). The position of hip is termed as \( P_H \). The position of center of two leg mass center are termed as \( P_{st} \) and \( P_{sw} \). The slope down direction is its axis \( x \) and slope vertical direction is its axis \( y \). So we have:

\[
\begin{bmatrix}
P_{st} = P_H + R_{st}[B - C]^T \\
\dot{P}_{st} = F_H + R_{sw}[^{3} B - C]^T
\end{bmatrix}
\]

where \( P_H = \begin{bmatrix} -R\theta_{st} - (L - R)\sin(\theta_{st}) \\ R + (L - R)\cos(\theta_{st}) \end{bmatrix} \). And \( R_{st}, R_{sw} \) are rotation matrix. \( R_a = \begin{bmatrix} \cos(\theta_a) & -\sin(\theta_a) \\ \sin(\theta_a) & \cos(\theta_a) \end{bmatrix} \), \( a \) is \( st \) or \( sw \).

We define the generalized coordinate and coordinate as follows: \( q = [\theta_{st}, \theta_{sw}] \), \( x = [P_{st}(1), P_{st}(2), \theta_{st}, P_{sw}(1), P_{sw}(2), \theta_{sw}] \), \( J = \partial x / \partial q \), where \( J \) is the Jacobian matrix. Based on Newton theorem and virtual power principle, we have:

\[
\begin{bmatrix}
M \ddot{x} = F^e + F^a \\
J^T \dot{F}^c = 0
\end{bmatrix}
\]

Combing all the above equations, we can get the following expression:

\[
J^T M J \ddot{q} + J^T M \dot{J} \dot{q} - J^T F^a = 0
\]

where \( M \) is the mass matrix, \( M = \text{diag}[m, m, I, m, m, I] \). \( F^a \) is the force vector working on the leg mass. The active force vector is different for each model.

To the PASSIVE model, the active force equals the gravity.

\[
F^a_{P} = F_g = mg[\sin(\gamma), -\cos(\gamma), 0, \sin(\gamma), -\cos(\gamma), 0]^T.
\]

To the HIP model, the active force contains gravity and hip torque. The hip torque is linear with the angle between two legs.

\[
F^a_{HIP} = F_g + F_T = [mgsin(\gamma), -mgcos(\gamma), k_{hip}(\theta_{sw} - \theta_{st}), mgsin(\gamma), -mgcos(\gamma), -k_{hip}(\theta_{sw} - \theta_{st})]^T.
\]
Fig. 4: Lateral view and front view of CoM model.

To the CoM model, active force contains gravity and elastic force, \( F_{\text{CoM}} = F_g + F_e \). As shown in Fig. 4, we denote that \( F = k_{\text{CoM}} (l - l_0) \), where \( l = \sqrt{||P_{\text{sw}} - P_{\text{st}}||^2 + a^2} \). \( l_0 \) is the natural length of spring. \( F \) is the elastic force. To get \( F_e \), we should project this force to the walking plane.

That is:

\[
F_e = F \cdot [P_{\text{sw}} - P_{\text{st}}] \frac{l}{||P_{\text{sw}} - P_{\text{st}}||} \left( \frac{P_{\text{sw}} - P_{\text{st}}}{||P_{\text{sw}} - P_{\text{st}}||}, 0 \right)^T
\]

(6)

2) Equation of Impact: The equation of impact is the same for the three models.

According to conservation of momentum theorem, we have

\[
(J^{\text{sw}})^T M (J^{\text{sw}}) \dot{q}^+ = (J^{\text{sw}})^T M (J^{\text{st}}) \dot{q}^-,
\]

(7)

where '-' and '+' denote the state before and after the impact respectively.

To the variables in equation of impact, they are almost the same as above except the Jacobian matrix. There are two Jacobian matrix, \( J^{\text{st}} \) and \( J^{\text{sw}} \). They are different because of the different states. \( J^{\text{st}} \) exists at the state of before impact, so \( J^{\text{st}} \) is the same with Jacobian matrix in the swing stage phase. \( J^{\text{sw}} \) exists at the state of after impact. So we should exchange the leg to calculate the Jacobian matrix. \( J^{\text{st}} \) and \( J^{\text{sw}} \) can be derived as follows:

\[
P_H^s = \begin{bmatrix}
-RB_s \sin(\theta_s) \\
(R + (L - R) \cos(\theta_s)) \\
\end{bmatrix}
\]

\[
P_{\text{sw}}^s = P_H^s + R_s [B, -C]^T
\]

\[
P_{\text{sw}}^s = P_H^s + R_{\text{sw}} [B, -C]^T
\]

\[
x^s = [P_{\text{sw}}^s (1), P_{\text{sw}}^s (2), \theta_{\text{sw}}, P_{\text{sw}}^s (1), P_{\text{sw}}^s (2), \theta_{\text{sw}}]
\]

\[
J^s = \frac{\partial x^s}{\partial \theta}
\]

where \( s \) is \( \text{st} \) or \( \text{sw} \).

III. THE OVERALL WALKING PERFORMANCE INDEX

To construct more capable bipedal robots, it is essential to find some values to measure the overall walking performance. The overall performance often contains many criteria, such as velocity and robustness. Sometimes those criteria are contradictory. Hass et al. found the maximal robustness and highest walking velocity are partly conflicting when they wanted to optimize mass distribution for passivity-based bipedal robots [4]. Hobbelten et al. found swing-leg retraction can improve limit cycle walkers’ disturbance rejection, which would result in lower speed [8]. It means we have to sacrifice one criterion to achieve the maximal optimization of other criteria. However, some criteria are essential for the stable walking. If we want to construct a capable robot with good comprehensive walking performance, we have to combine the essential walking criteria to assess the overall walking performance. In this study, the essential criteria contain walking velocity, step length and disturbance rejection ability.

Now we construct the overall walking performance measurement index \( P \) as follows.

\[
P = v s^n e^{-\frac{||x - x_0||}{\sigma}}
\]

(9)

where \( v \) is the walking velocity, \( s \) is the inverse of Gait Sensitivity Norm (GSN) [9], and \( \sigma \) is the walking step length, \( \sigma_0 \) is the desired step length.

Our aim is to construct a robot to walk fast, with fitted step length, good disturbance rejection ability and low energy expenditure. When a robot can walk down a small slope steadily, the gait must enter into a stable limit cycle. Here \( v \) and \( \sigma \) are referred to the velocity and step length at fixed point. But in this case, the step length at fixed point may not be what we desire. We want to obtain a proper step length. Then \( e^{-\frac{||x - x_0||}{\sigma}} \) represents the closeness of obtained step length and desired step length. \( s \) represents the disturbance rejection ability of passive walker. Often to a walker, it is much more important to keep walking against disturbance. So we multiply the number \( n \) power of \( s \) to add the importance of disturbance rejection ability to the overall walking performance index \( P \). \( n \) should not be too large or too small. too large would lead to that \( P \) is totally decided by \( s \) and too small would lead to that disturbance rejection ability has no effect on \( P \). The choice of \( n \) should be limited in the scope from 2.5 to 4, which can keep balance among those criteria. Here, we set \( n \) to be 3. To the limit cycle walking, the energy expenditure depends on the slope.

We will discuss it in section IV. Now we explain the Gait Sensitivity Norm in our research.

A. Gait Sensitivity Norm

Gait Sensitivity Norm is proposed by Hobbelen and Wisse in 2007, it is a good measurement to reflect the disturbance rejection ability. Another benefit is that its calculation time is shorter comparing to calculating BoA (Basin of Attraction) [10] and Floquet multiplier [11]. Therefore, we choose GSN as a criterion to measure a walker’s disturbance rejection ability.

The detail definition of GSN has been given in paper [9]. Now we give a brief introduction to it and how we get the GSN fitted to our models.

When a stable limit cycle walking is disturbed, there will be some deviations for the gait state from the limit cycle. We use \( e_n \) to represent the disturbance. The gait deviations are called gait indicators, which are noted by \( g_n \). The gait indicators can be manifold, but they should be directly related to the possible failure modes of the walker. We take the \( H_2 \) norm of those indicators to show the ability of gait rehabilitation. That is the sum of the squares of gait deviations.
indicators in all subsequent steps after the disturbance until the effect died out [9]. In short:

$$GSN = \| \frac{\partial g}{\partial e} \|_2 = \frac{1}{e} \sqrt{\Delta \dot{q}_1^2 + \Delta \dot{q}_2^2 + \ldots}$$  \hspace{1cm} (10)$$

From the above expression, we know a smaller $\| \partial g/\partial e \|_2$ means a better disturbance handling. So we define, $s = 1/\| \partial g/\partial e \|_2$, which means a bigger $s$ can represent a better disturbance rejection ability.

B. The Choice of Disturbance and Gait Indicators

To calculate the Gait Sensitivity Norm, one is to choose the type(s) of disturbances $e$ of interest. The disturbances $e$ should be proper and large enough to make the rejection ability visible in the simulation. In general, floor height variations is a good disturbance for most walkers. Because it is constantly present in real-world situations and it has a large influence on walker’s behavior. Then we choose a single step-up on the floor as a disturbance for our models. For the PASSIVE model, when the gait state enters into limit cycle on the typical slope of 0.0125, the maximum allowable height of step-up disturbance, $e_{\text{max}} = 0.0067$. However, we should choose a lower height disturbance because the maximum disturbance will make the walker fall in other slopes. Therefore we set $e = 0.006$. The gait state is, $[q^*, \dot{q}^*] = [0.2199, -0.2199, -0.3244, -0.2528]$ before disturbing. After disturbing, the gait state will become, $[q^?, \dot{q}^?] = [0.2374, -0.1977, -0.3116, -0.2520]$. From that, we find the disturbance is large enough to make the gait state deviate greatly from its limit cycle. And to other slopes and models, $e = 0.006$, is also proper.

Now we have to choose gait indicators. The gait indicators should be qualified to directly related to the failure modes. Various gait indicators can be used, such as step length, step time and so fourth. But it is important to choose the best quantified one. Hobbelen et al. argued the step time is a representative one for the simplest model [9]. To our models, we think the step time may or not be a good indicator. Therefore, we choose step time, step length, kinetic energy and total energy as indicator to calculate the Gait Sensitivity Norm and compared their correlation to the actual disturbance rejection ability. When the measure is highly correlated to the actual disturbance rejection, its gait indicators’ prediction is good. The actual disturbance rejection is the walker’s ability to prevent falling in the real presence of disturbances, and we choose a Gaussian white noise of floor height variations as its actual disturbance. If a walker falls 5 times in 100-step trial, we define the magnitude (95% confidence interval) of this Gaussian white noise as its actual disturbance rejection. To our three models, we should test the gait indicators’ prediction to actual disturbance rejection respectively. Shown in Fig. 5, we get the comparison graph (The detail simulation methods can be found in the paper [9]). For the comparison, we are only interested in the correlation rate with actual disturbance rejection, the absolute values are not important.

To the HIP and CoM model, the step length is unreasonably small with large spring stiffness in one fixed slope, it is not good to study the effect of spring stiffness. So we fix the step length by adjusting the slope to test the elastic effect. Table II shows that, to our three models, the best qualified gait indicator should be the kinetic energy comparing with other gait indicators. Then we use kinetic energy as gait indicator to calculate $1/GSN$.

TABLE II: Performance Gait Sensitivity With Various Gait Indicators

<table>
<thead>
<tr>
<th>model</th>
<th>Correlation Gait Sensitivity Norm to Actual Disturbance Rejection with Gait Indicator</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Step time</td>
</tr>
<tr>
<td>PASSIVE</td>
<td>95.27%</td>
</tr>
<tr>
<td>HIP</td>
<td>42.98%</td>
</tr>
<tr>
<td>CoM</td>
<td>61.48%</td>
</tr>
</tbody>
</table>

IV. Optimization of Spring Stiffness

In this section, we investigate the extent to which that the walking performance can be improved by the spring stiffness. First, we analyze the index $P$ of the PASSIVE model, which is much easier because of no elastic effect. Next, we explore
how to use the index $P$ to find optimal spring stiffness and the detailed properties of elastic element working on HIP and CoM models.

A. PASSIVE model

The Passive model is a type of compass-like walker with arc feet. Many scholars have studied it [1],[12]. One main character about it is that the gait will become bifurcated and enter into chaos after the slope exceeds a range [12]. But a hip spring with appropriate stiffness can increase this range to the largest value of $\frac{\pi}{2}$ in the ideal impact assumption. It is just like rimless wheel that has fixed step length and will not fall at the slope of $\frac{\pi}{2}$ [13]. To limit our slope in the scale that the gait won’t become bifurcated for the passive model, we set $0.0125 \text{ (rad)} < \gamma < 0.0919 \text{ (rad)}$. In order to compare with a natural gait like human, we define the desired step length is 0.45 folds of leg length, i.e. $\sigma_0 = 0.45$.

Figure 6 shows how velocity, step length, $1/|GSN|$ and $P$ vary across the range of slope. With the increasing of slope, $1/|\partial g/\partial e|_2$ and velocity increase. The closeness between obtained step length and desired step length decreases, which indicates the step length becomes much larger than desired step length after the slope pass about 0.0135 (rad). The curve of $P$ shows that, with the increase of slope, $P$ becomes larger. That is to say, the larger the slope is, the better the overall walking performance becomes among these slopes.

B. HIP and CoM model

To HIP and CoM model, the walking gait is influenced by slope and spring stiffness. To get the maximum $P$, it now becomes a three-dimension (3-D) problem. The problem can be solved as following steps. That is, we fix the slope, find the spring stiffness to make $P$ maximum. Then we get the corresponding spring stiffness for a maximum index $P$ to any slope. Now we discuss it respectively on the two models.

1) HIP model: To HIP model, we can derive a 3-D curved surface with slope and spring stiffness. A part of this curved surface is shown in Fig. 7.

The asterisks in the figure mark the apex $P$ to those slopes. If we project those asterisks into the bottom plane, we can obtain the relationship curve between the optimal spring stiffness and slope shown in Fig. 8. When $\gamma = 0.0459$ and $\gamma = 0.0519$, the spring stiffness decrease sharply, this is because the $1/|GSN|$ curve with increasing spring stiffness is not smooth in this model. As shown in Fig. 9, we find that the slope can be divided into three main intervals. In interval

Fig. 6: The effect of parameter variations with the slope.
Fig. 7: The curved surface with the slope and spring stiffness of HIP model.
Fig. 8: The curve of optimal spring stiffness with slope of HIP model.
Fig. 9: The effect of optimal spring stiffness in HIP model.
I and interval III, the step length increases with the increasing of slope. The increasing rate of velocity or \(1/\text{GSN}\) is almost the same in the two intervals. In the interval II, the step length equals the desired step length and keeps unchanged, which shows step length is dominant to get the maximum \(P\). This is resulted from the fact that the rate of \(e^{-\frac{|\sigma_0-\sigma|}{\sigma_0}}\) changing for the increasing slope decrease the quickest among those criteria contained in \(P\). To keep \(P\) max, the step length is maintained to be desired step length. In the entire intervals, \(1/\text{GSN}\), velocity and maximum \(P\) keep increasing. It shows the walker performs better for the increasing slope.

2) CoM model: What CoM model differs from HIP model lies in that it works in walker through extension force like gravity instead of torque. And this force is nonlinear when it is projected onto the walker’s walking plane. Based on the same procedure as HIP model, we also get the optimal spring stiffness curve and parameter variations to slope. As shown in Fig. 10, the curve of optimal spring stiffness with slope is irregular, which is mainly caused by the non-linear spring force. However, we still find the curve \(P\) is increasing with the increasing of slope shown in Fig. 11.

![Fig. 10: The curve of optimal spring stiffness with slope of CoM model.](image1)

![Fig. 11: The effect of optimal spring stiffness of CoM model.](image2)

Fig. 10: The curve of optimal spring stiffness with slope of CoM model.

Fig. 11: The effect of optimal spring stiffness of CoM model.
V. THE MODELS COMPARISON

From the above discussion and simulation, we have found the optimal spring stiffness and maximum $P$ to any slope in HIP and CoM models. Now we compare velocity, step length, disturbance rejection ability and overall walking performance in these models. HIP and CoM models have been applied to the optimal spring stiffness. The comparison results are shown in Fig. 12. From them, we can see the HIP and CoM model with optimal spring stiffness perform much better in each index. The velocity becomes much larger, the obtained step length is closer to the desired step length, and the disturbance rejection ability is greater than pure PASSIVE model. And the sharper the slope is, the more obvious the advantages become. There are other sides we should notice. From Fig. 12, we can see the step length can only be turned down by adding spring. This is because the elastic works as energy cache that can only store energy. Comparing with HIP model, CoM model has much greater disturbance rejection ability with almost consistent velocity and step length.

However, our measurement to overall walking performance doesn’t contain the energy efficiency. In fact, the energy efficiency is also important to construct a capable robot. In passive dynamic walking, the definition of energy efficiency is various. Here, we think the energy efficiency is the amount of energy expenditure per distance traveled per weight of the walker. That is:

$$c_t = \frac{Used \ Energy}{Distance \ Traveled \times Weight} \quad (11)$$

To the three models, the only energy source is the gravity. That is to say, the energy complement with a same step length depends on the slope. But to a limit cycle walking, the energy compensation is equal to the energy expenditure. Therefore, we have, $c_t = \frac{mgh}{mgl} = \frac{1}{\sin(\gamma)}$. If we consider the energy efficiency to the overall walking performance, we can use the index $P/\sin(\gamma)$ to evaluate its walking performance. Shown in Fig. 13, the PASSIVE curve has apex at $\gamma = 0.0591$, which illustrates the Passive prototype performs the best at this slope. The HIP and CoM curve increase with the increasing of slope.

VI. CONCLUSIONS

In this paper, we study the effect of spring stiffness on the overall walking performance of passive dynamic walker. First, we construct a measurement index $P$ to evaluate overall walking performance, which includes velocity, step length and gait sensitivity norm. Then, we apply this index to three models to evaluate their overall walking performance and find the optimal spring stiffness. Now, we conclude that:

- Elastic element in the hip can improve the overall walking performance, including the increase of walking velocity, disturbance rejection ability, and adjusting step length.
- The optimal extension spring placed on the leg mass center can improve walker’s disturbance rejection ability much better than hip torsion spring.

- The index $P$ can represent walker’s overall walking performance well. The method we use to obtain the optimal spring stiffness is very efficient.

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