Efficient Network Coding for Wireless Broadcasting

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Abstract—It has been shown in the literature that network coding can improve the transmission efficiency of wireless broadcasting as compared to traditional ARQ schemes. In this paper, we propose an improved network coding scheme that can asymptotically achieve the theoretical lower bound on transmission overhead for a sufficiently large number of information blocks. The proposed scheme makes use of an index allocation algorithm that distributes information blocks that have been erased during transmission into a minimum number of encoding sets, where each set represents the erased blocks to be jointly network encoded and retransmitted. Numerical results show that the proposed scheme enables higher transmission efficiencies than traditional ARQ, and previously proposed networks coding schemes for wireless broadcasting.

I. INTRODUCTION

Wireless digital broadcasting applications such as digital audio broadcast (DAB) and digital video broadcast (DVB) are becoming increasingly popular since the digital format allows for quality improvements as compared to traditional analogue broadcast. In a typical DAB/DVB scenario, a base station (BS) broadcasts information to a population of user terminals through wireless broadcasting channels. In common terminology a user terminal is referred to as user equipment (UE). With the digital format, error control strategies can be introduced to significantly improve the reliability of the broadcast at both physical and at higher network layers (data link, network and transport). The broadcast is based on packet (block) transmission, where packets are subject to channel noise, fading and interference at physical layer. Channel error correction may not be perfect; however, assuming perfect error detection at higher layers a received packet at an UE is either error-free or discarded as erroneous. Consequently, the higher-layer broadcast transmission from the BS to the set of UEs can be modeled as broadcast packet-erasure (or block-erasure) channel as shown in Fig. 1.

The block-erasure channel has been investigated in [1]–[3], where block-wise maximum-distance separable (MDS) codes are shown to be optimal in terms of error probability. Considering a block-wise MDS code over a block-wise alphabet, the Singleton bound is satisfied with equality. Therefore coding schemes applied across a sequence of broadcast information packets are typically considered for error control.

The prevailing approach to packet-level coding is automatic-repeat-request (ARQ) error control protocols, e.g., [4]. In a broadcasting scenario, a retransmission is requested by any UE with an erased packet; a strategy that becomes increasingly inefficient as the number of UEs in the broadcast increases, both in terms of feedback and in terms of retransmissions.

To improve the system efficiency, the use of network coding [5], [6], [7], [8] during the retransmission phase has been proposed for wireless broadcasting [9], [10], [11]. We show that the proposed network coding scheme asymptotically achieves the lower bound on transmission overhead as the number of transmitted blocks required for retransmission grows large. Numerical results are presented and discussed, demonstrating the performance improvements obtained by network coding, while a lower bound on the transmission overhead is developed in [10], [11]. A specific network coding scheme for two UEs is further proposed in [10], [11].

As our main contribution, we propose a more efficient network coding scheme for wireless broadcasting, applicable to the case of an arbitrary number of UEs. The proposed scheme offers performance improvements over traditional ARQ schemes and the network coding schemes proposed in [9], [10], [11]. We show that the proposed network coding scheme asymptotically achieves the lower bound on transmission overhead as the number of information blocks grows large. Numerical results are presented and discussed, demonstrating the performance improvements obtained by our proposed network coding scheme.

The organization of the paper is as follows. The system model is defined in Section II and the proposed network coding scheme is described in Section III. The efficiency of the proposed approach is analyzed in Section IV, and numerical results are presented and discussed in Section V.
II. System description

In a broadcast session a set of \( N \) information blocks \( \mathbf{I}_i \), \( i = 1, 2, ..., N \) is to be broadcast from a BS to a set of \( M \geq 2 \) UEs. We assume the \( M \) BS-to-UE block-erasure channels to be independent with block-erasure probabilities \( p_i \), \( i = 1, 2, ..., M \), respectively. The transmission process is divided into two phases: the information transmission phase and the retransmission phase. In the information transmission phase, the BS broadcasts \( N \) information blocks, and during the transmission, some blocks are lost over the respective BS-to-UE block-erasure channels. Each UE subsequently feeds back a packet with indices of the erased blocks, where we assume orthogonal and error-free feedback channels. A corresponding error matrix \( \mathbf{E} \) is generated at the BS to record the block-erasure status reported by the UEs. The size of the error matrix is \( M \times N \), where \( e_{i,j} = 1 \) if the \( j \)-th block of UE \( i \) is erased; otherwise, \( e_{i,j} = 0 \).

In the retransmission phase, the set of erased blocks is divided into subsets such that at most one erased block per UE is in any particular subset. The erased blocks in a subset are then encoded with a binary network code (modulo-2 addition)\(^1\) into one encoded block for retransmission. In contrast, when traditional ARQ is applied, each individual erased block for any UE will be retransmitted separately. To exemplify the differences, consider the following scenario. The error matrix is in any particular subset. The erased blocks in a subset are sorted in descending order, according to the respective channel conditions. In the original broadcast, correctly receiving \( \hat{e}_m \) blocks, \( \hat{e}_m \leq N \), information blocks \( \hat{e}_m \) can be easily retrieved. The retransmission process is summarized by the following three steps, while a more formal description is given in Algorithm 1.

1) Initialization: Determine \( \hat{n} \) and \( \hat{i} \), and denote the corresponding erased blocks as \( \mathbf{I}_{m_1}, \mathbf{I}_{m_2}, ..., \mathbf{I}_{m_{\hat{n}}} \). Initialize \( k = \hat{n} \) and \( k \) coding sets as \( C_k = \{m_{\ell}\}, \ell = 1, 2, ..., k \).
2) Index Allocation: In this step, the erased blocks of the remaining UEs are allocated into the encoding sets \( C_{\ell}, \ell = 1, 2, ..., k \) if possible. Obviously, if all erased blocks can be allocated into the \( \hat{n} \) encoding sets (and encoded into \( \hat{n} \) codewords), the lower bound is achieved if the first round of retransmissions is successful for all UEs. Otherwise, the algorithm attempts to minimize the number of encoding sets required. For each non-allocated UE, the algorithm attempts to minimize the number of encoding sets required. For each non-allocated UE, the remaining erased blocks are sorted in descending order, according to the number of encoding sets the particular erased block can be allocated in subject to the column grouping constraints. Starting from the top, the remaining erased blocks are allocated to an eligible encoding set, subject to the column grouping constraints. If there are no eligible encoding sets available for a particular block, a new encoding set is generated.

To measure the system efficiency, we define a normalized overhead \( \eta \), evaluated as

\[
\eta = \frac{X}{N}.
\]

Here \( X \) denotes the number of blocks sent from the BS until termination of the broadcast session. We can also interpret \( \eta^{-1} \) as the average throughput (or code rate) of the broadcast network coding scheme.

III. Proposed retransmission scheme

We assume that \( n_i \) blocks are erased on the BS-to-UE link, \( i = 1, 2, ..., M \), during the information transmission phase. Let \( \hat{n} = \max_i n_i \) and \( \hat{i} = \arg \max_i n_i \). It follows that if \( \hat{n} > 0 \) then a new round of retransmissions is required. Since UE \( \hat{i} \) must receive at least \( \hat{n} \) blocks in order to recover all \( N \) information blocks, \( \hat{n} \) is a lower bound on the total number of required retransmissions before the transmission condition is satisfied. The proposed network coding scheme is partly motivated by this observation.

As part of the coding scheme, we introduce the concept of column groupings of the error matrix, which inherently enforces a coding constraint. The columns of the error matrix are grouped into a minimum number, \( k \geq \hat{n} > 0 \), of sub-matrices such that each row in a column grouping has at most one \( 1 \). Since each column in the error matrix corresponds to an information block, the column groupings correspond to encoding sets, \( C_k, \ell = 1, 2, ..., k \), containing the indices of the respective information blocks in a grouping. The information blocks whose indices are in the same set will be jointly encoded into a block for retransmission. For example, if \( C_1 = \{1, 2\} \), then the encoded block is \( \mathbf{I}_1 \oplus \mathbf{I}_2 \). Since each UE has at most one erased information block within an encoding set, erased blocks can be easily retrieved. The retransmission process is summarized by the following three steps, while a more formal description is given in Algorithm 1.

1) Initialization: Determine \( \hat{n} \) and \( \hat{i} \), and denote the corresponding erased blocks as \( \mathbf{I}_{m_1}, \mathbf{I}_{m_2}, ..., \mathbf{I}_{m_{\hat{n}}} \). Initialize \( k = \hat{n} \) and \( k \) coding sets as \( C_k = \{m_{\ell}\}, \ell = 1, 2, ..., k \).
2) Index Allocation: In this step, the erased blocks of the remaining UEs are allocated into the encoding sets \( C_{\ell}, \ell = 1, 2, ..., k \) if possible. Obviously, if all erased blocks can be allocated into the \( \hat{n} \) encoding sets (and encoded into \( \hat{n} \) codewords), the lower bound is achieved if the first round of retransmissions is successful for all UEs. Otherwise, the algorithm attempts to minimize the number of encoding sets required. For each non-allocated UE, the algorithm attempts to minimize the number of encoding sets required. For each non-allocated UE, the remaining erased blocks are sorted in descending order, according to the number of encoding sets the particular erased block can be allocated in subject to the column grouping constraints. Starting from the top, the remaining erased blocks are allocated to an eligible encoding set, subject to the column grouping constraints. If there are no eligible encoding sets available for a particular block, a new encoding set is generated.

\(^1\)Due to complexity considerations, we only consider network coding schemes based on GF(2) (binary codes).
3) **Retransmission:** All blocks assigned to a particular encoding set are jointly network encoded through modulo-2 addition, and transmitted.

The encoding constraint enforced by the column groupings rule simplifies the decoding process and minimizes delay with no loss of throughput performance. An UE retrieves an erased block for each received retransmitted block through a simple modulo-2 addition. Without the encoding constraints, erased blocks may be encoded across multiple code blocks, leading to increased delay and complexity in the decoding process. The operation of the network coding scheme is illustrated with the example below used in [9]. The relevant error matrix is as follows,

\[
E = \begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\
\end{pmatrix}
\]

Using the proposed encoding approach, the encoding sets are \(C_1 = \{1\}, C_2 = \{2, 3, 4\}, C_3 = \{5, 6\}, C_4 = \{8, 10\}\) and \(C_5 = \{9\}\), leading to five encoded blocks, \(I_1, I_2 \oplus I_3 \oplus I_4, I_5 \oplus I_6, I_7 \oplus I_{10}\) and \(I_8\) for retransmission. Under the assumption that the retransmissions are received correctly at all UEs, the number of retransmissions required is five, whereas six retransmissions are required for the scheme in [9], and nine retransmissions are required for traditional ARQ. Of the three approaches considered in this example, only our proposed scheme meets the lower bound on the number of required retransmissions.

Since retransmitted blocks may be erased during transmission, additional rounds of retransmissions may be required. Following a round of retransmissions, the UEs therefore repeat the process of feeding back indices of remaining erased blocks. In turn, the BS updates the error matrix and determines a new set of encoded packets for retransmission. New retransmission rounds are initiated until the termination condition is met, or until a predetermined maximum number of retransmission rounds is reached.

**IV. PERFORMANCE ANALYSIS**

In this section we show that our proposed scheme with an arbitrary number of UEs is asymptotically able to meet the lower bound on transmission overhead under the assumption that no blocks are erased during the retransmission phase. We begin with the special case where \(M = 2\), before generalizing to \(M > 2\).

**Proposition 1:** Consider a BS broadcast of \(N\) information blocks to UE\(_1\) and UE\(_2\), using the network coding scheme in Algorithm 1. Assume that the BS-to-UE links are block-erasure channels during the information transmission phase, but error-free during the retransmission phase. The indices of the erased blocks of UE\(_1\) are collected in the set \(L_1 = \{l_{11}, l_{12}, ..., l_{1n_1}\}\), and the indices of the erased blocks of UE\(_2\) are collected in the set \(L_2 = \{l_{21}, l_{22}, ..., l_{2n_2}\}\). Without loss of generality, assume further that \(n_1 \geq n_2\). In this case it is sufficient to retransmit \(n_1\) network coded blocks for both UE\(_1\) and UE\(_2\) to be able to recover all information blocks.

**Proof:** As illustrated in Fig. 2 there are three cases to consider, depending on the potential overlap between the elements of \(L_1\) and \(L_2\).

**Case 1:** \(L_1 \cap L_2 = \emptyset\). Retransmit \(I_{l_{11}} \oplus I_{l_{12}}, i = 1, 2, ..., n_2\) and \(I_{l_{11}} (n_2 + 1 \leq i \leq n_1)\).

**Case 2:** \(L_2 \subseteq L_1\), retransmit \(I_{l_{1i}}, i \in L_2\) without coding.

**Case 3:** \(\{L_1 \cap L_2 \neq \emptyset\} \cap \{L_2 \cup L_1 \neq L_1\}\). In this case, the indices can be divided into two new sets as follows.

- **Case A:** Set \(L_3 = L_1 \cap L_2\). This is the same as Case 2.
- **Case B:** Assume there are \(n_{L_3}\) indices in \(L_3\), then \(n_{L_3}\) blocks

<table>
<thead>
<tr>
<th>Algorithm 1 Encoding Algorithm</th>
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| **Initialization:** Set \(T = \{j | \exists E_{i,j} = 1, i = 1, 2, ..., M\}\) (indices of packets to be retransmitted). \(A = \{1, 2, ..., N\}\). Determine \(i, i = \arg \max_{a \in A} \left(\sum_{j=1}^{N} E_{a,j}\right)\) (the index of the UE which erased most number of packets). Set \(\hat{n} = \sum_{j=1}^{N} E_{i,j}\) and \(k = \hat{n}\). (The number of packets erased by UE\(_i\)). Set \(J = \{j | E_{i,j} = 1\}\) (indices of packets erased by UE\(_i\)). Create \(k\) different encoding sets \(C_\ell, \ell = 1, 2, ..., k\), allocating one distinct element from \(J\) to each set. \(T = T - J\). **Index Allocation:** \(\begin{array}{l}
\text{repeat} \\
\text{Assume } T = \{t_1, t_2, ..., t_m\}. \text{ Set } x_b, b = 1, ..., m \text{ to denote the number of choices for } I_{b_k} \text{ to be allocated into initial coding sets. } x_1 = x_2 = ... = x_m = 0. \\
\text{for } b = 1 \text{ to } m \text{ do} \\
\text{for } \ell = 1 \text{ to } k \text{ do} \\
\text{flag} = 0 \text{ (use a flag to denote whether the encoding constraint is violated or not. If flag == 1, the encoding constraint is violated. Otherwise, not). }
\text{for } r = 1 \text{ to } M \text{ do} \\
\text{if } \sum_{q \in C_{\ell,b}} E_{r,q} \geq 2 \text{ then} \\
\text{flag} = 1. \\
\text{end if} \\
\text{end for} \\
\text{if flag} = 0 \text{ then} \\
\text{ Set } x_b = x_b + 1. \text{ Sent the index } \ell\text{ to List}_{b_k}. \\
\text{end if} \\
\text{end for} \\
\text{end for} \\
\text{end if} \\
c = \arg \min x_b, \text{ which denotes the index of the block which has least choices to be allocated into the initial encoding sets.} \\
\text{if } x_c = 0 \text{ then} \\
\text{ Set } k = k + 1, C_k = \{t_c\}. \text{ C_d = C_d \cup \ell_c}\text{.} \\
\text{else} \\
\text{ Choose any index from List}_{c}, \text{ denotes as } d. \text{ Then, } C_d = C_d \cup \ell_c. \\
\text{end if} \\
\text{end if} \\
\text{until } T = \emptyset. \\
\text{Encoding: Jointly network encode blocks assigned to encoding set } C_\ell, \ell = 1, 2, ..., k, \text{ through modulo-2 addition (XOR), and transmits the } k\text{ encoded blocks.} 
\end{array}\)
will be retransmitted.

- **Case B:** Set $L_o = (L_2 \cup L_1) - L_c$. This is the same as Case 1. Since $n_1 \geq n_2$, we have $n_1 - n_L \geq n_2 - n_L$.

  Then, $n_1 - n_L$, encoded blocks will be retransmitted.

The total number of retransmissions is the sum of Case A and B, leading to $n_1$ distinct blocks being retransmitted. Thus, in all three cases, the number of retransmitted blocks is $n_1$. Q.E.D.

![Venn diagram of L1 and L2](image)

Fig. 2. The Venn diagram of $L_1$ and $L_2$.

Note that for ARQ the index set of retransmitted blocks are $L_1 \cup L_2$. Since $L_1 \subseteq (L_1 \cup L_2)$, the number of retransmission blocks will be equal to or larger than our proposed approach. Equality can be achieved if and only if $L_2 \subseteq L_1$ (Case 2).

Proposition 2: Consider a BS broadcast of $N$ information blocks to $M \geq 2$, using the network coding scheme in Algorithm 1. Assume that the BS-to-UE links are independent block-erasure channels with erasure probabilities $p_i$, $i = 1, 2, \ldots, M$ during both transmission phases. For sufficiently large $N$ the normalized overhead is then

$$\eta = \frac{1}{1 - \max_i \{p_i\}}. \quad (8)$$

Proof: Without loss of generality, we assume that $p_1 \geq p_2 \geq \ldots \geq p_M$. The number of blocks erased at UE$_i$ is denoted by $L_i$. For sufficiently large $N$, $L_1 \geq L_2, \ldots, \geq L_M$ and $L_i = Np_i$ on average. We show by induction that the blocks erased at the remaining UEs can be allocated to the $L_1$ encoding sets required for retransmissions of the blocks erased at UE$_1$ without violating the encoding constraints. From Proposition 1, we know that the statement is true when $M = 2$. We now assume that the statement is true for $M - 1$, and show that it is also true for $M$.

Consider the first $M - 1$ UEs out of $M$. From our induction assumption, the erased blocks at the first $M - 1$ UEs can be allocated to the $L_1$ encoding sets required for UE$_1$. At UE$_M$ there are $L_M$ blocks erased on average. With the assumption of independent block-erasure channels, the number of blocks erased at UE$_M$ which are also erased at one or more other UEs is $T_1 = L_M - L_M \prod_{i=1}^{M-1} (1 - p_i)$. These blocks are already allocated into the $L_1$ encoding sets. The remaining $T_2 = L_M \prod_{i=1}^{M-1} (1 - p_i)$ blocks are only erased at UE$_M$. These $T_2$ erased blocks can only be encoded with $T_3 = L_1 - T_1$ blocks without violating the coding rule. Since $T_3 - T_2 = L_1 - L_M \geq 0$, the remaining erased blocks of UE$_M$ can be encoded jointly with the erased blocks of UE$_1$. Thus, the statement is true for $M$.

It follows that when $N$ is sufficiently large, all erased blocks can be allocated to the encoding sets of UE$_1$ without violating the encoding constraints. The normalized overhead for the broadcast is then the same as the normalized overhead for UE$_1$, and thus

$$\eta = \frac{1}{1 - p_1}. \quad (9)$$

Clearly, the same reasoning applies for the case for any other UE suffering the worst block-erasure channel. Q.E.D.

For comparison, we consider the normalized overhead for traditional ARQ. When $N$ is sufficiently large, we have the overhead as [11]

$$\eta_{\text{ARQ}} = \sum_{i_1, i_2, \ldots, i_M} \left(\frac{\sum_{j=1}^{M-1} i_j - 1}{1 - \prod_{j=1}^{M} p_j^{i_j}}\right), \quad (10)$$

where $i_j \in \{0, 1\}$, $j = 1, 2, \ldots, M$ and $\exists i_j \neq 0$. As expected $\eta_{\text{ARQ}} > \eta$. 

From Proposition 2 we conclude that our proposed scheme for $M = 2$ achieves the theoretical lower bound [10]. In Proposition 3 below we generalize this result to arbitrary $M \geq 2$. 

**Proposition 3**
V. NUMERICAL RESULTS

The performance of the proposed scheme is contrasted against the performance of traditional ARQ and the network coding scheme in [9]. We specifically consider the impact of $N$, $M$ and $p_i$ on the normalized overhead $\eta$. Two scenarios are considered separately: (1) identical erasure rate links; (2) erasure rate on link 1, $p_1$, and identical erasure rate $p_i = p$, $i = 2, 3, \ldots, M$ on all other links.

A. Equal Erasure Probability

1) The impact of $N$: In Fig. 3 the normalized overhead $\eta$ is shown as a function of $N$ for the case of $M = 5$ and $p = 0.05$. The results confirm the analysis in Section IV that the proposed network coding scheme approaches the theoretical lower bound asymptotically when $N$ is sufficiently large. At $N = 1800$ the gap to the lower bound is less than 0.001. When the erasure probabilities are the same for all links, it is likely that the assumptions of Proportion 3 is violated, and thus the theoretical lower bound may only be achieved in the limit as $N$ grows. From Fig. 3, we also observe that the proposed approach performs better than the scheme proposed in [9] for all $N$. The performance gap between our proposed approach and the scheme in [9] decreases with increasing $N$, since it was proven to approach the theoretical lower bound when $N$ is sufficiently large [9].

2) The impact of $M$: In Fig. 4 the normalized overhead $\eta$ is shown as a function of the number of UEs for the case of $N = 100$ and $p = 0.1$. With increasing $M$ the normalized overhead increases slightly for the proposed approach while the rate of increase is considerably larger for traditional ARQ. For $M = 9$ ARQ requires more than 70% higher overhead as compared to the proposed approach. The advantage relative to the scheme suggested in [9] are roughly 9% for $M = 9$. The gap between the proposed scheme and the theoretical lower bound is due to the relatively small $N$ in this example. A larger $N$ will cause the gap to decrease as we show in Fig. 3.

3) The impact of erasure probabilities: In Fig. 5 the normalized overhead $\eta$ is shown as a function of the erasure probability, which is identical for all UEs. We consider the case of $M = 5$ and $N = 100$. Conclusions similar to those in the previous sections can be drawn.

B. Unequal Erasure Probability

In Figs. 6, 7 and 8 the normalized overhead is shown as a function of $N$, $M$, and $p_i$ for cases with unequal erasure probabilities. Similar to the cases with equal erasure probabilities we observe the proposed scheme enjoys substantially better performance as compared to ARQ and the scheme proposed in [9]. As before the proposed method approaches the lower bound as $N$ increases. When $N = 350$ the gap between our approach and the lower bound is less than 0.001, which is a smaller $N$ than for the case of equal erasure probabilities. This behavior is due to the erasure probability of UE1 being considerably larger than for other UE links. It is therefore more likely that erased blocks of other UEs can be allocated to the encoding sets formed by UE1.

VI. CONCLUSIONS

In this paper, we have investigated efficient network coding schemes for wireless broadcasting. In our proposed scheme we use an index allocation algorithm to form encoding sets, which effectively allocate erased information blocks of all UEs to be joint network encoded for retransmission. Our scheme aims to minimize the number of encoding sets, which in turn also minimizes the number of retransmission and subsequently the transmission overhead. Theoretical analysis shows that our scheme can asymptotically achieve the theoretical lower bound when the number of information blocks are sufficiently
large. Numerical results are presented to demonstrate that our scheme offers better performance for wireless broadcasting than traditional ARQ and previously suggested schemes based on network coding.

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