On changing points of mean residual life and failure rate function for some generalized Weibull distributions

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Abstract

The failure rate function and mean residual life function are two important characteristics in reliability analysis. Although many papers have studied distributions with bathtub-shaped failure rate and their properties, few have focused on the underlying associations between the mean residual life and failure rate function of these distributions, especially with respect to their changing points. It is known that the change point for mean residual life can be much earlier than that of failure rate function. In fact, the failure rate function should be flat for a long period of time for a distribution to be useful in practice. When the difference between the change points is large, the flat portion tends to be longer. This paper investigates the change points and focuses on the difference of the changing points. The exponentiated Weibull, a modified Weibull, and an extended Weibull distribution, all with bathtub-shaped failure rate function will be used. Some other issues related to the flatness of the bathtub curve are discussed.

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1. Introduction

Let $F(t)$ and $f(t)$ be the cumulative distribution function (cdf) and probability density function (pdf) of a random lifetime distribution, respectively. The reliability function and the FR function are then:

$$R(t) = 1 - F(t), \quad h(t) = f(t)/R(t).$$

In case of continuous distributions, the following simple relationship exists

$$R(t) = \exp\left[-\int_0^t h(x) dx\right].$$

The MRL function $\mu(t)$ is defined as the expected remaining lifetime given that the product has survived to time $t$. In other words, $\mu(t)$ can be expressed as:

$$\mu(t) = \begin{cases} E[X - t | X \geq t] = \int_t^\infty R(x)dx/R(t), & \text{if } R(t) > 0, \\ 0, & \text{if } R(t) = 0 \end{cases}$$

Therefore, $\mu(0)$ is the mean time to failure. It is also clear that, as discussed in Ref. [1], there exists a simple relationship between MRL function $\mu(t)$ and $h(t)$ given as

$$h(t) = \frac{\mu'(t) + 1}{\mu(t)}.$$  \hspace{1cm} (2)

It can be shown that $\mu'(t) \equiv -1$. Both FR and MRL functions can uniquely determine the underlying lifetime distribution as pointed out in Refs. [2,3]:

$$R(t) = \exp\left[-\int_0^t h(x) dx\right] = \frac{\mu(0)}{\mu(t)} \exp\left[-\int_0^t \frac{1}{\mu(x)} dx\right].$$  \hspace{1cm} (3)

There also exists relationship between FR and MRL as:

IFR(DFR) $\Rightarrow$ DMRL(IMRL)

However, it is known that the products usually experience three main life phases, i.e.

1. The infant mortality region with a high failure rate when the product is newly introduced.
2. The constant failure rate region when the product is stable.
3. The wear-out region when the failure rate is significantly increased and this occurs towards the end of the life cycle time of a product.
Such product is said to exhibit a bathtub-shaped failure rate (BFR) property. BFR distributions are important in reliability applications and study. Rajarsi and Rajarsi [4] gave a systematic account on BFR definitions, model constructing techniques, distributions and their properties. Lai et al. [5] provided a detailed survey on some recent models.

Relationship between the FR function and mean MRL function of BFR distribution has been investigated by several authors. Gupta and Akman [6] showed that the MRL of a component is an upside-down bathtub shape on condition that FR is bathtub-shaped and \( h(0) > 1/\mu \), where \( \mu \) is the mean time to failure. MRL of a BFR distribution is a decreasing function if \( h(0) \leq 1/\mu \). Mi [7] also concluded that, assuming the model has a differentiable bathtub-shaped FR function with a unique change point \( 0 < b^* < +\infty \), the MRL function \( \mu(t) \) has an upside-down bathtub shape with a unique changing point \( 0 < t^* < b^* \).


Furthermore, the change points of the two functions would be critical for the application of BFR distributions. There are some papers dealing with the estimations of the change points of FR and MRL as well. Chen et al. [12] provided constructing methods of approximate interval estimation of the change point for FR function. Guess et al. [13] discussed the FR and MRL with trend changes and analyzed the change points of FR and MRL of the IDB distribution proposed by Hjorth [14].


Although FR and MRL are studied in many papers, few deal with the change point in association with the flatness of bathtub curve in BFR distributions. In this paper, we analyze the FR and MRL functions and their change points for some BFR distributions. Section 2 investigates the change points of MRL and FR functions of exponentiated Weibull family [17], the Weibull extension distribution [18], and the new modified Weibull distribution [19]. Section 3 focuses on the flatness of bathtub curve of BFR models. We investigate the flat portion of bathtub curve in terms of the parameter selections and the underlying relationship with the change points of FR and MRL.

2. FR and MRL for some Weibull BFR distributions

In this section, FR and MRL functions and especially their critical change points of some BFR models are studied. These models studied include the exponentiated Weibull family [17], the Weibull extension model [18], and the new modified Weibull model [19]. Note that the focus is on the comparing the change points as, for example, in burn-in, the optimum burn-in time that maximizes the residual life could be much shorter than the time at which the failure rate reaches the minimum.

2.1. Exponentiated Weibull family by Mudholkar and Srivastava [17]

The distribution has this form:

\[
F(t) = \left[1 - \exp\left(-\left(t/\alpha\right)^\theta\right)\right]^{\beta}, \quad t > 0, \alpha, \theta, \sigma > 0.
\]  

(4)

The density function, failure rate function and MRL function are as follows

\[
f(t) = \frac{\alpha \theta}{\sigma} \left[1 - \exp\left(-\left(t/\alpha\right)^\theta\right)\right]^{\beta-1} \exp\left(-\left(t/\alpha\right)^\theta\right) \left(t/\alpha\right)^{\theta-1},
\]  

(5)

\[
h(t) = \frac{\alpha \theta \left[1 - \exp\left(-\left(t/\alpha\right)^\theta\right)\right]^{\beta-1} \exp\left(-\left(t/\alpha\right)^\theta\right) \left(t/\alpha\right)^{\theta-1}}{\sigma \left[1 - \left(1 - \exp\left(-\left(t/\alpha\right)^\theta\right)\right)^\beta\right]},
\]  

(6)

\[
\mu(t) = \int_0^t \frac{\left[1 - \left(1 - \exp\left(-\left(s/\alpha\right)^\theta\right)\right)^\beta\right]}{1 - \left[1 - \exp\left(-\left(t/\alpha\right)^\theta\right)\right]} ds.
\]  

(7)

When \( \alpha > 1 \) and \( \alpha \theta < 1 \), the model exhibits bathtub-shaped failure rate function and \( \lim_{t \to \infty} h(t) = +\infty \). Hence, the distribution also has an upside-down bathtub MRL with a unique change point [6,7].

Denote the exponentiated Weibull model with \( \sigma = 1 \), which has the cdf given as

\[
F(t) = \left[1 - \exp\left(-t^{\theta}\right)\right]^\beta,
\]  

as the standard exponentiated Weibull family model. It is noted that the change points of FR and MRL function of
the exponentiated Weibull model are proportional to the values of the scale parameter \(\alpha\). Therefore, the scale parameter \(\alpha\) will not determine the shapes of the FR and MRL curves and the change points are proportional to the change points of standard exponentiated Weibull family. In order to study the change point, take the derivative of the failure rate function of the standard exponentiated Weibull model when \(\alpha > 1\) and \(\alpha \theta < 1\). Denote \(y = 1 - \exp(-t^\alpha)\). We have

\[
\frac{dh(t)}{dt} = \frac{y^{\theta-2}a^{-2}\exp(-t^\alpha)}{(1-y^\theta)^2}\{\alpha(\theta-1)t^\alpha\exp(-t^\alpha) + (\alpha-1)y + \alpha y^\theta t^\alpha\exp(-t^\alpha) + \alpha y^{\theta+1}t^\alpha - (\alpha-1)y^{\theta+1} - \alpha y^\alpha\}
\]

After some simplifications, solving the equation of \(\frac{dh(t)}{dt} = 0\) becomes equivalent to the solution to the equation of

\[
(\alpha-1)y(1-y^\theta) + \alpha \log(1-y)[1 + \theta y - \theta - y^\theta] = 0,
\]

(8)

The change point of FR can be obtained and

\[
b^* = \left[\log(1-y_1)\right]^{1/\alpha},
\]

(9)

where \(y_1\) is the solution to Eq. (8).

Similarly, taking the derivatives of MRL function we can find the change point \(t^*\). The change point can be obtained using the following equation:

\[
\frac{d}{dt} \left[\log(1-y)\right] = \left[1 - (1-e^{-t^\alpha})^\theta\right]dx - (1-y^\theta)^2 = 0
\]

(10)

and \(t^* = \left[-\log(1-y_2)\right]^{1/\alpha}\), where \(y_2\) is the solution to the above equation.

Although the change points of FR and MRL functions do not have close form, they are unique and can be obtained numerically. Fig. 1 shows some typical curves of the failure rate and mean residual life function. Note that the difference between the change points can be very large for this distribution.

2.2. Weibull extension model by Xie et al. [18]

The cdf and pdf functions of the Weibull extension distribution have the following forms:

\[
F(t) = 1 - R(t) = 1 - \exp\{b\alpha(1 - e^{t/\alpha})\},
\]

(11)

\[
f(t) = \lambda b t^{(\theta-1)} \exp(\theta t/\alpha) + \lambda \alpha(1 - e^{t/\alpha}),
\]

(12)

where \(\alpha > 0\) is the scale parameter and \(\beta, \lambda > 0\) are shape parameter. Re-parameter the parameters as \(\alpha, \beta\) and let \(\eta = \lambda \alpha > 0\), we get

\[
F(t) = 1 - \exp\{\eta(1 - e^{t/\alpha})\},
\]

(13)

\[
h(t) = \frac{\eta \beta}{\alpha} t^{(\beta-1)} \exp(t/\alpha)^\beta,
\]

(14)

\[
\mu(t) = \exp(\eta e^{t/\alpha}) \int_t^\infty [\exp(\eta e^{x/\alpha})]dx.
\]

(15)

It is shown in Ref. [18] that, when \(\beta < 1\), FR function has bathtub shape; FR is increasing as \(\beta \geq 1\). As \(\lim_{t \to 0^-} \times h(t) = +\infty > 1/\mu\), the distribution has an upside-down bathtub MRL with unique change point.

Since \(\alpha\) is a scale parameter, it can be easily shown that the change points of FR and MRL of the Weibull extension model are positively proportional to \(\alpha\). The minimum point of the failure rate could be obtained from the derivative function of FR, which is

\[
h^*(t) = \frac{\eta \beta}{\alpha} (t/\alpha)^{\beta-2} \exp(t/\alpha)^\beta + (\beta - 1) \exp\{t/\alpha\}^\beta.
\]

(16)

Hence, the change point of failure rate is at

\[
b^* = \alpha(1/\beta - 1)^{1/\beta},
\]

and

\[
h(b^*) = \frac{\eta \beta}{\alpha} (1/\beta - 1)^{1-1/\beta} \exp(1/\beta - 1).
\]

(17)

The change point is proportional to \(\alpha\) and inversely related to \(\beta\). It has no relationship with the value of \(\eta\).

On the other hand, change point of MRL is \(\alpha\) times the change point of standard Weibull extension model,

\[
\mu(t) = \exp(\eta e^{t/\alpha}) \int_t^\infty [\exp(\eta e^{x/\alpha})]dx,
\]

i.e. when \(\alpha = 1\) which is also the form of Chen’s model [12].

The maximum of MRL point of standard Weibull extension model can be solved by taking the derivatives of MRL function. It reduces to the following equation:

\[
\eta \beta^{\beta-1} \exp(t/\alpha + \eta t/\alpha) \int_t^\infty [\exp(\eta e^{x/\alpha})]dx - 1 = 0.
\]

(18)

Again, the changing points of FR and MRL functions do not have close form, but it can be obtained numerically. Fig. 2 shows some typical curves of the failure rate and mean residual life function.
2.3. Modified Weibull distribution (MWD) by Lai et al. [19]

A modified Weibull distribution by Lai et al. [19] is a three-parameter model with the following characteristic functions:

\[ R(t) = \exp(-at^b e^{\lambda t}), \]  
\[ f(t) = a(b + \lambda t)^{b-1} e^{\lambda t} \exp(-at^b e^{\lambda t}), \]  
\[ h(t) = a(b + \lambda t)^{b-1} e^{\lambda t}, \]
\[ \mu(t) = \exp(at^b e^{\lambda t}) \int_0^t \exp(at^b e^{\lambda t}) dt, \]

where \( a > 0, b \geq 0 \) and \( \lambda \geq 0 \). It is shown in Ref. [19] that, when \( 0 < b < 1 \) and \( \lambda > 0 \), the distribution can be used to estimate bathtub-shaped failure rate data. When \( \lambda = 0 \), the modified Weibull distribution has the form of Weibull distribution.

When \( 0 < b < 1 \) and \( \lambda > 0 \), the minimum failure rate point could be easily obtained as

\[ b^* = \frac{\sqrt{b} - b}{\lambda}, \]

and

\[ h(b^*) = a \sqrt{b} e^{\sqrt{b}/b} \left( \frac{\sqrt{b} - b}{\lambda} \right)^{b-1}. \]  

Similarly, taking the derivative of the MRL function, which is

\[ \frac{d\mu(t)}{dt} = a[b^* e^{\lambda t} + \lambda t^b e^{\lambda t}] \mu(t) - \exp(2at^b e^{\lambda t}) \]

and equating it to zero, the point of maximum MRL can be obtained. The equation is:

\[ a(b + \lambda t)^{b-1} e^{\lambda t} \int_0^t \exp(-at^b e^{\lambda t}) dt - \exp(at^b e^{\lambda t}) = 0. \]

The change points have to be obtained numerically. Fig. 3 shows some typical curves of the FR and MRL function.

3. Length of the flat portion of the bathtub curve

In practice, the region of constant failure rate is especially important for application. However, most BFR models have a V-shape failure rate under most parameter combinations. In this section, we will discuss the issue of a longer constant phrase in the bathtub curve. It also deals with the underlying relationship between the change points and flatness of the bathtub curve.

A method to determine optimum burn-in time is to find the time that maximizes the mean residual life function [20]. There are many papers, which consider other burn-in criteria such as reliability characteristics and costs. For example, Yun et al. [21] proposed a new methodology of determining optimal time by minimizing the total mean cost. If the maximum MRL criterion is used as the burn-in time, it is natural to study the length of useful period of the bathtub after the burn-in is introduced. Mi [22] studied the bounds for burn-in time and replacement time which can be used for identifying the useful period between burn-in and replacement time. It is also of interest to know the duration for which the failure rate is expected to continue to decrease as the change point of MRL is earlier than that of FR function.

Define the absolute difference and relative difference between the change points of FR and MRL as \( D = b^* - t^* \) and \( d = Db^* \) respectively. In fact, these differences can be used to measure the flatness of the bathtub curve. This is because when the difference is large, the change in the trend of FR function cannot be very rapid since the trend of MRL function does not change around that time.

To further study the flatness of the bathtub curve, a more specific criterion to determine and consequently to compare
the length of the constant phase in the bathtub curve is needed. The following is a reasonable one.

Criterion A. The constant life period within a maximum FR tolerance is defined as:

\[ h(t) \leq (1 + k)h(b^s). \]  

Let \( K = 1 + k \), then the criterion above is written as \( h(t) \leq Kh(b^s) \).

It follows that two critical points will be obtained from above inequality. The length of constant life period that satisfies the inequality can be found and further applied to compare the period of flat portion in the bathtub curve.

4. Some numerical values

The exponentiated Weibull model, modified Weibull model and Weibull extension model are studied for their change points of FR and MRL in terms of the individual model parameters. Both change points and differences are compared for each model.

4.1. Exponentiated Weibull model

Take an example of parameters with \( \alpha = 5, \theta = 0.1 \) and \( \sigma = 100 \), which satisfy the condition of BFR property. Table 1 summarizes some numerical results of change points when \( \alpha \) ranges from 1.5 to 8; \( \theta \) ranges from 0.01 to 0.18; \( \sigma \) ranges from 1 to 450. Here \( K = 5\% \) is used to compute the corresponding length of bathtub curve. We can see that the length of the flat portion is strongly related to the difference, \( D \), between the change point of FR and MRL functions.

It can be seen that the change points of FR and MRL functions are highly related to the selection of parameters. The change points decrease as any of the shape parameter \( \alpha \) or \( \theta \) increases. The larger \( \alpha \) or \( \theta \) is, the greater the relative deviation will be. The absolute difference between change points decreases when either increases parameter \( \alpha \) or \( \theta \). As indicated in Fig. 1, the larger the shape parameter, the shaper the bathtub FR curve will be. Since \( \sigma \) is a scale parameter, change points and absolute difference are positively proportional to the values of \( \sigma \).

Besides, the length of the constant region in the bathtub curve decreases when increasing any of the shape parameters \( \alpha \) or \( \theta \). It is also observed that the absolute difference between the change points is exhibiting the same pattern of changes in terms of the parameters. There exists the tendency that the portion of flatness is longer if the difference between the change points is larger. Finally, it is noted that the product can achieve a lower failure rate, if \( \alpha \) is small or \( \sigma \) is large. In contrast, a longer flat portion can be obtained in case of models with smaller \( \theta \), but the minimal failure rate, \( h(b^s) \), is also increasing. Hence it is more preferable for a model with parameters selections of smaller \( \alpha \) or greater \( \sigma \), which will achieve both a longer flat portion of bathtub curve and a lower random failure rate in the useful life period.

4.2. Weibull extension distribution

To maintain the bathtub hazard shape property, this model requires the condition that \( 0 < \beta < 1 \). As an example, use \( \alpha = 100, \beta = 0.5 \) and \( \eta = 2 \) as the basic combination of parameters and investigate the effects by only change one parameter, respectively. The results are summarized for different parameters in Table 2.

The change point of FR is related to the selection of \( \alpha \) and \( \beta \) and each parameter can affect the change point of MRL. The change point of MRL is positively related to scale parameter \( \alpha \) and \( \eta \) and is negatively related to shape parameter \( \beta \). The absolute difference between the change points is greater if scale parameter \( \alpha \) or \( \eta \) is larger. \( D \) is also increasing when decreasing shape parameter \( \beta \), which has a flatter bathtub curve as observed from Fig. 2.

The length of the second phase in the bathtub curve of Weibull extension model increases when decreasing \( \beta \). It can be shown that the FR is proportional to scale parameter, hence the length is proportional to \( \alpha \). It is also known that minimal FR is not related to parameter \( \eta \) and \( \eta \) has no influence on the flatness as defined for the constant failure rate region. There exists a pattern that the portion of flatness is longer if the difference between the change points is larger.
Changes points of FR and MRL for Weibull extension distribution

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<th>$\alpha$</th>
<th>$\beta$</th>
<th>$\eta$</th>
<th>$a^*$</th>
<th>$r^*$</th>
<th>$d$</th>
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Change points of FR and MRL for MWD

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</table>

Weibull extension model with larger $\alpha$ has a longer flat portion of constant life phrase with a constant failure rate level. Although models with parameter setting of smaller $\beta$ can achieve a longer constant failure rate region as well, the FR at the change point varies as the value of $\beta$ changes.

4.3. Modified Weibull distribution (MWD)

To maintain the bathtub hazard shape property, this model needs to satisfy the condition that $0 < b < 1$. Moreover, the distribution is BFR distribution with unique change point when $0 < b < 1$. As $\lim_{t \to +\infty} h(t) = +\infty > 1/\mu$, the distribution has an upside-down bathtub MRL with unique change point. Table 3 summarizes the changing points, deviations by varying $0.5 \leq a \leq 5$, $0.1 < b < 0.9$, and $0.005 < \Lambda < 0.1$, respectively.

For the shape parameter, it can be proved that change point of FR always reaches maximum at $b = 0.25$, which is consistent to the table above. The general tendency of increasing shape parameter is to obtain smaller change points and absolute difference. The change point of MRL is higher for model with larger $a$ or smaller $\Lambda$. The absolute difference between change points is decreasing for a bathtub model with small parameter values.

The length of the flat portion is generally larger if the change points greatly deviate from each other for the shape parameter $b$, and parameter $\Lambda$. The length of flatness has no relation to the parameter $a$, and reaches maximum for $b = 0.25$. Last, MWD with either smaller $b$ or $\Lambda$ can possesses a comparatively flatter bathtub curve with a lower failure rate at the constant life period at the same time.

5. Conclusion

This paper has dealt with an interesting issue of comparing the change points of failure rate and mean residual life function. The difference is important from a number of points of view. First, after a burn-in period that maximizes the mean residual life, the failure rate can be expected to continue to decrease for quite sometime. Hence, if the initial failure rate is of concern, burn-in time could be extended. Also, a model that can be used to describe the bathtub curve should have a flat portion that is long. If the different between change points is large, we also expect this to be the case. In general, some criteria for flatness are needed.

A simple criterion to measure the length of flat portion is used and studied in this paper. In fact, the difference of change points of the FR function and MRL function is shown to be indicative of the flatness of bathtub curve. A general conclusion is that the bathtub is flatter if the difference is larger. The current criterion is not always effective to differentiate flat and shape bathtub curve. Other criteria can be considered as a combination for the decision-making.
We have focused on models with bathtub-shaped failure rate function, and specifically on some three parameter generalization of Weibull distribution. Note that Weibull distribution can only be used to model monotonic failure rate function. The approach and problem statement can be easily generalized to other distributions.

References