Generalized linear fractional programming under interval uncertainty

Milan Hladík

Department of Applied Mathematics, Charles University, Malostranské nám. 25, 11800, Prague, Czech Republic

Abstract

Data in many real-life engineering and economical problems suffer from inexactness. Herein we assume that we are given some intervals in which the data can simultaneously and independently perturb. We consider a generalized linear fractional programming problem with interval data and present an efficient method for computing the range of optimal values. The method reduces the problem to solving from two to four real-valued generalized linear fractional programs, which can be computed in polynomial time using an appropriate interior point method solver.

We consider also the inverse problem: How much can data of a real generalized linear fractional program vary such that the optimal values do not exceed some prescribed bounds. We propose a method for calculating (often the largest possible) ranges of admissible variations; it needs to solve only two real-valued generalized linear fractional programs. We illustrate the approach on a simple von Neumann economic growth model.

Key words: Generalized linear fractional programming, Interval analysis, Tolerance analysis, Sensitivity analysis, Economic growth model

1. Introduction

Uncertainties in data measurement and observation is a common phenomenon in practice. Considering their interval envelopes is one way to tackle these uncertainties. Computing with interval values has many useful properties, e.g., it ensures that all possible instances of interval data are

Email address: milan.hladik@matfyz.cz (Milan Hladík)
URL: http://kam.mff.cuni.cz/~hladik (Milan Hladík)
taken into account. Contrary to the traditional sensitivity analysis, this approach can handle simultaneous and independent perturbations of selected parameters.

Mathematical programming problems with interval data have been investigated for several decades. Many papers studied the problem of computing the range of optimal values of linear programming problem with data varying inside intervals, see [5, 7, 14, 21] among others. Less people were involved nonlinear programming with data perturbing inside intervals. For instance, interval convex quadratic programming was studied in [13, 20], posynomial geometric programming in [13, 17, 18, 19], and a specific nonlinear programming problem with linear constraints in [29].

In this paper, we focus on a generalized linear fractional programming problem the data of which vary inside some given intervals. To the best of our knowledge, this problem itself has never been investigated. In the essence, it can be solved by the general method from [13], where a unified method for dealing with interval nonlinear programming problems was proposed. That approach was based on duality theory in nonlinear programming, and for generalized linear fractional programming we have a developed duality [6, 15] to use. Nevertheless, the approach is a bit cumbersome: we have to derive characterization of primal and dual interval solution sets and the results will be restricted by some assumptions. Stronger results with no needless assumptions are obtained by direct inspection, which is exactly what we do in Section 2.

In Section 2, we show that the exact range of optimal values can be calculated by solving up to four real-valued mathematical programs. Moreover, the method is easily adapted for solving the inverse problem (Section 3): We are given real-valued a generalized linear fractional programming problem and some bounds on the optimal value function, and we calculate tolerances (intervals) for all required parameters such that the optimal values do not exceed the bounds while the parameters are perturbing inside their intervals.

Many applications of generalized linear fractional programming arise in the field of economics and optimization. For instance, von Neumann growth model of expanding economy [25], goal programming with rational criteria [3, 4], or Chebyshev discrete rational approximation. For another applications, see e.g. [23, 24]. Since the economical parameters of the real life problems (including the mentioned applications) are often imprecise, the results developed in this paper form a useful and efficient tool in decision making and analysis.
2. Range of optimal values

Consider a generalized linear fractional programming problem

\[ f(A, B, C, c) := \inf \lambda \text{ subject to } Ax \leq \lambda Bx, Cx \leq c, x \geq 0, \]  

where \( A, B \in \mathbb{R}^{m \times n}, \ C \in \mathbb{R}^{l \times n} \) and \( c \in \mathbb{R}^l \). Moreover, we assume that \( Bx \geq 0 \) holds for all \( x \) satisfying \( Cx \leq c, x \geq 0 \). Such problems are solvable in polynomial time using an interior point method [8, 22].

Now suppose that the input data are not known exactly, and we are given only lower and upper bounds on their values. Formally, the matrix \( A \) varies in some interval matrix \( \bar{A} := \{ A \in \mathbb{R}^{m \times n} \mid a_{ij} \leq a_{ij} \leq \overline{a}_{ij} \} \), where \( \underline{A}, \bar{A} \in \mathbb{R}^{m \times n} \) are given matrices. In a similar way we consider interval matrices \( B \) and \( C \) and interval vector \( c \) in which \( B, C \) and \( c \) may perturb, respectively. Thus we have a family of problems (1) with \( A \in \underline{A}, B \in \bar{B}, C \in \bar{C} \) and \( c \in \bar{c} \). Any problem belonging to this family is referred as an instance.

To ensure that each instance is a proper generalized linear fractional programming problem we have to assume:

(A1) For every \( B \in \bar{B}, C \in \bar{C} \) and \( c \in \bar{c} \) any solution to \( Cx \geq c, x \geq 0 \) solves also \( Bx \geq 0 \).

Proposition 1 shows that to verify this assumption; it suffices to verify only one instance with \( B = \underline{B}, C = \underline{C} \) and \( c = \underline{c} \).

Proposition 1. Assumption (A1) is true if and only if \( \underline{B}x \geq 0 \) holds for all \( x \) satisfying \( \underline{C}x \leq \underline{c}, x \geq 0 \).

Proof. One implication is easily seen as \( B = \underline{B}, C = \underline{C} \) and \( c = \underline{c} \) is an instance of our family of problems.

Conversely, let \( B \in \bar{B}, C \in \bar{C} \) and \( c \in \bar{c} \) and suppose that any \( x \) satisfying \( \underline{C}x \leq \underline{c}, x \geq 0 \) is also a solution of \( \bar{B}x \geq 0 \). Now, let \( x^* \) be any solution to \( Cx \leq c, x \geq 0 \). Then

\[ \underline{C}x^* \leq Cx^* \leq c \leq \overline{c}. \]

Thus \( x^* \) is a solution to \( \overline{C}x \leq \overline{c}, \) and by our supposition \( x^* \) solves also \( \underline{B}x \geq 0 \). Hence

\[ Bx^* \geq \bar{B}x^* \geq 0. \]

Therefore \( x^* \) is a solution to \( Bx \geq 0 \).
As data are perturbing in their intervals, the optimal value \( f(A, B, C, c) \) ranges in some interval as well. Our aim is to determine the exact lower and upper bound on the optimal value. They are respectively defined as

\[
\underline{f} := \inf f(A, B, C, c) \text{ subject to } A \in A, B \in B, C \in C, c \in c,
\]

\[
\overline{f} := \sup f(A, B, C, c) \text{ subject to } A \in A, B \in B, C \in C, c \in c.
\]

The following theorem says that both bounds can be calculated by solving 1 to 2 real-valued generalized linear fractional programming problems.

**Theorem 1.**

1. (Lower bound) Let

\[
f_1 := \inf \lambda \text{ subject to } Ax \leq \lambda Bx, \lambda \leq 0, Cx \leq c, x \geq 0.
\]

If \( f_1 < 0 \) then \( \underline{f} = f_1 \), otherwise \( \underline{f} = f_2 \) with

\[
f_2 := \inf \lambda \text{ subject to } Ax \leq \lambda Bx, \lambda \geq 0, Cx \leq c, x \geq 0.
\]

2. (Upper bound) Let

\[
f_3 := \inf \lambda \text{ subject to } Ax \leq \lambda Bx, \lambda \geq 0, Cx \leq c, x \geq 0.
\]

If \( f_3 > 0 \) then \( \overline{f} = f_3 \), otherwise \( \overline{f} = f_4 \) with

\[
f_4 := \inf \lambda \text{ subject to } Ax \leq \lambda Bx, \lambda \leq 0, Cx \leq c, x \geq 0.
\]

**Proof.** 1. (Lower bound) First we consider the case when \( \underline{f} < 0 \). There is at least one instance of (1) with negative optimal value, so we can restrict our considerations to the family

\[
\inf \lambda \text{ subject to } Ax \leq \lambda Bx, \lambda \leq 0, Cx \leq c, x \geq 0
\]

with \( A \in A, B \in B, C \in C, c \in c \). For any instance and any feasible point \( \lambda, x \) we have \( \lambda x \leq Ax \leq \lambda Bx \leq \lambda Bx \), and \( Cx \leq c \leq \overline{c} \). It means that \( \lambda, x \) is also a feasible solution to the problem

\[
\inf \lambda \text{ subject to } Ax \leq \lambda Bx, \lambda \leq 0, Cx \leq \overline{c}, x \geq 0.
\]

That is, the feasible set to (6) covers feasible sets of all instances of problems (5). Therefore the lower bound \( \underline{f} \) will be achieved for this instance.
Suppose now that $f \geq 0$. In this case, all instances of (1) have non-negative optimal values, and all their feasible solutions $\lambda, x$ have $\lambda \geq 0$. That is why (2) is either infeasible or its optimal value is zero. So it remains to show that $f \geq 0$ implies $f = f_2$. Herein, (1) takes the equivalent form

$$\inf \lambda \text{ subject to } Ax \leq \lambda Bx, \lambda \geq 0, Cx \leq c, x \geq 0.$$ 

For any instance and every feasible solution $\lambda, x$ we have $Ax \leq \lambda Bx \leq \lambda Bx$, and $Cx \leq c \leq \bar{c}$. It means, $\lambda, x$ is also a feasible solution to the problem

$$\inf \lambda \text{ subject to } Ax \leq \lambda Bx, \lambda \geq 0, Cx \leq \bar{c}, x \geq 0.$$ 

Hence the feasible set to (7) covers feasible sets of all instances of problems (1), and the lower bound $f$ will be achieved for this instance.

2. (Upper bound) First we assume that $f > 0$. Then there is at least one instance of (1) with positive optimal value, so we can restrict our considerations to the family

$$\inf \lambda \text{ subject to } Ax \leq \lambda Bx, \lambda \geq 0, Cx \leq c, x \geq 0 \quad (8)$$

with $A \in A$, $B \in B$, $C \in C$, $c \in c$. If (3) is infeasible then $f = f_3 = \infty$ and we are finished. So let $\lambda, x$ be any feasible solution of (3). Then for any instance of (8) we have $Ax \leq Ax \leq \lambda Bx \leq \lambda Bx$, and $Cx \leq c \leq \bar{c}$. Thus $\lambda, x$ is a feasible solution to any instance of (8). In other words, the feasible set to (3) is included in a feasible set of any instance of (8). Therefore the highest optimal value will be achieved for $A = \bar{A}, B = \bar{B}, C = \bar{C}, c = \bar{c}.$

Suppose now that $f \leq 0$. In this case, all instances of (1) have non-positive optimal values, and all their feasible solutions $\lambda, x$ have $\lambda \leq 0$. That is why (3) is either infeasible or its optimal value is zero. It remains to show that $f = f_4$. We rewrite (1) in the equivalent form

$$\inf \lambda \text{ subject to } Ax \leq \lambda Bx, \lambda \leq 0, Cx \leq c, x \geq 0.$$ 

Let $\lambda, x$ be any feasible solution of (4); if (4) is infeasible then $f = f_4 = \infty$ contradicting our assumption. For any instance of (9) we have $Ax \leq \bar{A}x \leq \lambda Bx \leq \lambda Bx$, and $Cx \leq \bar{c} \leq c$. Thus the feasible set to (4) is included in the feasible set of any instance of (9). Therefore the highest optimal value will be achieved in the setting $A = \bar{A}, B = \bar{B}, C = \bar{C}, c = \bar{c}.$

$\square$
3. Tolerances of variations

In this section we consider the inverse problem to the previous one. We start with some real-valued problem and want to extend the reals to intervals such that the optimal value of all instances ranges in some prescribed bounds. Analogous problems were studied in linear programming [12], but—to the best of our knowledge—no one discussed any nonlinear case.

Similar kinds of problems are called tolerance analysis, and we usually study how much may certain parameters perturb while preserving some characteristics, e.g. optimality of some point or basis. They were dealt with mainly in linear programming [1, 2, 11, 26, 27, 28] concerning only selected parameters (in the objective function or in the right-hand side of constraints). Tolerance analysis for all objective function coefficients in multiobjective linear programming was done in [9, 10].

Consider a real-valued generalized linear fractional programming problem (1) with $A := A^0$, $B := B^0$, $C := C^0$, $c := c^0$, and let $f^*$ be its optimal value. Next, $f$ and $\tilde{f}$, $f \leq f^* \leq \tilde{f}$, is a given lower and upper bound on the optimal value function, respectively. Our aim is to extend the real data of (1) to tolerance intervals such that optimal values of all instances ranges in $[f, \tilde{f}]$. For that purpose we introduce so called tolerance rates:

Let $A_{\Delta} \in \mathbb{R}^{m \times n}$, $B_{\Delta} \in \mathbb{R}^{l \times n}$, and $C_{\Delta} \in \mathbb{R}^{l \times n}$ be non-negative matrices and $c_{\Delta} \in \mathbb{R}^l$ be the absolute tolerance, and put $a_{\Delta} := |a^0|$ for the relative (percentage) tolerance. Likewise for $B_{\Delta}$, $C_{\Delta}$, and $c_{\Delta}$.

Formally, our problem states as follows. Denote $A_{\delta} := [A^0 - \delta A_{\Delta}, A^0 + \delta A_{\Delta}]$, $B_{\delta} := [B^0 - \delta B_{\Delta}, B^0 + \delta B_{\Delta}]$, $C_{\delta} := [C^0 - \delta C_{\Delta}, C^0 + \delta C_{\Delta}]$, and $c_{\delta} := [c^0 - \delta c_{\Delta}, c^0 + \delta c_{\Delta}]$. Find (possibly maximal) tolerance $\delta^* > 0$ such that $f(A, B, C, c) \in [f, \tilde{f}]$ for all $A \in A_{\delta^*}$, $B \in B_{\delta^*}$, $C \in C_{\delta^*}$, and $c \in c_{\delta^*}$. Any $\delta^* > 0$ with this property is called admissible tolerance.

We divide this problem onto two smaller sub-problems: Find tolerances $\delta_1, \delta_2 > 0$ such that $f(A, B, C, c) \geq f$ for all $A \in A_{\delta_1}$, $B \in B_{\delta_1}$, $C \in C_{\delta_1}$ and $c \in c_{\delta_1}$, and $f(A, B, C, c) \leq \tilde{f}$ for all $A \in A_{\delta_2}$, $B \in B_{\delta_2}$, $C \in C_{\delta_2}$ and $c \in c_{\delta_2}$. The overall tolerance then simply equals $\delta^* = \min(\delta_1, \delta_2)$. We call $\delta_1$ a lower tolerance and $\delta_2$ an upper tolerance.

First, we verify that under some assumptions the maximal admissible tolerance is well defined.
Lemma 1. Let
\[ \delta_1 := \sup \delta \text{ subject to } f \leq (A, B, C, c) \forall A \in A_\delta, B \in B_\delta, C \in C_\delta, c \in c_\delta, \]
\[ \delta_2 := \sup \delta \text{ subject to } f \geq (A, B, C, c) \forall A \in A_\delta, B \in B_\delta, C \in C_\delta, c \in c_\delta, \]
and denote \( \delta^* = \min(\delta_1, \delta_2) \). Assume that
\[ (A2) \quad (B^0 - \delta^* B^\Delta)x > 0 \quad \text{for all solutions of } (C^0 - \delta^* C^\Delta)x \leq c + \delta^* c^\Delta, x \geq 0. \]
\[ (A3) \quad (C^0 + \delta^* C^\Delta)x \leq c - \delta^* c^\Delta, x \geq 0 \text{ is solvable.} \]

Then \( \delta^* \) is the maximal admissible tolerance.

Proof. 1. (Lower tolerance) We prove that \( \delta_1^* \) is an admissible tolerance for the lower bound. For contradiction suppose that \( \delta_1^* \) is not an admissible tolerance, i.e.,
\[ f > \inf f(A, B, C, c), A \in A_{\delta_1^*}, B \in B_{\delta_1^*}, C \in C_{\delta_1^*}, c \in c_{\delta_1^*}. \]
Thus there is some \( \lambda', x', A \in A_{\delta_1^*}, B \in B_{\delta_1^*}, C \in C_{\delta_1^*} \) and \( c \in c_{\delta_1^*} \) such that
\[ f > \lambda', Ax' \leq \lambda' Bx', Cx' \leq c, x' \geq 0. \]
Without loss of generality suppose that \( \lambda' \geq 0 \); the converse situation is dealt analogously. In this case, \( \lambda', x' \) solves also the system
\[ f > \lambda, (A^0 - \delta_1^* A^\Delta)x \leq \lambda (B^0 + \delta_1^* B^\Delta)x, (C^0 - \delta_1^* C^\Delta)x \leq c^0 + \delta_1^* c^\Delta, x \geq 0. \]
Let \( \eta > 0 \) be arbitrarily small, and let \( x^0 \) be a point in the relative interior of polyhedra described by \( C^0x \leq c^0 \). Define \( x^\eta := \eta x^0 + (1 - \eta)x' \). This point lies in the relative interior of \( (C^0 - \delta_1^* C^\Delta)x \leq c^0 + \delta_1^* c^\Delta \), so for sufficiently small \( \varepsilon > 0 \) there is \( \eta = \eta(\varepsilon) \) such that \( x^\eta \) lies also in the relative interior of \( (C^0 - \delta_\varepsilon C^\Delta)x \leq c^0 + \delta_\varepsilon c^\Delta \) with \( \delta_\varepsilon := \delta_1^* - \varepsilon. \)

Define \( \lambda^\varepsilon \) in this way:
\[ \lambda^\varepsilon := \max_{i=1,...,m} \frac{(A_{i,1}, -\delta_\varepsilon A_{i,\Delta})x^\eta(i)}{(B_{i,1}^0 + \delta_\varepsilon B_{i,\Delta}^\Delta)x^\eta(i)}, \]
where \( M_{i,i} \) denotes the \( i \)-th row of a matrix \( M \). The value \( \lambda^\varepsilon \) is well defined as the denominators are positive due to assumption (A2). This \( \lambda^\varepsilon \) goes to \( \lambda' \) as \( \varepsilon \) goes to zero. Hence \( f > \lambda^\varepsilon \) when \( \varepsilon > 0 \) is small enough. It means that \( \delta_\varepsilon \) (and any larger value) is not admissible tolerance, which contradicts the definition of \( \delta_1^* \).

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2. (Upper tolerance) By contradiction suppose that
\[ \bar{f} < \sup f(A, B, C, c) \quad A \in A_{\delta^2}, \quad B \in B_{\delta^2}, \quad C \in C_{\delta^2}, \quad c \in c_{\delta^2}. \]
That is,
\[ \bar{f} < \sup f(A, B, C, c) \]
for some \( \varepsilon > 0 \), \( \bar{f}^\varepsilon := \bar{f} + \varepsilon \), and for some \( A \in A_{\delta^2}, \quad B \in B_{\delta^2}, \quad C \in C_{\delta^2}, \quad c \in c_{\delta^2} \). Thus
\[ \bar{f}^\varepsilon < \lambda \]
for any \( \lambda \) and \( x \) being a solution to \( Ax \leq \lambda Bx, \quad Cx \leq c, \quad x \geq 0 \). That is why the system
\[ \bar{f}^\varepsilon \geq \lambda, \quad Ax \leq \lambda Bx, \quad Cx \leq c, \quad x \geq 0 \]
is not solvable. Without loss of generality assume that \( \lambda \geq 0 \). Then also the system
\[ \bar{f}^\varepsilon \geq \lambda, \quad (A^0 + \delta^2_2 A^\Delta)x \leq \lambda(B^0 - \delta^2_2 B^\Delta)x, \quad (C^0 + \delta^2_2 C^\Delta)x \leq c - \delta^2_2 c^\Delta, \quad x \geq 0 \]
has no solution. By assumptions (A2) and (A3), the sub-system
\[ (A^0 + \delta^2_2 A^\Delta)x \leq \lambda(B^0 - \delta^2_2 B^\Delta)x, \quad (C^0 + \delta^2_2 C^\Delta)x \leq c - \delta^2_2 c^\Delta, \quad x \geq 0 \]
is solvable; we just find a solution to the second and third inequality and then put \( \lambda \) large enough. Therefore
\[ \bar{f} < \bar{f}^\varepsilon \leq \inf \lambda \text{ subject to } (A^0 + \delta^2_2 A^\Delta)x \leq \lambda(B^0 - \delta^2_2 B^\Delta)x, \]
\[ (C^0 + \delta^2_2 C^\Delta)x \leq c - \delta^2_2 c^\Delta, \quad x \geq 0. \]
For similar reasons as in the part 1. a sufficiently small decrease of \( \delta^2_2 \) implies that the above optimization problem still has optimal value greater than \( \bar{f} \). This contradicts the definition of \( \bar{f} \).
\[ \square \]
Note that assumption (A2) in Lemma 1 is an analogy of assumption (A1) in Section 2. The strict inequality is necessary. For example, consider the problem
\[ \inf \lambda \text{ subject to } x \leq \lambda x, \quad x \geq 1, \]
i.e., \( A^0 = 1, \quad B^0 = 1, \quad C^0 = -1, \quad c^0 = -1 \). Put tolerance rates \( A^\Delta = 1, \quad B^\Delta = 1, \quad C^\Delta = 0, \quad c^\Delta = 0 \). For \( \delta \in (0, 1) \) the optimal value ranges in \( [\frac{1-\delta}{1+\delta}, \frac{1+\delta}{1-\delta}] \), but for \( \delta = 1 \) the optimal value can achieve \(-\infty\).
Also assumption (A3) is necessary. Consider the example

\[
\inf \lambda \text{ subject to } x_1 + x_2 \leq \lambda(x_1 + x_2), \ x_2 = 1, \ x_1 + x_2 \geq 2,
\]
i.e.,
\[
A^0 = (1 \ 1), \ B^0 = (1 \ 1), \ C^0 = \begin{pmatrix} 0 & 1 \\ 0 & -1 \\ -1 & -1 \end{pmatrix}, \ c^0 = (1, -1, -2)^T.
\]

Put tolerance rates as follows
\[
A^\Delta = (0 \ 0), \ B^\Delta = (0 \ 0), \ C^\Delta = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}, \ c^\Delta = (0, 0, 0)^T.
\]

For \(\delta \in (0, 1)\) the optimal value is constantly one, but for \(\delta = 1\) the optimal value for any instance is either one or \(\infty\).

**Theorem 2.** Under assumption (A2) and (A3) the following procedure computes an admissible lower and upper tolerance:

1. (Lower tolerance) If \(f \geq 0\) then
   \[
   \delta_1 := \inf \delta \text{ subject to } (A^0 - fB^0)x \leq \delta(A^\Delta + fB^\Delta)x,
   \]
   \[
   C^0x - c^0 \leq \delta(C^\Delta x + c^\Delta), \ x \geq 0,
   \]
   otherwise
   \[
   \delta_1 := \inf \delta \text{ subject to } (A^0 - fB^0)x \leq \delta(A^\Delta - fB^\Delta)x,
   \]
   \[
   C^0x - c^0 \leq \delta(C^\Delta x + c^\Delta), \ x \geq 0.
   \]

2. (Upper tolerance) If \(\overline{T} \geq 0\) then
   \[
   \delta_2 := \sup \delta \text{ subject to } (-A^0 + \overline{T}B^0)x \geq \delta(A^\Delta + \overline{T}B^\Delta)x,
   \]
   \[
   -C^0x + c^0 \geq \delta(C^\Delta x + c^\Delta), \ x \geq 0,
   \]
   otherwise
   \[
   \delta_2 := \sup \delta \text{ subject to } (-A^0 + \overline{T}B^0)x \geq \delta(A^\Delta - \overline{T}B^\Delta)x,
   \]
   \[
   -C^0x + c^0 \geq \delta(C^\Delta x + c^\Delta), \ x \geq 0.
   \]
Proof. 1. (Lower tolerance) Due to Lemma 1 the maximal lower tolerance is computed by
\[ \delta^*_1 = \sup \delta \text{ subject to } f \leq f(A, B, C, c) \forall A \in A_\delta, B \in B_\delta, C \in C_\delta, c \in c_\delta. \]
(10)

Its optimal value is greater or equal to
\[ \inf \delta \text{ subject to } f \geq f(A, B, C, c), A \in A_\delta, B \in B_\delta, C \in C_\delta, c \in c_\delta, \]
or
\[ \inf \delta \text{ subject to } f \geq \lambda, Ax \leq \lambda Bx, Cx \leq c, x \geq 0, \]
\[ A \in A_\delta, B \in B_\delta, C \in C_\delta, c \in c_\delta. \]
(11)

Due to our assumption, \( Bx \geq 0 \) for any feasible solution \( x \). Thus the optimal solution will be achieved for \( \lambda = f \). It this setting the problem draws
\[ \inf \delta \text{ subject to } Ax \leq \lambda Bx, Cx \leq c, x \geq 0, \]
\[ A \in A_\delta, B \in B_\delta, C \in C_\delta, c \in c_\delta. \]

Similarly, the can put \( A := A^0 - \delta A^\Delta, C := C^0 - \delta C^\Delta, c := c^0 + \delta c^\Delta \) and \( B := B^0 - \delta B^\Delta \) if \( f < 0 \) and \( B := B^0 + \delta B^\Delta \) if \( f \geq 0 \). Therefore we conclude:
If \( f \geq 0 \) then
\[ \delta^*_1 \geq \delta_1 = \inf \delta \text{ subject to } (A^0 - \delta A^\Delta)x \leq f(B^0 + \delta B^\Delta)x, \]
\[ (C^0 - \delta C^\Delta)x \leq c^0 + \delta c^\Delta, x \geq 0, \]
\[ = \inf \delta \text{ subject to } (A^0 - \delta B^0)x \leq f(A^\Delta + fB^\Delta)x, \]
\[ C^0x - c^0 \leq \delta(C^\Delta x + c^\Delta), x \geq 0. \]

If \( f < 0 \) then
\[ \delta^*_1 \geq \delta_1 = \inf \delta \text{ subject to } (A^0 - \delta A^\Delta)x \leq f(B^0 - \delta B^\Delta)x, \]
\[ (C^0 - \delta C^\Delta)x \leq c^0 + \delta c^\Delta, x \geq 0, \]
\[ = \inf \delta \text{ subject to } (A^0 - \delta B^0)x \leq f(A^\Delta - fB^\Delta)x, \]
\[ C^0x - c^0 \leq \delta(C^\Delta x + c^\Delta), x \geq 0. \]

2. (Upper tolerance) Due to Lemma 1,
\[ \delta^*_2 = \sup \delta \text{ subject to } \overline{f} \geq f(A, B, C, c) \forall A \in A_\delta, B \in B_\delta, C \in C_\delta, c \in c_\delta, \]
\[ \geq \sup \delta \text{ subject to } \forall A \in A_\delta, B \in B_\delta, C \in C_\delta, c \in c_\delta \exists \lambda, x; \]
\[ \overline{f} \geq \lambda, Ax \leq \lambda Bx, Cx \leq c, x \geq 0. \]
Again, we put \( \lambda := f \) yielding

\[
\delta^*_{2} \geq \sup \delta \text{ subject to } \forall A \in A_\delta, \ B \in B_\delta, \ C \in C_\delta, \ c \in c_\delta \ \exists x; \quad Ax \leq \overline{f}Bx, \ Cx \leq c, \ x \geq 0.
\]

For the same reasons as in the proof of Theorem 2 it suffices to consider

\[
A := A^0 + \delta A^\Delta, \ C := C^0 + \delta C^\Delta, \ c := c^0 - \delta c^\Delta, \ \text{and} \ B := B^0 - \delta B^\Delta \ \text{if} \ \overline{f} \geq 0 \ \text{and} \ \text{B} := B^0 + \delta B^\Delta \ \text{if} \ \overline{f} < 0. \ \text{Therefore, if} \ \overline{f} \geq 0 \ \text{then}
\]

\[
\delta^*_2 \geq \delta_2 \ \sup \delta \text{ subject to } (A^0 + \delta A^\Delta)x \leq \overline{f}(B^0 - \delta B^\Delta)x,
\]

\[
(C^0 + \delta C^\Delta)x \leq c^0 - \delta c^\Delta, \ x \geq 0,
\]

\[
= \sup \delta \text{ subject to } (-A^0 + \overline{f}B^0)x \geq \delta(A^\Delta + \overline{f}B^\Delta)x,
\]

\[
-C^0x + c^0 \geq \delta(C^\Delta x + c^\Delta), \ x \geq 0.
\]

If \( \overline{f} < 0 \) then

\[
\delta_2 = \sup \delta \text{ subject to } (A^0 + \delta A^\Delta)x \leq \overline{f}(B^0 + \delta B^\Delta)x,
\]

\[
(C^0 + \delta C^\Delta)x \leq c^0 - \delta c^\Delta, \ x \geq 0,
\]

\[
= \sup \delta \text{ subject to } (-A^0 + \overline{f}B^0)x \geq \delta(A^\Delta - \overline{f}B^\Delta)x,
\]

\[
-C^0x + c^0 \geq \delta(C^\Delta x + c^\Delta), \ x \geq 0.
\]

\( \Box \)

Note that for calculating the admissible tolerance along Theorem 2 we have to solve only two ordinary generalized linear fractional programming problems. The resulting tolerances \( \delta_1 \) and \( \delta_2 \) are often the maximal ones, but they needn’t be maximal as (10) and (11) may differ. Indeed, if \( \delta_1 \) is not maximal, i.e \( \delta_1 < \delta^*_1 \), then necessarily (10) and (11) differ. It means that the optimal value to

\[
\inf f(A, B, C, c) \text{ subject to } A \in A_\delta, \ B \in B_\delta, \ C \in C_\delta, \ c \in c_\delta
\]

is constant for all \( \delta \in [\delta_1, \delta^*_1] \). Such behaviour is very unusual even though it may happen.

The next remark concerns assumptions (A2) and (A3). It is uncomfortable to check their validity as the maximal tolerance \( \delta^* \) is unknown. However, an insight into proof of Theorem 2 reveals that—from the practical standpoint—it suffices to determine admissible tolerances \( \delta_1 \) and \( \delta_2 \) along the statement, and afterwards to check validity of assumptions (A2)-(A3) with \( \delta^* = \min(\delta_1, \delta_2) \).
Note that our results are easily extendable to the non-symmetric case where interval matrices are defined as $A_\delta := [A^0 - \delta A^1, A^0 + \delta A^2]$ and $A^1, A^2$ are non-negative matrices. This case is more general, but for the sake of simplicity we don’t discuss it here in detail.

Example 1. We illustrate our approach on a von Neumann economic growth model. Such a model reads

$$\max \lambda \text{ subject to } \lambda A x \leq B x, \ x \geq 1,$$

where variables $x_i, \ i = 1, \ldots, n$ denote activity of sector $i$, and the constraints correspond to particular commodities. Matrix $A \in \mathbb{R}^{m \times n}$ consists of input coefficients and matrix $B \in \mathbb{R}^{m \times n}$ consists of output coefficients. The optimal $\lambda$ gives the resulting economic growth factor. Clearly, this model is easily transformed into the standard form (1) of generalized linear fractional programming problems.

Many economical parameters are subject to uncertainties, especially that ones involving prediction of future values. The methods developed in Sections 2 and 3 enable us to analyse impact of such uncertainties for selected (possibly all) parameters. Moreover, our approach handles simultaneous and independent perturbations of that parameters and thus overcome the main drawback of the frequently used sensitivity analysis.

Given interval estimates of the selected parameters, the method from Section 2 calculates the range of optimal values to (12), i.e., the interval in which the real economic growth factor may vary. The problem studied in Section 3 is the inverse one. For prescribed bounds on the growth factor, our method computes ranges of admissible variations for given parameters. For concreteness, the latter method is illustrated below by a numerical example.

Consider an example by Li [16] describing the dynamics of a simple two-country economy. The commodities considered are respectively: Labor power of country 1, Consumer good, Capital good and Labor power of country 2. The sectors considered are respectively: Household of country 1, Consumer good producer of country 1, Capital good producer of country 1, Capital good producer of country 2, Consumer good producer of country 2,
and Household of country 2. The technology matrices are as follows:

\[
A = \begin{pmatrix}
0.28 & 0.50 & 0.53 & 0 & 0 & 0 \\
0.84 & 0 & 0 & 0 & 0 & 0.77 \\
0 & 0.49 & 0.45 & 0.50 & 0.48 & 0 \\
0 & 0 & 0 & 0.51 & 0.57 & 0.29
\end{pmatrix},
\]

\[
B = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0.25 & 1 & 1 & 0.25 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\]

The optimal value is \( \lambda^* = 1.049 \), so the economy is slightly expanding.

We compute tolerances for all coefficients of matrix \( A \) and \( B \) such that the growth factors for all admissible perturbations lie within the interval \([f, \bar{f}] := [1, 1.2] \). We consider percentage tolerance, i.e., we put tolerance factors as \( A^\Delta := |A| \) and \( B^\Delta := |B| \). Calling Theorem 2 we obtain an admissible lower tolerance \( \delta_1 = 0.024 \) and upper tolerance \( \delta_2 = 0.067 \). Assumptions (A2) and (A3) are satisfied. We conclude that all entries of \( A \) and \( B \) may vary simultaneously and independently within 2.4\% tolerance of their nominal value while the economy is guaranteed to be expanding with the growth factor lying inside \([1, 1.2] \).

Naturally, larger tolerances can be obtained when we consider less parameters for perturbing. For instance, say we are interested in tolerances for entries of \( B \) only. In this case, we put \( A^\Delta := 0 \) and \( B^\Delta := |B| \) and call Theorem 2 again. We get an admissible lower tolerance \( \delta_1 = 0.046 \) and an upper tolerance \( \delta_2 = 0.143 \) and the corresponding assumptions (A2)–(A3) are satisfied, too. Now, the resulting percentage tolerance is 4.6\%, meaning that as long as the coefficients in \( A \) are constant and that ones from \( B \) vary within 4.6\% tolerance then the economic growth factor lies between 1 and 1.2.

4. Conclusion

The proposed method measures variations of the optimal values in a generalized linear fractional programming problem when input coefficients perturb within compact intervals. Advantage to the traditional sensitivity analysis consists in considering simultaneous and independent perturbation of selected (possibly all) coefficients.

The inverse problem is studied, too. Given bounds on the optimal values we calculate tolerance intervals for the coefficients such that anyhow we
perturbing the coefficients within the tolerances the corresponding optimal value never exceeds the prescribed bounds. Usually, the tolerances are the maximal ones.

References


