On Parametric Inference for Step-Stress Models

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Key Words - Accelerated life testing; Accelerated failure time; Additive accumulation of damages; Cox model; Generalized proportional hazards model; Step-stress; Tampered failure rate.

Summary & Conclusions - Applications of the additive accumulation of damages (AAD) or the accelerated failure time (AFT) and the proportional hazards (PH) models in accelerated life testing with step-stresses are discussed. A new model including both of them is proposed. It is more natural then PH model and wider then the AAD model. Maximum likelihood function construction is discussed.

1. INTRODUCTION

Acronyms
In ALT units are tested at higher-than-usual levels of stress to induce early failures. The results are extrapolated to estimate the life distribution at the design stress using models which relate the lifetime to the stress. More about the analysis of accelerated life-testing experiments one can see, for example, in Miner (1945), Bagdonavičius (1978), Shaked and Singpurwalla (1982), Nelson (1990), Meeker and Escobar (1998), Bagdonavičius and Nikulin (1999, 2000) etc.

In step-stress testing a unit is placed on test at an initial low stress and if it does not fail in a predetermined time $t_1$, the stress is increased and so on.
Denote by \(x(\cdot)\) a step-stress of the form

\[
x(\tau) = \begin{cases} 
x_1, & t_0 \leq \tau < t_1, \\
x_2, & t_1 \leq \tau < t_2, \\
\cdots \cdots \\
x_k, & t_{k-1} \leq \tau < t_k,
\end{cases}
\]

(1)

where \(t_0 = 0, t_k = \infty\).

The two most known general models of ALT with time varying stresses are: the AAD model (Bagdonavičius [1]) and the PH model (Cox [2]).

The AAD model:

\[
F_{x(\cdot)}(t) = F_0(\int_0^t r[x(\tau)]d\tau),
\]

where \(r(x)\) is a positive function.

The PH model:

\[
\lambda_{x(\cdot)}(t) = r[x(t)]\lambda_0(t).
\]

The AAD and PH models are rather restrictive. In the case of AAD model the stress changes locally only the scale. If the PH model holds, \(x_0\) is a design stress, \(x_1\) is an accelerated with respect to \(x_0\) stress, \(x_0, x_1 \in \mathcal{E}\), i.e. \(S_{x_0}(t) \geq S_{x_1}(t)\) for all \(t \geq 0\), and

\[
x(t) = \begin{cases} 
x_1, & 0 \leq t < t_1, \\
x_0, & t \geq t_1,
\end{cases}
\]

is a step-stress, then \(\alpha_{x(\cdot)}(t) = \alpha_{x_0}(t)\) for all \(t_1 > 0, t > t_1\). So if one group of items is tested under the design stress and the second group - under an accelerated stress \(x_1\) until the moment \(t_1\) and after this moment both groups are observed under the same design stress \(x_0\), the failure rate after the moment \(t_1\) is the same for both groups. If items are aging, it is not natural.
Notations
$x_0$ design stress

$k$ number of test stresses

$j$ index for test-stresses: $j = 1, \ldots, k$ unless otherwise specified

$x_j$ ordered test-stresses

$x(\cdot)$ time varying stress, particularly a step-stress

$n$ number of units placed on test

$T_{(i)}$ ordered failure times

$t_i$ stress-change times, $i = 1, \ldots, k - 1$

$t_f$ censoring time at $x_k$

$r, \varphi$ functions of a stress

$r_j r(x_j)$

$\Delta t_j t_j - t_{j-1}$

$s_i \sum_{j=1}^{i} r_j \Delta t_j$

$\rho r(x_1)/r(x_0)$

$F_{x(\cdot)}$ Cdf of times-to-failure under the stress $x(\cdot)$

$F_0$ baseline Cdf

$f_{x(\cdot)}$ pdf of times-to-failure under the stress $x(\cdot)$

$\lambda_{x(\cdot)}$ hazard rate under the stress $x(\cdot)$

$\Lambda_{x(\cdot)} \int_{0}^{t} \lambda_{x(\cdot)}(u) du$ ahr under $x(\cdot)$

$\lambda_0$ baseline hazard rate

$\Lambda_0 \int_{0}^{t} \lambda_0(u) du$ baseline ahr

$sp_{x(\cdot)}$ the right end of the support of distribution under $x(\cdot)$

$\beta, \delta, \gamma, \alpha_0$ unknown parameters

$k(t) min\{ j : t_j \geq t \}$

$\theta_i r_0 \alpha_0 / r_i$

$\hat{\beta}, \hat{\delta}, \hat{\gamma}, \hat{\alpha}_0$ ML estimators of $\beta, \delta, \gamma, \alpha_0$
2. FORMULATION OF AAD AND PH MODELS FOR CONSTANT AND STEP-STRESSES

In terms of Cdf the PH model (3) is written:

\[ F_{x(t)}(t) = 1 - \exp\{- \int_0^t r[x(\tau)]d\Lambda_0(\tau)\}. \]  \hfill (4)

In the particular case of constant in time stress \( x_j \) and the step-stress (1) the AAD model (2) implies

\[ F_{x_j}(t) = F_0(r_jt), \]

and for \( t \in [t_{i-1}, t_i) \), \((i = 1, 2, \ldots, k)\)

\[ F_{x(t)}(t) = F_0 \left( \sum_{j=1}^{i-1} r_j \Delta t_j + r_i(t - t_{i-1}) \right). \]

In terms of \( F_{x_j} \),

\[ F_{x(t)}(t) = F_{x_i}(s_{i-1} + t - t_{i-1}), \]  \hfill (5)

where \( s_i \) can be determined recurrently from the equations

\[ F_{x_{i+1}}(s_i) = F_{x_i}(s_{i-1} + t_i - t_{i-1}). \]  \hfill (6)

For the step-stress (1) and \( k = 2 \) the PH model (3) implies

\[ \lambda_{x(t)}(\tau) = \begin{cases} \lambda_{x_0}(\tau), & 0 \leq \tau \leq t, \\ \rho \lambda_{x_0}(\tau), & \tau > t. \end{cases} \]

In this particular case of step-stresses the Cox model is sometimes called the TFR model (Bhattacharyya and Stoejoeit (1989)). If \( k \geq 2 \), the formula (4) implies

\[ F_{x_j}(t) = 1 - e^{-r_j \Lambda_0(t)} \] and for \( t \in [t_{i-1}, t_i) \)

\[ F_{x(t)}(t) = 1 - \exp\{- \sum_{j=1}^{i-1} r_j(\Lambda_0(t_j) - \Lambda_0(t_{j-1})) - r_i(\Lambda_0(t) - \Lambda_0(t_{i-1}))\}. \]  \hfill (7)
In the case when the distribution under the design stress \(x_0\) is Weibull:

\[
F_{x_0}(t) = 1 - e^{-t^\delta/\alpha_0}, \quad \Lambda_0(t) = \frac{t^\delta}{r_0 \alpha_0},
\]

we have

\[
F_{x_i}(t) = 1 - e^{-t^\delta/\theta_i},
\]

and for \(t \in [t_{i-1}, t_i)\)

\[
F_{x(\cdot)}(t) = 1 - \exp \left\{ - \sum_{j=1}^{i-1} \frac{t^\delta_j - t^\delta_{j-1}}{\theta_j} - \frac{t^\delta_i - t^\delta_{i-1}}{\theta_i} \right\}.
\]

This particular case of the PH model was considered by Khamis and Higgins [4].

3. NEW MODEL

Following Bagdonavičius and Nikulin (1999) consider generalisation of both the AAD and PH models: GPH model. For this purpose we formulate these models in other terms. For any \(x(\cdot)\) the random variable \(R = \Lambda_{x(\cdot)}(T_{x(\cdot)})\) has the standard exponential distribution with the survival function \(S_R(t) = e^{-t}, \ t \geq 0\), and is called the exponential resource, the number \(\Lambda_{x(\cdot)}(t)\) being the exponential resource used until the moment \(t\) under the stress \(x(\cdot)\). So the failure rate \(\lambda_{x(\cdot)}(t)\) is the rate of exponential resource usage.

Under the PH model we have

\[
\Lambda'_{x(\cdot)}(t) = r[x(t)]\lambda_0(t),
\]

i.e. the rate of resource usage is proportional to some baseline rate and the constant of proportionality is a function of a stress.

GPH model holds on \(E\) if for all \(x(\cdot) \in E\)

\[
\lambda_{x(\cdot)}(t) = r\{x(t)\}q\{\Lambda_{x(\cdot)}(t)\}\lambda_0(t).
\]
The rate of resource usage at the moment \( t \) depends not only on a stress applied at this moment but also on the resource used until \( t \).

The particular cases of the GPH model are the PH model \((q(u) \equiv 1)\) and the AAD model \((\lambda_0(t) \equiv \lambda_0 = \text{const})\).

The function \( r \) in the GPH model (as in the PH model) can be parametrized in the following way:

\[
r[x(\tau)] = e^{\beta T \varphi(x(\tau))},
\]

where \( \varphi \) is some known function of the stress. For example, if \( \varphi(x) = x, \ln x, 1/x \), we have generalizations of the log-linear, power rule, Arrhenius models, respectively, see, for example, Nelson (1990). We’ll write \( x(\cdot) = \varphi(x(\cdot)) \) in what follows.

Consider the case when the rate of exponential resource usage with respect to the baseline rate is proportional to a monotone function of the used resource. One of the most natural parametrizations of the function \( q \) in this case would be

\[
q(u) = e^{\gamma u}, \quad \gamma \in \mathbb{R}.
\]

So we consider

**GLPH model :**

\[
\lambda_{x(\cdot)}(t) = e^{\beta x(t) + \gamma \Lambda_{x(\cdot)}(t)} \lambda_0(t).
\] (8)

This model implies

\[
F_{x(\cdot)}(t) = \begin{cases} 
1 - \left\{ 1 - \gamma \int_0^t e^{\beta x(u)} d\Lambda_0(u) \right\}^{1/\gamma}, & \gamma \neq 0 \\
\exp\left\{ - \int_0^t e^{\beta x(u)} d\Lambda_0(u) \right\}, & \gamma = 0
\end{cases}
\]

If \( \gamma \leq 0 \), the support of distribution is \([0, \infty)\), if \( \gamma > 0 \), it is \([0, sp_{x(\cdot)})\), where \( sp_{x(\cdot)} \) verifies the equation

\[
\int_0^{sp_{x(\cdot)}} e^{\beta x(u)} d\Lambda_0(u) = 1.
\]
In the particular cases of the step-stress (1) we have the formula (7) when $\gamma = 0$; if $\gamma \neq 0$ then for all $t \in [t_{i-1}, t_i)$

$$F_{x_i}(t) = 1 - \{1 - \gamma \sum_{j=1}^{i-1} e^{\beta x_j (\Lambda_0(t_j) - \Lambda_0(t_{j-1}))} - \gamma e^{\beta x_i (\Lambda_0(t_i) - \Lambda_0(t_{i-1}))}\}^{1/\gamma}, \quad (9)$$

and for the constant stress $x_j$

$$F_{x_j}(t) = 1 - \{1 - \gamma e^{\beta x_j \Lambda_0(t)}\}^{1/\gamma}.$$

If the distribution of time-to-failure under the design stress $x_0$ is Weibull, i.e.

$$F_{x_0}(t) = 1 - e^{-t^{\delta}/\alpha_0}, \quad \lambda_0(t) = \frac{1}{\gamma} e^{-\beta x_0} \left(1 - e^{-\frac{2}{\alpha_0} + \delta}\right),$$

then under the GLPH model (8) and the step-stress (1)

$$F_{x_i}(t) = 1 - \{1 + \sum_{j=1}^{i-1} e^{\beta (x_j - x_0)} [e^{-\gamma t^{\delta}/\alpha_0} - e^{-\gamma t^{\delta}_j/\alpha_0}] + e^{\beta (x_i - x_0)} [e^{-\gamma t^{\delta}/\alpha_0} - e^{-\gamma t^{\delta}_{i-1}/\alpha_0}]\}^{1/\gamma}. \quad (10)$$

Note that if the rate of resource usage depends on used resource (i.e. $\gamma \neq 0$), the distribution of the time-to-failure under constant in time stresses $x_j \neq x_0$ is not Weibull:

$$F_{x_j}(t) = 1 - \{1 - e^{\beta (x_j - x_0)} [1 - e^{-\gamma t^{\delta}/\alpha_0}]\}^{1/\gamma}.$$

Under the GLPH model distributions of times-to-failure under constant in time stresses are from the same family of distributions if we take one of the following two families of distributions:

The family $K_1$:

$$F_{x_0}(t) = 1 - (1 + \alpha_0 t^{\delta})^{1/\gamma}, \quad t \geq 0 \quad (\alpha_0 > 0, \delta > 0, \gamma < 0). \quad (11)$$

If $0 < \delta \leq 1$, the failure rate $\lambda_{x_0}(t)$ is decreasing. If $\delta > 1$, then the failure rate is increasing until the moment $t = \left(\frac{\delta-1}{\delta\alpha_0}\right)^{1/\delta}$ and decreasing after this moment, i.e.
has the $\cap$-shape.

The family $K_2$:

$$F_{x_0}(t) = 1 - (1 + \alpha_0 t^\delta)^{1/\gamma}, \quad t \in [0, \left(\frac{1}{\alpha_0}\right)^{1/\delta}) \quad (\alpha_0 < 0, \delta > 0, \gamma > 0).$$

If $0 < \delta < 1$, the hazard rate is decreasing in the interval $[0, \left(\frac{\delta-1}{\alpha_0}\right)^{1/\delta})$ and increasing in the interval $(\left(\frac{\delta-1}{\alpha_0}\right)^{1/\delta}, \left(-\frac{1}{\alpha_0}\right)^{1/\delta})$. The hazard rate has $\cup$-shape. So this family is very appealing. If $\delta > 1$, the hazard rate is increasing on $(0, \left(-\frac{1}{\alpha_0}\right)^{1/\delta})$.

Suppose that the time-to-failure distribution is from one of the above mentioned families. If $F_{x_0}(t) \in K_i$ ($i=1,2$) and the GLPH model holds, then

$$\Lambda_0(t) = -\frac{\alpha_0}{\gamma} e^{-\beta x_0 t^\delta}$$

and

$$F_{x_j}(t) = 1 - (1 + \alpha_j t^\delta)^{1/\gamma},$$

where $\alpha_j = \alpha_0 e^{\beta (x_j - x_0)}$. If $F_{x_0} \in K_2$, the support of the distribution is finite: $(0, \left(\frac{1}{\alpha_0}\right)^{1/\delta})$.

If $x(\cdot)$ is a step stress of the form (1) then

$$F_{x(\cdot)}(t) = 1 - \left\{1 + \alpha_0 \varphi(t, \beta, \delta)\right\}^{1/\gamma}, \quad t < sp_{x(\cdot)},\quad (12)$$

where

$$\varphi(t, \beta, \delta) = \begin{cases} \sum_{j=1}^{k(t)-1} e^{\beta (x_j - x_0)} (t_j^\delta - t_{j-1}^\delta) + e^{\beta (x_{k(t)} - x_0)} (t_{k(t)}^\delta - t_{k(t)-1}^\delta), & k(t) > 1 \\ e^{\beta (x_1 - x_0) t^\delta}, & k(t) = 1. \end{cases}$$

If $\alpha_0 > 0, \gamma < 0$, then $sp_{x(\cdot)} = +\infty$. If $\alpha_0 < 0, \gamma > 0$, the right limit of the support $(0, sp_{x(\cdot)})$ verifies the equation $\varphi(sp_{x(\cdot)}, \beta, \delta) = -1/\alpha_0$, i.e.

$$sp_{x(\cdot)} = \left\{t_{k(t)-1}^\delta - \frac{1}{\alpha_0} e^{\beta (x_{k(t)} - x_0)} - \sum_{j=1}^{k(t)-1} e^{\beta (x_j - x_0)} (t_j^\delta - t_{j-1}^\delta).\right.$$
The probability density is

\[ f_{x(\cdot)}(t) = -\alpha_0^\delta \frac{e^{\beta (x_{k(\cdot)} - x_0)} t^{\delta - 1} \{1 + \alpha_0 \varphi(t, \beta, \delta)\}^{1/\gamma - 1}}{\gamma}, \quad t < sp_{x(\cdot)}, \]

and the failure rate

\[ \lambda_{x(\cdot)}(t) = -\alpha_0^\delta \frac{e^{\beta (x_{k(\cdot)} - x_0)} t^{\delta - 1}}{(1 + \alpha_0 \varphi(t, \beta, \delta))}, \]

has the same shape as in the case of constant in time stresses (decreasing or \(\cap\)-shape for the family \(K_1\), increasing or \(\cup\)-shape for the family \(K_2\).

4. ESTIMATION

Suppose that all units are initially placed on test at \(x_1\) and run until \(t_1\) when the stress is changed to \(x_2 > x_1\). At \(x_2\), testing continues until \(t_2\) when stress is changed to \(x_3 > x_2\), and so on, until \(x_k\). Under \(x_k\), testing continues until all remaining units fail or until \(t_f\) whichever comes first.

Denote by \(D\) the number of failures during the experiment, \(T_{(1)} \leq ... \leq T_{(D)}\) the observed moments of failures. In the case of a finite support the following reparametrization can be introduced: \(\sigma = sp_{x(\cdot)}\), \(\beta, \gamma, \delta\). Then \(\alpha_0 = -1/\varphi(\sigma, \beta, \delta)\).

The likelihood function:

for the family \(K_1\)

\[ L(\alpha_0, \beta, \gamma, \delta) = \left( -\alpha_0^\delta \right)^D e^{\beta \sum_{i=1}^{D} (x_{k(T_{(i)})} - x_0)} \times \prod_{i=1}^{D} T_{(i)}^{\delta - 1} \{1 + \alpha_0 \varphi(T_{(i)}; \beta, \delta)\}^{1/\gamma - 1} \{1 + \alpha_0 \varphi(t_f; \beta, \delta)\}^{n - D} \]

and for the family \(K_2\)

\[ L(\sigma, \beta, \gamma, \delta) = \left( \frac{\delta}{\gamma} \right)^D e^{\beta \sum_{i=1}^{D} (x_{k(T_{(i)})} - x_0)} \left( \frac{1}{\varphi(\sigma, \beta, \delta)} \right)^D \times \]

11
\[
\prod_{i=1}^{n} T_{(i)}^{\gamma - 1} \left( 1 - \frac{\varphi(T_{(i)}, \beta, \delta)}{\varphi(\sigma, \beta, \delta)} \right)^{\frac{1}{\gamma} - 1} \left( 1 - \frac{\varphi(t_f, \beta, \delta)}{\varphi(\sigma, \beta, \delta)} \right) \frac{n - D}{\gamma} 1\{ T_{(i)} < \sigma \}.
\]

The cdf \( F_{x_0}(t) \) is estimated by

\[
\hat{F}_{x_0}(t) = 1 - (1 + \hat{\alpha}_0 t^\hat{\delta})^{\frac{1}{\gamma}}.
\] (13)

In the case of the class \( K_1 \) approximate confidence intervals for \( F_{x_0}(t) \) and other reliability characteristics under the design stress \( x_0 \) are obtained by standard ML methods using the estimated Fisher information matrix. In the case of the class \( K_2 \) when the support is finite, rigorous asymptotic theory is to be established. If the time-to-failure under the design stress \( x_0 \) has the Weibull (or another) distribution, then estimation under the GLPH model can be done similarly taking the formula (10) (or (9) with respective \( \Lambda_0 \)) for \( F_x(t) \) instead of (12) in all formulas.

5. EXAMPLES

Suppose that the cdf \( F_{x_0}(t) \) is from the family \( K_1 \), defined by (11). To do a small simulation study we took \( \beta = 1, \gamma = -0.2, \delta = 1 \) and \( \alpha_0 = 1.25 \times 10^{-4} \).

The graph of the function \( F_{x_0}(t) \) is given in the Fig.1.

The following plan of experiments is considered: \( n \) units are tested time \( t_f \) under the step-stress (1), \( k = 4 \). The graph of \( x(t), t \leq t_f \) is given in the Fig.2.

We simulated 20 samples of size \( n = 100 \) corresponding to the experiment when units are observed only under the design stress \( x_0 \) time \( t_f \).

Realizations of the ML estimators of the cdf \( F_{x_0}(t) \) are given in the Fig.3. The discontinuous line is the graph of the true function \( F_{x_0}(t) \). The line with squares is obtained by taking arithmetic means of of all twenty realizations.
It is evident that the estimator is very bad for $t > t_f$. It confirms the well known fact that under heavy censoring ML estimators are bad for finite samples. Thus using of an experiment with accelerated stresses is necessary, because in this case the number of failure times can be increased.

We simulated 20 samples of size $n = 100$ (using the same uniform random variables as in the previous case), corresponding to the experiment given in this paper, i.e. under the step-stress (1).

Realizations of the estimator (13) are given in the Fig.4. As before the discontinuous line is the graph of the true function $F_{x_0}(t)$. The line with squares is obtained by taking arithmetic means of of all twenty realizations.

The results are obviously better. Even taking the mean of a small number of realizations (twenty), we obtained that the graph of this mean is close to the the graph of the true function.

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Figure 1. Distribution function $F_{x_0}(t)$. 
Figure 2. Plan of experiments
Figure 3. Realizations of the estimator $\hat{F}_{x_0}(t)$ obtained from the experiment under the design stress.
Figure 4. Realizations of the estimator $\hat{F}_{x_0}(t)$ obtained from the accelerated experiments.
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