Small vertex-transitive directed strongly regular graphs

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Abstract

We consider directed strongly regular graphs defined in 1988 by Duval. All such graphs with \( n \) vertices, \( n \leq 20 \), having a vertex-transitive automorphism group, are determined with the aid of a computer. As a consequence, we prove the existence of directed strongly regular graphs for three feasible parameter sets listed by Duval. For one parameter set a computer-free proof of the nonexistence is presented. This, together with a recent result by Jørgensen, gives a complete answer on Duval's question about the existence of directed strongly regular graphs with \( n \leq 20 \). The paper includes catalogues of all generated graphs and certain theoretical generalizations based on some known and new graphs. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

The notion of a directed strongly regular graph (s.r.g.), which was suggested by Duval [11], still has a short history. It is a generalization of a classical definition of a strongly regular graph to the class of graphs which have both undirected edges and directed arcs. Each directed strongly regular graph has natural parameters \((n, k, \lambda, \mu, t)\)
which are defined below. Sometimes, we will call such a graph briefly an \((n,k,\mu,\lambda,t)\)-
graph (here \(n\) is the number of vertices). Duval found a number of necessary con-
ditions for the existence of an \((n,k,\mu,\lambda,t)\)-graph; a set of parameters which satisfy
all these conditions is called a feasible set. Several infinite classes as well as a
few sporadic examples of directed strongly regular graphs were introduced

Duval’s paper also contains a list of all feasible parameter sets with \(n \leq 20\) and
information available to Duval about the existence of graphs with the parameters from
this list.

A next step towards the systematic investigation of “Duval’s graphs” was made in
[31], where coherent (cellular) algebras in the sense of Higman [21], Weisfeiler and
Leman [41,44], respectively, were used. We continue in the spirit of this work.

Duval’s list contains 23 sets of parameters, for 7 of which he was not able to answer
the question about the existence of an \((n,k,\mu,\lambda,t)\)-graph. This list was revisited in [31].
In particular, one possible set of parameters was eliminated, and for one more set the
existence of a graph was established. However, for 5 parameter sets the problem of
existence still remained open. This open problem served for us as the main challenge
to start the investigation of directed strongly regular graphs with the aid of a computer.

In this paper, we present the results of the computer enumeration of all “small”
vertex-transitive directed strongly regular graphs, that is, with \(n \leq 20\). All but one of the
graphs mentioned in [31] have transitive automorphism groups. Therefore, the problem
considered by us seems to be rather natural. We give a complete list of all vertex-
transitive directed strongly regular graphs with \(n \leq 20\). It turns out there exist up to
isomorphism 51 such graphs which are distributed into 35 so-called equivalence classes
(see below). Ten classes of graphs are new; in particular, we prove the existence of
\((n,k,\mu,\lambda,t)\)-graphs for 3 of 5 sets from Duval’s list where the problem remained open.
For one of such sets we prove the nonexistence of a graph without the use of a com-
puter. Our results together with numerous \((15,5,2,1,2)\)-graphs having an intransitive
automorphism group, recently discovered by Jørgensen [28], give a complete answer
on the question of the existence of small directed strongly regular graphs from Duval’s
list.

This paper consists of 8 sections. All necessary preliminaries related to the main
definitions, the use of cellular algebras, and exploited computer tools are introduced
in Section 2. Section 3 includes a brief summary of the construction of directed
strongly regular graphs. Part of the results presented in Section 3 are new. They
may be of independent theoretical interest. The techniques of computer enumera-
tion are discussed in Section 4; the main results of the enumeration are presented
in Section 5. Section 6 includes the catalogue of graphs; in Section 7 we give a
proof of the nonexistence of a \((16,6,3,1,3)\)-graph. Finally, in Section 8 we dis-
cuss with more details some of our results and a few questions worthy for further
investigations.

We refer to [15,19,42] for the standard definitions related to graphs, permutation
groups, and cellular algebras.
2. Preliminaries

2.1. Strongly regular graphs

Let $\Omega$ be a finite set and $R \subseteq \Omega^2$ be given. $\Gamma = (\Omega, R)$ is a directed graph with vertex set $\Omega$ and arc set $R$. If $R$ is an antireflexive (binary) relation then $\Gamma$ is a graph without loops, i.e., simple (only such graphs are considered below). $\Gamma$ is called an undirected graph if $R$ is symmetric. A graph with a constant output and constant input valency $k$ is called regular of valency $k$.

A strongly regular graph (see [6,7,8,20,24,38]) briefly s.r.g., is the classical notion of an undirected graph $\Gamma$ on $n$ vertices with parameters $(n, k, \lambda, \mu)$ whose adjacency matrix $A = A(\Gamma)$ satisfies

\[ A^2 = kI + \lambda A + \mu(J - I - A), \]  
(2.1) 
\[ AJ = JA = kJ. \]  
(2.2)

Here $J = J_n$ is the matrix whose entries are all equal to 1. It is known (see, e.g., [18]) that an undirected graph is an s.r.g. if and only if it is regular and has 3 distinct eigenvalues.

If $\Gamma$ is an s.r.g. then its complement $\bar{\Gamma} = (\Omega, \bar{R})$ also is an s.r.g. with parameters $(n, n - k - 1, n - 2k + \mu - 2, n - 2k + \lambda)$. Such a pair is equivalent to a symmetric association scheme with two classes.

Let $R^\top = \{(b, a) \mid (a, b) \in R\}$. The existence of a two-class antisymmetric association scheme $(V, \{R, \bar{R}\})$ with $R^\top = \bar{R}$ is equivalent to the existence of a tournament $T = (V, R)$ whose adjacency matrix $A$ satisfies (2.2) and

\[ A^2 = \lambda A + \mu(J - I - A). \]  
(2.3)

In [9] $T$ is called a doubly regular tournament.

2.2. Directed strongly regular graphs

Duval suggested a generalization of the notion of strongly regular graphs in [11]. A simple directed graph $\Gamma$ is called a directed strongly regular graph (d.s.r.g.) with parameters $(n, k, \lambda, \mu, t)$ if its adjacency matrix $A = A(\Gamma)$ satisfies (2.2) and

\[ A^2 = tI + \lambda A + \mu(J - I - A). \]  
(2.4)

This equation can be rewritten as

\[ A^2 + (\mu - \lambda)A - (t - \mu)I = \mu J. \]  
(2.5)

Let $\Omega$ be the vertex set of $\Gamma$. The arc set $R$ of a d.s.r.g. $\Gamma$ can be decomposed uniquely into a symmetric binary relation $R_s$ and an antisymmetric binary relation $R_a$, i.e., $R = R_s \cup R_a$, $R_s \cap R_a = \emptyset$. Here $\Gamma_s = (\Omega, R_s)$ is an undirected graph of valency $t$, while $\Gamma_a = (\Omega, R_a)$ is a directed graph of valency $k - t$. 
It is clear that a d.s.r.g. with \( t = k \) is an s.r.g. and a d.s.r.g. with \( t = 0 \) is a doubly regular tournament. Since there exist d.s.r.g.’s with \( 0 < t < k \) this notion really is a generalization of both s.r.g.’s and doubly regular tournaments. We will call a d.s.r.g. with \( 0 < t < k \) a genuine d.s.r.g. following [31].

Duval has proved that the complement of an \((n,k,\mu,\lambda,t)\)-graph also is a d.s.r.g. with the parameters

\[
\begin{align*}
k' &= (n - 2k) + k - 1, \\
\mu' &= (n - 2k) + \lambda, \\
\lambda' &= (n - 2k) + \mu - 2, \\
t' &= (n - 2k) + t - 1.
\end{align*}
\]

By this reason, in what follows, we will consider only genuine d.s.r.g.’s with \( k \leq (n - 1)/2 \). There is one more type of graphs which can be associated to a given d.s.r.g. which also proved to be a d.s.r.g. as the following observation made by Ch. Pech shows.

**Proposition** (Klin et al. [31, Proposition 5.6]). Let \( \Gamma \) be a d.s.r.g., \( A = A(\Gamma) \) be its adjacency matrix. Let \( \Gamma^\top \) be a graph such that \( A(\Gamma^\top) = A^\top \). Then \( \Gamma^\top \) also is a d.s.r.g. with the same parameters as \( \Gamma \).

In what follows, \( \Gamma^\top \) will be called transposed to d.s.r.g. \( \Gamma \). In general, \( \Gamma \) and \( \Gamma^\top \) are not isomorphic, cf. parameter set #3a in Section 6.1. Nevertheless, we will not distinguish \( \Gamma \) and \( \Gamma^\top \), considering them always to belong to the same equivalence class.

**Theorem** (Duval [11, Main Theorem]). Let \( \Gamma \) be a d.s.r.g. with parameters \((n,k,\mu,\lambda,t)\). Then one of the following holds:

(a) \( \Gamma \) is complete \((A = J - I)\).

(b) \( \Gamma \) is an s.r.g. \((t = k)\).

(c) \( \Gamma \) is a doubly regular tournament \((t = 0)\).

(d) \( \Gamma \) is a genuine d.s.r.g. \((0 < t < k)\), and there exists some positive integer \( d \) for which the following requirements are satisfied:

- \( k(k + (\mu - \lambda)) = t + (n - 1)\mu, \)
- \( (\mu - \lambda)^2 + 4(t - \mu) = d^2, \)
- \( d \mid (2k - (\mu - \lambda)(n - 1)), \)
- \( (2k - (\mu - \lambda)(n - 1))/d \equiv n - 1 \pmod{2}, \)
- \( |(2k - (\mu - \lambda)(n - 1))/d| \leq n - 1. \)

Duval’s paper contains a lot of theoretical results about genuine d.s.r.g.’s and also a list of all feasible parameter sets for genuine directed strongly regular graphs with \( n \leq 20 \) vertices. Table 1 is based on this list.
Table 1
List of feasible parameters and number of nonequivalent realizations

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Duval [11] was a starting point for [31]. Using the notion of coherent algebra, this paper gave examples for almost all feasible parameter sets from Duval’s list. However, some problems remained open, such as the existence of d.s.r.g. with parameter set #10, #11, #13, #15, or #17.

2.3. Coherent algebras

Here, we give a short outline of necessary background from algebraic combinatorics. For more details the reader is referred to [15,31].

The main subject of interest in this paper is the enumeration of d.s.r.g.’s with “high” symmetry. More exactly, we are looking for all d.s.r.g.’s Γ such that the automorphism group Aut(Γ) acts transitively on the vertex set of Γ. Such d.s.r.g.’s are usually called vertex-transitive d.s.r.g.’s.

The adjacency matrix of such a d.s.r.g. necessarily belongs to some class of matrix algebras. Potentially suitable matrix algebras may be defined axiomatically and then efficiently enumerated with the aid of a computer. This is, roughly speaking, the groundwork of our approach. Some extra details are given below.
Let \((G, \Omega)\) be a permutation group and let \((G, \Omega^2)\) be the natural (componentwise) induced action of \(G\) on the set \(\Omega^2\). The orbits of this induced action are called 2-orbits of \((G, \Omega)\) [29,43]. Let 2-orb\((G, \Omega)\) denote the set of all 2-orbits of \((G, \Omega)\).

Suppose that a graph \(I=(\Omega,R)\) is invariant with respect to \((G, \Omega)\), that is \((G, \Omega)\) is a subgroup of the automorphism group Aut\((I)\) of \(I\). Then it is easy to see that \(R\), the set of arcs of \(I\), can be obtained as a union of suitable (disjoint) 2-orbits of \((G, \Omega)\). This simple claim will be reformulated below in terms of matrices.

Let \(K\) be a field (ring), \((G, \Omega)\) be a permutation group of degree \(n=|\Omega|\), and let \(M(g)\) denote the permutation matrix corresponding to \(g \in (G, \Omega)\).

\[\mathcal{V}_k(G, \Omega) := \{A \in K^{n \times n} | \forall g \in (G, \Omega) \ AM(g) = M(g)A\}.\]

Obviously \(\mathcal{V}_k(G, \Omega)\) is a matrix algebra. \(\mathcal{V}_C(G, \Omega)\) is called the centralizer algebra of the permutation group \((G, \Omega)\). \(\mathcal{V}_Z(G, \Omega)\) is called the centralizer ring (or V-ring).

Let 2-orb\((G, \Omega)\) = \(\{R_0, \ldots, R_d\}\). Consider directed graphs \(I_i=(\Omega, R_i)\) and their adjacency matrices \(A_i=A(I_i)\), \(0 \leq i \leq d\). The set \(\{A_i\}_{i=0}^d\) forms a basis of \(\mathcal{V}_k(G, \Omega)\). These graphs (matrices) are called basis graphs (matrices) of \(\mathcal{V}_k(G, \Omega)\), respectively.

Some evident properties of basis matrices are generalized with the use of the following definition.

Let \(\mathcal{W}\) be a matrix algebra of order \(n\) over \(C\) which satisfies

(a) As a vector space \(\mathcal{W}\) has a basis \(\{A_1, \ldots, A_r\}\) of \(\{0,1\}\)-matrices;
(b) \(\forall i \in \{1, \ldots, r\} \exists i' \in \{1, \ldots, r\}: A_i^\top = A_{i'}\);
(c) \(\sum_{i=1}^r A_i = I\).

\(\mathcal{W}\) is called a cellular algebra of order \(n\) and rank \(r\). Usually, it is denoted by \(\langle A_1, \ldots, A_r\rangle\). In addition to the usual matrix operations, each cellular algebra is closed with respect to the Schur–Hadamard product. A submodule of \(\mathcal{W}\) over \(Z\) which is formed by all integer matrices of \(\mathcal{W}\) is called a cellular ring. Each basis matrix \(A_i\) naturally can be interpreted as an adjacency matrix of a suitable (directed) graph \(A_i=A(I_i)\).

If the unit matrix \(I = I_1\) belongs to \(\mathcal{W}\) then it is called a coherent algebra. These definitions were suggested in [21,22,41,44], respectively.

It is clear that centralizer algebras are coherent algebras. However, there are examples of coherent algebras which are not centralizer algebras. Such algebras are called non-Schurian.

A coherent algebra is called a homogeneous coherent algebra (or a cell) if the identity matrix is one of its basis matrices. A homogeneous coherent algebra is the same as a Bose–Mesner algebra (briefly BM-algebra) of an association scheme. It is easy to see that the centralizer algebra of a transitive permutation group is a BM-algebra.

Finally, we emphasize that all above definitions may be reformulated in terms of relations. Then coherent algebras correspond to coherent configurations while cells correspond to association schemes (see, e.g., [1]), or to homogeneous coherent configurations.

Let \((G_1, \Omega)\) and \((G_2, \Omega)\) be permutation groups acting on the same set \(\Omega\). These groups are called 2-equivalent if 2-orb\((G_1, \Omega)\) = 2-orb\((G_2, \Omega)\). In this case the centralizer rings \(\mathcal{V}(G_1, \Omega)\) and \(\mathcal{V}(G_2, \Omega)\) coincide. The maximal group in the class of groups
that are 2-equivalent to \((G, \Omega)\) is called the 2-closure of these groups and is denoted by \((G^{(2)}, \Omega)\). The following formula shows a way to compute the 2-closure. Here for a relation \(R\) on \(\Omega\), \(\text{Aut}(R)\) is the same as \(\text{Aut}(\Gamma)\), where \(\Gamma = (\Omega, R)\) is the (directed) graph with arc set \(R\):

\[
(G^{(2)}, \Omega) = \bigcap_{\Phi \in \text{2-orb}(G, \Omega)} \text{Aut}(\Phi).
\]

It is easy to see that the automorphism groups of all graphs are 2-closed. (Sometimes such groups are called König groups. There exist examples of 2-closed permutation groups which are not König, see [32].)

The pair of functors \(\text{Aut} \dashv \mathfrak{S}\) defines a Galois correspondence between coherent algebras and permutation groups (see [13,15,32] or [33] for details). The Galois-closed objects are centralizer rings and 2-closed permutation groups.

The following paradigm explains how this Galois correspondence may be used for the enumeration of all vertex-transitive graphs with \(n\) vertices:

- Consider all minimal transitive permutation groups of degree \(n\) up to similarity.
- Describe the centralizer algebras of these groups.
- Enumerate all coherent subalgebras of the above centralizer algebras.
- Select a transversal \(T\) from these subalgebras which consists of the set of all isomorphism classes of coherent subalgebras.
- Select a subset \(T'\) from \(T\) consisting of all Schurian algebras.
- Find the automorphism groups of all coherent algebras in \(T'\).

We will get a complete set \(T''\) of all 2-closed transitive permutation groups of degree \(n\) (up to similarity). For each vertex-transitive graph (up to similarity) its adjacency matrix belongs to at least one of the coherent algebras in \(T'\).

It is important to note that this approach requires the preliminary knowledge of minimal transitive permutation groups only. Information about all transitive groups is redundant and yields extra computations.

Let \(H\) be a finite group with the identity element \(e\). Let \(X \subseteq H \setminus \{e\}\). The graph \(\Gamma = \Gamma(H, X) = (H, R)\), where \(R = \{(h, hx) \mid h \in H, x \in X\}\), is called a (directed) Cayley graph over \(H\). The subset \(X\) usually is called the connection set of \(\Gamma\). A Cayley graph \(\Gamma\) may be regarded as a usual (undirected) graph if \(x \in X \Rightarrow x^{-1} \in X\). The automorphism group of any Cayley graph over \(H\) evidently contains a regular subgroup \((H, H)\), that is, \(H\) acts on \(H\) by left shifts. Thus each Cayley graph is vertex-transitive.

The famous Petersen graph may serve as an example of a vertex-transitive graph with the smallest number of vertices which is not a Cayley graph over a suitable group. We shall see later on that the result of our investigations in this paper will consist of mostly Cayley graphs (just a few exceptions will appear).

Let again \(H\) be a finite group and \((H, H)\) a left regular action. Then the centralizer algebra (ring) \(\mathfrak{C}(H, H)\) is antiisomorphic to the group algebra (ring) of \(H\). Hence, there exists a bijection \(\eta : \mathfrak{C}(H, H) \rightarrow \mathbb{Z}[H]\) of the centralizer ring onto the group ring of \(H\). Let \((G, \Omega)\) be a (transitive) permutation group which contains a regular subgroup \((H, \Omega)\). In this situation, we can identify \(H\) with \(\Omega\) in such a way that \((G, H)\) is an
overgroup of \((H,H)\). Then \(\mathfrak{U}(G,H)\) is a subring of \(\mathfrak{U}(H,H)\). Thus \(\mathfrak{S} = \eta(\mathfrak{U}(G,H))\) is a \(\mathbb{Z}\)-submodule of \(\mathbb{Z}[H]\). Due to the structure of \(\mathfrak{U}(G,H)\) it is generated by the images of 2-orb\((G,H)\). It turns out that these images are well-known subrings of \(\mathbb{Z}[H]\) called Schur-rings, or briefly S-rings. An S-ring \(\mathfrak{S}\) over \(H\) satisfies

(S1) \(\mathfrak{S}\) has a basis \(\{T_1, \ldots, T_r\}\) formed of simple quantities.
(S2) \(T_i = e, \sum_{i=1}^{r} T_i = H\).
(S3) \(T_i^{-1} = \{g^{-1} | g \in T_i\} = T'_i\) for some \(i' \in \{1, \ldots, r\}\).

Regarding S-rings, we follow the notation and terminology suggested by Wielandt in his classical book [42]. In particular, for \(X \subseteq H\), a simple quantity \(X\) is the element \(X = \sum_{x \in X} x\) of \(\mathbb{Z}[H]\). There exists an evident one-to-one correspondence between S-rings over \(H\) and cellular subalgebras (subcells) of \(\mathfrak{U}(H,H)\). To each simple quantity \(X\) in \(\mathbb{Z}[H]\) we bijectively associate a Cayley graph \(\Gamma = \Gamma(H,X)\). Sometimes we will use the same notation for a simple quantity and the connection set of the corresponding Cayley graph.

The set of all coherent algebras of the same order \(n\) is closed with respect to intersection. It follows that for each square matrix \(A\) there exists a minimal coherent algebra \(\mathfrak{W}(A)\) which includes \(A\), see, e.g., [33].

**Theorem 2.3.1.** Let \(\Gamma\) be a genuine d.s.r.g. with the adjacency matrix \(A = A(\Gamma)\). Let \(\mathfrak{W}(A)\) be the minimal coherent algebra which includes \(A\). Then

(a) \(\mathfrak{W}(A)\) is noncommutative;
(b) \(\text{rank}(\mathfrak{W}(A)) \geq 6\).

The proof of this theorem can be found in [31] as well as a careful discussion of the notion of \(\mathfrak{W}(A)\), see also [17].

In what follows, this theorem will play a very important role in our computational approach, serving as a criterion for the selection of coherent algebras that are candidates to include an adjacency matrix of a suitable d.s.r.g.

We denote by \(D_n\) the transitive representation of degree \(n\) of the dihedral group of order \(2n\). \(D_n\) has a unique cyclic subgroup \(C_n\); let \(c\) be a generator of \(C_n\). In what follows, \(d\) will denote an involution from \(D_n \setminus C_n\).

It turns out that a lot of d.s.r.g’s can be interpreted as Cayley graphs over dihedral groups. This explains a significant role of dihedral groups in our presentation.

**Example 2.3.2.** According to [11], there exists only one possible set of parameters for a d.s.r.g with 6 vertices (see the restrictions introduced in Section 2.2), namely the set \((6,2,1,0,1)\). We are looking for all vertex-transitive d.s.r.g’s which have this set of parameters.

We need the following elementary information from the theory of permutation groups (see [5,10]): each transitive permutation group of degree 6 contains a regular subgroup.

It is evident that there exist just two (nonsimilar) regular groups of degree 6, representations of \(C_6\) and \(D_3\). The cyclic group is out of our interest (because it is com-
mutative), thus we will only start with the regular dihedral group \((D_3, D_3)\), which has rank 6. All overgroups of \((D_3, D_3)\) have rank < 6 and therefore (due to Theorem 2.3.1) cannot appear as the automorphism group of a genuine d.s.r.g. Nevertheless, implicitly we will use some information about the S-rings over \(D_3\).

Let us set \((D_3, D_3) = \langle (0, 1, 2)(3, 5, 4), (0, 3)(1, 4)(2, 5) \rangle\). For simplicity, we will denote elements of \(D_3\) by numbers 0, 1, ..., 5, referring to the identification of an element of the group with the image of 0 with respect to the action of the element. Note that with this identification elements 3, 4, 5 correspond to the involutions of \(D_3\).

It turns out that there exist the following 6 S-rings over \(D_3\) (up to isomorphism; the total number is 1 + 1 + 3 + 1 + 3 + 1 = 10):

\[
\begin{align*}
\mathcal{S}_1 &= \langle 0, 1, 2, 3, 4, 5 \rangle, \\
\mathcal{S}_2 &= \langle 0, 1, 2, 3, 4, 5 \rangle, \\
\mathcal{S}_3 &= \langle 0, 1, 2, 3, 4, 5 \rangle, \\
\mathcal{S}_4 &= \langle 0, 1, 2, 3, 4, 5 \rangle, \\
\mathcal{S}_5 &= \langle 0, 1, 2, 3, 4, 5 \rangle, \\
\mathcal{S}_6 &= \langle 0, 1, 2, 3, 4, 5 \rangle.
\end{align*}
\]

Now, we are looking for a Cayley graph over \(D_3\) of valency 2 such that symmetric and antisymmetric parts of it have valency 1. There are exactly 6 options to get such a graph, namely the Cayley graphs over \(D_3\) \(\Gamma = \Gamma(D_3, X)\), where \(X = \{a, b\}, a \in \{3, 4, 5\}, b \in \{1, 2\}\). None of the corresponding simple quantities belongs to a proper S-ring over \(D_3\). Thus, the adjacency matrix of each such graph generates the same coherent algebra of rank 6 and still remains to be regarded as “suspicious”.

Finally, simple computations show that all 6 graphs are isomorphic and each of them is really a required d.s.r.g.

One of these 6 graphs, namely \(\Gamma(D_3, \{1, 3\})\), is depicted in Fig. 1. This is the smallest genuine d.s.r.g. We will refer to it a few times later on in our presentation.

2.4. Computer tools

The computer package COCO for the computations with coherent configurations ([14,15]; UNIX implementation by Brouwer) allows, in principle, to carry out the sequence of computations according to our main computational paradigm as it was outlined in Section 2.3.

We used this package in conjunction with groups, algorithms and programming (GAP), see [37], which turns out to be very suitable for the computations related to combinatorial objects and their symmetries. In particular, we were using GRAPE (a share package of GAP, see [39]) and nauty [35] in its interface with GRAPE.
Some recent innovations to COCO, mostly created by Pech, which still exist on the level of experimental programs, were very helpful, especially on the initial stage of the constructive enumeration of all vertex-transitive association schemes of degree \( \leq 20 \).

The use of all these tools will be discussed with more details in Section 4.

3. Some methods of construction of d.s.r.g.’s

The main result of this paper is a complete catalogue of all pairwise nonequivalent d.s.r.g.’s with \( n \leq 20 \), which is presented in Section 6. This catalogue was obtained with the aid of a computer, without use of any preliminary knowledge of concrete d.s.r.g.’s.

Nevertheless, when the catalogue is ready, it is natural to create some computer-independent explanation of the results being obtained. Usually, we are speaking about the interpretation of the graphs which were enumerated. In our case it is convenient to distinguish three different modes of the results:

- the interpretation of some graphs in terms of known constructions of d.s.r.g.’s;
- an insight that some of the generated graphs are, in fact, members of new classes of d.s.r.g.’s;
- the treatment of some graphs as sporadic ones.

We include below brief outlines of the constructions of d.s.r.g.’s which will be exploited in the course of the interpretation of d.s.r.g.’s in our catalogue. For details about known classes of d.s.r.g.’s we refer to [11,17,31]. In our opinion, the results outlined in this section are of independent theoretical interest.
3.1. **Known methods**

3.1.1. **D.s.r.g.’s constructed from Paley graphs**

In [11, Theorem 5.6] Duval proved that a Paley graph $P(q)$, $q$ a prime power with $q \equiv 1 \pmod{4}$, always yields a d.s.r.g. with parameters $(2q, q; q^{-1}, 1/2(q - 1), 1/2(q - 1) - 1, 1/2(q - 1))$.

3.1.2. **D.s.r.g.’s constructed from doubly regular tournaments**

**Lemma** (Duval [11, Theorem 6.1]). Let $T$ denote the adjacency matrix (of order $2\mu + 1$) of a doubly regular tournament, $P$ be the permutation matrix of an involution in $S_{2\mu + 1}$. Then $\Gamma$ with

$$
A(\Gamma) = \begin{pmatrix} T & PT \\ (PT)^\top & T \end{pmatrix}
$$

is a d.s.r.g. with parameters $(2(2\mu + 1), 2\mu, \mu - 1, \mu)$ if and only if $PT = (PT)^\top$.

3.1.3. **Cayley graphs over dihedral groups**

We recall that $C_n$ is the normal cyclic subgroup of order $n$ of a dihedral group $D_n$.

**Lemma** (Klin et al. [31, Lemma 6.1]). Let $n$ be odd and $X, Y \subseteq C_n$ such that

(a) $X + X^{(-1)} = C_n - e$,
(b) $Y \cdot Y^{(-1)} - X \cdot X^{(-1)} = eC_n$, $e \in \{0, 1\}$.

Let $d \in D_n \setminus C_n$. Then the Cayley graph $\Gamma$ which corresponds to $\gamma = X + dY$ is a d.s.r.g. with parameters $(2n, n - 1 + e, (n - 1)/2 + e, (n - 3)/2 + e, (n - 1)/2 + e)$.

In the next subsection a new analogue of this lemma will be formulated and proved also for even $n$. This proof is similar to the proof in [31].

3.1.4. **Subalgebras of flag algebras**

An incidence structure $\mathcal{I} = (\mathcal{P}, \mathcal{B}, \mathcal{F})$ is a triple which consists of a set $\mathcal{P}$ of points, a set $\mathcal{B}$ of blocks, and of a set $\mathcal{F} \subseteq \mathcal{P} \times \mathcal{B}$ of flags. We consider uniform regular incidence structures without repeated blocks, see [25] for further definitions. A uniform regular incidence structure $\mathcal{I}$ is called a balanced incomplete block design (BIBD) if each pair of distinct points is incident with the same number $\lambda$ of blocks.

In what follows, we shall consider the special case of Steiner designs where $\lambda = 1$. A few infinite classes of d.s.r.g.’s were introduced in [31, Sections 7,8]. The vertices of these graphs are flags of Steiner designs. The exposition is based on the use of so-called flag coherent algebras. Note that our Example 2.3.2 appears as the first member of one of such classes if we consider a degenerate case of the Steiner design with 3 points.
3.1.5. Wreath products of coherent algebras

For given permutation groups \((G_1, \Omega_1)\) and \((G_2, \Omega_2)\), we denote the wreath product of \((G_1, \Omega_1)\) and \((G_2, \Omega_2)\) by \((G_1 \wr G_2, \Omega_1 \times \Omega_2)\). By definition (see [32]), this is a group of order \(|G_1||G_2|^{|\Omega_1|\}\. It is isomorphic to the semidirect product of \(G_2^{|\Omega_1|}\) and \(G_1\). Let \(\times\) denote the Kronecker product of matrices.

Proposition (Klin et al. [31, Proposition 9.1]). Let \(\Gamma\) be a genuine d.s.r.g. with parameters \((n, k, \mu, \lambda, t)\) and adjacency matrix \(A\). Let \(\mathcal{M}_1 = \mathcal{M}(A)\) and let \((G_1, \Omega_1)\) be the automorphism group of \(\mathcal{NUL}\). Then

(a) \(B' = A \times J_m\) is the adjacency matrix of a genuine d.s.r.g. \(\Gamma^*\) if and only if \(t = \mu\); in this case \(\Gamma^*\) is an \((nm, km, \mu m, \lambda m, tm)\)-graph;

(b) \(B'' = A \times J_m + I_n \times (J_m - I_m)\) is the adjacency matrix of a genuine d.s.r.g. \(\Gamma^{**}\) if and only if \(\lambda = t - 1\); in this case \(\Gamma^{**}\) is an \((nm, (k+1)m - 1, \mu m, (t+1)m - 2, (t+1)m - 1)\)-graph;

(c) \(\Gamma^*\) and \(\Gamma^{**}\) are invariant with respect to the wreath product \((G_1 \wr S_m, \Omega_1 \times \Omega_2)\) where \(\Omega_2 = \{1, 2, \ldots, m\}\);

(d) The adjacency matrices of graphs \(\Gamma^*\) and \(\Gamma^{**}\) generate the same cellular algebra, namely the wreath product \(\mathcal{M}_1[\mathcal{M}_2]\), where \(\mathcal{M}_2 = \langle I_m, J_m - I_m \rangle\).

The proof of (a) and (b) was given in [11]. We refer to [31] for more details and definitions related to this proposition.

3.1.6. “Repetition” of graphs

The following lemma goes back to Duval’s considerations in [11, Section 7]. Sometimes such “repeated graphs” appear again in our table.

Lemma. Let \(\Gamma\) be an \((n, k, \mu, \lambda, t)\)-graph with \(t = \lambda + 1\) and \(n = 2(2k - \mu - \lambda)\). Let \(A = A(\Gamma)\) be its adjacency matrix and \(\tilde{A} = J - I - A\). Then the graph \(\Delta\) with adjacency matrix

\[
\begin{pmatrix}
A & \tilde{A} \\
\tilde{A} & A
\end{pmatrix}
\]

is a d.s.r.g. with parameters \((2n, n - 1, 2(k - t), 2(k - \mu), 2(k - \mu) + 1)\).

Proof. Let \(B = A(\Delta)\). Then with \(A\tilde{A} = \tilde{A}A\)

\[
B^2 = \begin{pmatrix}
A^2 + \tilde{A}^2 & 2A\tilde{A} \\
2A\tilde{A} & A^2 + \tilde{A}^2
\end{pmatrix}.
\]

Let \((n, k', \mu', \lambda', t')\) denote the parameters of \(\tilde{A}\). Then

\[
A^2 + \tilde{A}^2 = (t - \mu)I + (\lambda - \mu)A + \mu J
\]

\[
+ (t' - \mu')I + (\lambda' - \mu')(J - I - A) + \mu' J
\]

\[
= (2t - 2\mu + 1)I + 2(\lambda - \mu + 1)A + (n - 2k + 2\mu - 2)J
\]
and

\[ 2A\tilde{A} = 2(J - I - \tilde{A})\tilde{A} \]
\[ = 2k'J - 2\tilde{A} - 2((t' - \mu')I + (\lambda' - \mu')\tilde{A} + \mu'J) \]
\[ = (\mu' - t')I + (2\mu' - 2\lambda' - 2)\tilde{A} + (2k' - 2\mu')J \]
\[ = (\lambda - t + 1)I + (2\lambda - 2\mu + 2)\tilde{A} + (2k - 2\lambda - 2)J. \]

With \( t = \lambda + 1 \) the coefficient of \( I \) is zero and

\[ 2A\tilde{A} = (2\lambda - 2\mu + 2)\tilde{A} + 2(k - \lambda - 1)J. \]

\( n = 2(2k - \mu - \lambda) \) implies that \( n - 2k + 2\mu - 2 = 2(k - \lambda - 1) \), hence

\[ B^2 = (2t - 2\mu + 1)I_{2n} + 2(\lambda - \mu + 1)B + 2(k - \lambda - 1)I_{2n}, \]

i.e., using (2.5) and \( t = \lambda + 1 \), \( B^2 = t_BI + \lambda_BB + \mu_B\tilde{B} \) where

- \( t_B - \mu_B = 2(t - \mu) + 1 \)
- \( \lambda_B - \mu_B = 2(t - \mu) \)
- \( \mu_B = 2(k - t) \).

Thus, \( \Delta \) is a d.s.r.g. with parameters \( (2n, n - 1, 2k - \mu, 2(k - \mu) + 1) \). \( \square \)

Note that \( \Delta \) also fulfills the premises of Lemma 3.1.6. Hence, we actually get a series. Quite many parameter sets in the Duval’s list fulfill these requirements, such as \#1, \#2, \#3, \#6, \#8, \#12, \#13, \#18, \#19, and \#23.

### 3.2. New methods

#### 3.2.1. Cayley graphs over dihedral groups

In this section we present a new result about Cayley d.s.r.g.’s over \( D_n \), which is an analogue of Lemma 3.1.3. A particular case of it first appeared in [23].

**Lemma.** Let \( n \) be even, \( c \in C_n, c \neq e \), be an involution and \( X, Y \subseteq C_n \) such that

1. \( X + X^{(-1)} = C_n - e - e \)
2. \( Y = X \) or \( Y = X^{(-1)} \)
3. \( Xc = \{xc \mid x \in X\} = X^{(-1)} \).

Let \( d \in D_n \setminus C_n \). Then the Cayley graph \( \Gamma \) which corresponds to \( \gamma = X + dY + d \) is a d.s.r.g. with parameters \( (2n, n - 1, n/2 - 1, n/2 - 1, n/2) \).

**Proof.** Since \( \gamma' = X^{(-1)} + dX + d \) defines the graph with the transposed adjacency matrix of \( \gamma = X + dX + d \) it suffices to prove the lemma for the case \( Y = X \):

\[ \gamma^2 = (X + dX + d)^2 \]
\[ \begin{align*}
&= X \cdot X + d \cdot X^{(-1)} \cdot X + d \cdot X \cdot X + X^{(-1)} \cdot X + X^{(-1)} \\
&+ d \cdot X + X + \epsilon \\
&= (e + d) (X^2 + X^{(-1)} \cdot X + X + X^{(-1)}) + \epsilon \\
&= (e + d) (X + \epsilon) (X + X^{(-1)}) + \epsilon \\
&= (e + d) (X + \epsilon) (C_n - \epsilon - \epsilon) + \epsilon \\
&= (e + d) \left( \frac{n}{2} C_n - (X + \epsilon) - (X^{(-1)} + \epsilon) \right) + \epsilon \\
&= (e + d) \left( \frac{n}{2} - 1 \right) C_n + \epsilon \\
&= \left( \frac{n}{2} - 1 \right) D_n + \epsilon.
\end{align*} \]

Hence with (2.5)

\[ \begin{align*}
\mu &= \frac{n}{2} - 1, \\
\mu - \lambda &= 0, \\
t - \mu &= 1
\end{align*} \]

which proves the lemma. \( \square \)

**Remark.** For each even \( n \leq 10 \) the graphs constructed according to the above lemma form just one equivalence class. This changes for larger \( n \), i.e., the graph on 24 vertices corresponding to \( \gamma = X + d + dX, \ X = \{c, c^2, c^4, c^5\} \), has automorphism group \( D_{12} \)

wheras \( \gamma' = X' + d + dX', \ X' = \{c^3, c^7, c^8, c^{10}, c^{11}\} \), has an automorphism group of order 72.

### 3.2.2. Graphs derived from double Paley designs

In this subsection we shed a new light onto some links among a few known series of combinatorial objects. This is the reason why we regard the material submitted here as relevant to new methods.

Let \( p \) be an odd prime number, \( q = p^m, \ m \geq 1 \). Let \( F = \mathbb{F}_q \) be a finite field consisting of \( q \) elements, \( \mathbb{F}^* = \mathbb{F} \setminus \{0\} \). Consider \( G = AGL(1, q) = \{ \mu_{a,b} \mid a \in \mathbb{F}^*, \ b \in \mathbb{F} \} \), where \( \mu_{a,b} : \mathbb{F} \to \mathbb{F} \) is the function such that for \( x \in \mathbb{F} \) \( x^{\mu_{a,b}} = ax + b \). It is well-known that \( (G, \mathbb{F}) \) is a doubly transitive permutation group of order \( q(q - 1) \) and of degree \( q \). This group is usually called the one-dimensional *affine group* over \( \mathbb{F}_q \).

Let \( S = \{ x^2 \mid x \in \mathbb{F}, \ x \neq 0 \} \) and \( N = \mathbb{F} \setminus (S \cup \{0\}) \) be the sets of nonzero squares and nonsquares in \( \mathbb{F} \), respectively. For \( X \subseteq \mathbb{F} \) and \( c \in \mathbb{F} \) let \( X + c = \{ x + c \mid x \in X \} \).

Let \( \mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \), where \( \mathcal{B}_1 = \{ S + c \mid c \in \mathbb{F} \}, \ \mathcal{B}_2 = \{ N + c \mid c \in \mathbb{F} \} \). Consider an incidence structure \( \mathcal{D} = (\mathbb{F}, \mathcal{B}) \) with inclusion as incidence relation. (We again refer to [25] for all definitions related to the incidence structures.)
Proposition. (a) \( D \) is a 2-design with parameters \((v, b, r, k, \lambda) = (q, 2q, q - 1, (q - 1)/2, (q - 3)/2)\); 
(b) \( G \leq \text{Aut}(D)\); 
(c) \( G \) acts as a transitive rank 6 permutation group on the set \( B \) of blocks of the design \( D \).

Proof. (a) The proof of (a) easily follows from (b) and from 2-transitivity of \((G, \mathbb{F})\).
(b) The proof of (b) is evident.
(c) It is easy to see that \( G \) acts transitively on \( B \). Let \( G_S = G_N \) denote the stabilizer of \( S \) in \( G \). Obviously, \( \mu_t \in G_S \) for all nonzero squares \( t \) and \( \mu_{t, 0} \notin G_S \) for nonsquares \( t \). Hence \( G_S \) acts transitively on squares and on nonsquares. Thus, \( G_S \) has six orbits on \( B \), which bijectively correspond to the following 2-orbits of \((G, \mathbb{F})\): \( \{(S, S)\}^{(G, \mathbb{F})}, \{(S, N)\}^{(G, \mathbb{F})}, \{(S, S + x) \mid x \in S\}^{(G, \mathbb{F})}, \{(S, N + x) \mid x \in S\}^{(G, \mathbb{F})}, \{(S, S + x) \mid x \in N\}^{(G, \mathbb{F})}, \) and \( \{(S, N + x) \mid x \in N\}^{(G, \mathbb{F})} \). □

Let us now consider directed graphs \( I_i = (\mathbb{B}, \mathbb{R}_i) \), \( 1 \leq i \leq 4 \), with the vertex set \( \mathbb{B} \) and the arc set \( \mathbb{R}_i \), where

\begin{itemize}
  \item \( \mathbb{R}_1 = \{(S, S + x) \mid x \in S\}^{(G, \mathbb{F})} \cup \{(S, N + x) \mid x \in S\}^{(G, \mathbb{F})} \);
  \item \( \mathbb{R}_2 = \{(S, S + x) \mid x \in S\}^{(G, \mathbb{F})} \cup \{(S, N + x) \mid x \in N\}^{(G, \mathbb{F})} \);
  \item \( \mathbb{R}_3 = \{(S, S + x) \mid x \in N\}^{(G, \mathbb{F})} \cup \{(S, N + x) \mid x \in S\}^{(G, \mathbb{F})} \);
  \item \( \mathbb{R}_4 = \{(S, S + x) \mid x \in N\}^{(G, \mathbb{F})} \cup \{(S, N + x) \mid x \in N\}^{(G, \mathbb{F})} \).
\end{itemize}

We stress that \( \mathbb{R}_i \) consists of two of the six 2-orbits of the transitive action of \((G, \mathbb{F})\). Therefore, by definition our graphs are invariant with respect to \((G, \mathbb{F})\).

It turns out that \( I_1 \) is the transposed graph of \( I_2 \), and \( I_3 \) is transposed to graph \( I_4 \).

Proposition. \( I_i \) are \((2q, q - 1, (q - 1)/2, (q - 3)/2, (q - 1)/2)\)-graphs.

Proof. We prove the proposition for \( I_1 \).

It is easily verified that \((S + x, S + y) \in \mathbb{R}_1 \), \((S + x, N + y) \in \mathbb{R}_1 \) for \( x, y \in \mathbb{F} \) if and only if \( y - x \in S \). Similarly, \((N + x, S + y) \in \mathbb{R}_1 \), \((N + x, N + y) \in \mathbb{R}_1 \) if and only if \( y - x \in N \). Since \( G \) acts transitively on \( B \) we can consider \( S \) as a starting point for counting the values of \( k \), \( t \), \( \lambda \), and \( \mu \). With the well-known fact that \(|S| = (q - 1)/2\) it follows \( k = 2(q - 1)/2 = q - 1 \).

Let \( x \in S \). With an arc \((S, S + x) \in \mathbb{R}_1 \) also \((S + x, S) \in \mathbb{R}_1 \) if and only if \(-x \in S \). On the other hand, both \((S, N + x) \in \mathbb{R}_1 \) and \((N + x, S) \in \mathbb{R}_1 \) if and only if \(-x \notin S \). Hence for each \( x \in S \) either \((S, S + x)\) or \((S, N + x)\) is directed. Thus \( t = |S| = (q - 1)/2 \).

Let \((S, S + y) \in \mathbb{R}_1 \). There is a one-to-one correspondence between \( x \in S \) with \( x \neq y \) and vertices \( P \) such that \((S, P) \in \mathbb{R}_1 \), \((P, S + y) \in \mathbb{R}_1 \). For if \( y - x \in S \) then \((S, S + x) \in \mathbb{R}_1 \) and \((S + x, S + y) \in \mathbb{R}_1 \); also \((S, N + x) \in \mathbb{R}_1 \) but \((N + x, S + y) \notin \mathbb{R}_1 \). On the other hand, if \( y - x \in N \) then \((S, N + x) \in \mathbb{R}_1 \) and \((N + x, S + y) \in \mathbb{R}_1 \), and since \((S + x, S + y) \notin \mathbb{R}_1 \) the former is the only path of length two from \( S \) to \( S + y \) via \( x \).

Now consider an arc \((S, N + y) \in \mathbb{R}_1 \). In the same way as above, it follows that the paths of length two from \( S \) to \( N + y \) are in one-to-one correspondence with squares \( x \in S \), \( x \neq y \). Hence \( \lambda = (q - 1)/2 - 1 \).
Let \((S, S + y) \notin \mathcal{R}_1\) (or, similarly, \((S, N + y) \notin \mathcal{R}_1\)). Hence \(y \in N\). Using the same reasoning as above it follows that there is a one-to-one correspondence between squares \(x \in S\) and vertices \(P \in \mathcal{B}\) such that \((S, P) \in \mathcal{R}_1\) and \((P, S + y) \in \mathcal{R}_1\). Thus \(\mu = \lambda + 1 = (q - 1)/2\). □

**Remark.** We did not investigate whether the above proposition gives examples which are not covered by other general constructions, cf. Sections 3.1.1 and 3.1.2. However, one of the advantages of this method is providing an automorphism group of the resulting graph.

### 3.2.3. Series on \((q - 1)(q^2 - 1)\) points

Let \(q\) be a prime power, \(q > 2\). \(\mathbb{F} = \mathbb{F}_q\) be the finite field with \(q\) elements and \(V\) be the two-dimensional vector space over \(\mathbb{F}\). 0 be the zero vector of \(V\). Let \(L\) denote all affine lines in \(V\). Let \(V^* = V \setminus \{0\}\), and \(L^*\) be the set of all affine lines in \(V\) which do not include 0. We consider the incidence structure \(\mathcal{D} = (V^*, L^*)\) with natural incidence relation, namely inclusion. Let \(\mathcal{A}\) be the set of antiflags \((l, P)\) of \(\mathcal{D}\) such that \(l\) is an affine line not including the zero vector, and \(P \neq 0\) is a point on the line parallel to \(l\) which includes the zero vector, that is

\[
\mathcal{A} = \{(l, P) \mid l \in L^* \land P \neq 0 \land \exists y \in L \setminus L^* (0 \in l' \land l' \parallel l \land P \in l')\}.
\]

Evidently, \(|\mathcal{A}| = (q - 1)(q^2 - 1)\). On \(\mathcal{A}\) we introduce eight natural relations. We emphasize that all relations \(R_i\) are regular, \(\text{val}(R_i)\) will denote the valency of \(R_i\). Let \(f = (p, P)\) and \(g = (r, R)\) be antiflags in \(\mathcal{A}\). \(PR\) will denote the affine line determined by \(P\) and \(R\), \(P \neq R\):

- \(R_0 = \{(f, g) \mid P = R \land p = r\}\) and \(\text{val}(R_0) = 1\);
- \(R_1 = \{(f, g) \mid P = R \land p \parallel r\}\) and \(\text{val}(R_1) = q - 2\);
- \(R_2 = \{(f, g) \mid P \neq R \land 0 \in \text{PR} \land p = r\}\) and \(\text{val}(R_2) = q - 2\);
- \(R_3 = \{(f, g) \mid P \neq R \land 0 \in \text{PR} \land p \parallel r\}\) and \(\text{val}(R_3) = (q - 2)^2\);
- \(R_4 = \{(f, g) \mid P \neq R \land 0 \notin \text{PR} \land P \in r \land R \notin p\}\) and \(\text{val}(R_4) = q(q - 2)^2\);
- \(R_5 = \{(f, g) \mid P \neq R \land 0 \notin \text{PR} \land P \notin r \land R \in p\}\) and \(\text{val}(R_5) = q(q - 2)\);
- \(R_6 = \{(f, g) \mid P \neq R \land 0 \notin \text{PR} \land P \in r \land R \in p\}\) and \(\text{val}(R_6) = q(q - 2)\);
- \(R_7 = \{(f, g) \mid P \neq R \land 0 \notin \text{PR} \land P \in r \land R \notin p\}\) and \(\text{val}(R_7) = q\).

Now we consider the directed graphs \(\Gamma_i = (\mathcal{A}, \mathcal{R}_i)\) where

- \(\mathcal{R}_1 = R_1 \cup R_5 \cup R_7\);
- \(\mathcal{R}_2 = R_1 \cup R_6 \cup R_7\);
- \(\mathcal{R}_3 = R_2 \cup R_5 \cup R_7\);
- \(\mathcal{R}_4 = R_2 \cup R_6 \cup R_7\).

We note that \(A(\Gamma_1) = A(\Gamma_2)^\top\) and \(A(\Gamma_3) = A(\Gamma_4)^\top\).

**Proposition.** \(\Gamma_i\) are \(((q - 1)(q^2 - 1), q^2 - 2, q, 2q - 3, 2q - 2)\)-graphs.
We omit the proof which is based on a routine inspection of representatives of the seven 2-orbits $R_1, \ldots, R_7$.

**Remark.** The method we use is similar to the methods developed in [36].

### 3.2.4. Series on $2q^2$ points

Let $V$ be a two-dimensional vector space over $GF(q)$. $C_1$, $C_2$, and $C_3$ be distinct classes of parallel affine lines in $V$. We assume some fixed ordering on the lines in each class. Let $\pi$ be a permutation on the (indices of) lines in a class, i.e., $\pi \in S_q$.

We now define six relations on $\mathcal{V} = V \times \{1, 2\}$. Let $x^{(i)}$ denote $x \times \{i\} \in V \times \{1, 2\}$ for $x \in V$, $i \in \{1, 2\}$.

- $R_1 = \{(x^{(1)}, y^{(1)}) \mid x, y \in V \land \exists l \in C_1, x, y \in l\}$;
- $R_2 = \{(x^{(2)}, y^{(2)}) \mid x, y \in V \land \exists l \in C_1, x, y \in l\}$;
- $R_3 = \{(x^{(1)}, y^{(1)}) \mid x, y \in V \land \exists l \in C_2, x, y \in l\}$;
- $R_4 = \{(x^{(2)}, y^{(2)}) \mid x, y \in V \land \exists l \in C_2, x, y \in l\}$;
- $R_5 = \{(x^{(1)}, y^{(1)}) \mid x, y \in V \land \exists i \in \{1, \ldots, q\}, (x \in l_i \in C_1 \land y \in l_i \in C_2)\}$;
- $R_6 = \{(x^{(2)}, y^{(2)}) \mid x, y \in V \land \exists i \in \{1, \ldots, q\}, (x \in l_i \in C_1 \land y \in l_i \in C_2)\}$.

In other words, $R_1$ and $R_3$ denote the adjacency relation of the affine lines from $C_1$ and $C_2$ in $V \times \{1\}$. $R_2$ and $R_4$ denote this adjacency for $V \times \{2\}$. $R_5$ and $R_6$ define adjacencies from points in $V \times \{1\}$ with points in $V \times \{2\}$ and vice versa. $R_5$ describes that all points in $V \times \{1\}$ on line number $i$ of class $C_1$ are adjacent to all points in $V \times \{2\}$ on line number $i$ of class $C_2$. Similarly, $R_6$ means that all points in $V \times \{2\}$ on line $i$ of class $C_1$ are adjacent to all points in $V \times \{1\}$ on line number $i$ of class $C_2$. Finally, the graph depends on the choice of $\pi$, see graphs #17a, #17b.

**Proposition.** Let $\mathcal{R} = R_1 \cup R_2 \cup R_3 \cup R_4 \cup R_5 \cup R_6$. Then $\Gamma = (\mathcal{V}, \mathcal{R})$ is a d.s.r.g. with parameters $(2q^2, 3q - 2, 3, q - 1, 2q - 1)$.

We omit the rather routine proof of this proposition.

### 4. Strategy of enumeration

Group-theoretical background of our results is the knowledge of all minimal transitive permutation groups of degree $\leq 20$. In fact, all transitive groups of small degrees are known, see [26] and the corresponding GAP catalogues (GAP 3.4.4 or later).

We used these groups in the course of the enumeration of all vertex-transitive association schemes of small degrees. The latter problem is of independent wider interest; its solution will be presented in works in progress (one of them jointly with Pech, see also [16, 17]). There, all S-rings over groups of order $\leq 31$ are classified, and all other vertex-transitive association schemes of order $\leq 24$ (that is those, that do not have at least one S-ring interpretation) are determined.

Knowledge of all vertex-transitive association schemes of degree $n$ results in the enumeration of all directed vertex-transitive graphs with $n$ vertices. The process of
such enumeration, which uses the principle of inclusion and exclusion, was suggested in [30], see also [15]. By constructive enumeration, following [12], we understand the generation of a transversal of all isomorphism classes of vertex-transitive graphs. Here we use a constructive version of the similar inclusion/exclusion procedure (see also [34]).

In principle, we may look through all vertex-transitive graphs of prescribed valency $k$ with $n$ vertices, and check whether a graph is a d.s.r.g. with prescribed parameters $(n,k,\mu,\lambda,t)$. In fact, we restricted ourselves only to the consideration of noncommutative association schemes. (The knowledge of all other schemes was used only implicitly, in order to get the graphs which generate a given scheme.) A transversal of pairwise non-isomorphic graphs was constructed with the aid of a special normalizer algorithm (by Pech), which allows the determination of the normalizer of a 2-closed permutation group $(G,\Omega)$ in the symmetric group $S(\Omega)$. A more comprehensive treatment of this problem can be found in [40], which also is incorporated into GAP.

5. General results of enumeration

In [11], Duval presented a list of all possible feasible parameters of d.s.r.g.’s with $n \leq 20$. This list includes 23 sets. In his list Duval mentioned all cases of existence known to him; the problem of the complete enumeration of graphs (up to isomorphism) was not considered. For 7 sets the problem of the existence of a d.s.r.g. remained open.

Duval’s list was revised in [31]. For one set the authors proved the existence of a d.s.r.g.; for one more set the nonexistence was proved, using a combination of combinatorial arguments with techniques from linear algebra.

In [31] a first step was also made towards the enumeration of d.s.r.g.’s up to isomorphism: namely, the authors listed all cases known to them, where there exist at least two nonisomorphic d.s.r.g.’s. Because the use of computers was beyond the scope of [31], more detailed information about the isomorphism classes was not obtained. Finally, for 5 feasible sets of parameters the problem of existence was left open in [31].

These open cases served for us as a challenge to start the investigations presented in this paper. As a result, we proved the existence of a d.s.r.g. for further three open cases and established nonexistence for one case. In fact, we were able (using the methodology outlined in Section 4) to determine all equivalence classes of vertex-transitive d.s.r.g.’s with $n \leq 20$. It turns out that there exist exactly 35 such classes. Altogether these classes include 51 pairwise non-isomorphic d.s.r.g.’s.

Duval’s list of feasible parameters, now revisited for the second time after [31], is presented in Table 1. Each line of the table includes: the number of a set (according to Duval), the corresponding parameters, information about the existence/nonexistence in the three papers, and remarks. “+” means existence of at least one d.s.r.g. in [11]; while the sign “?” means that the question of existence still remained open in the corresponding paper. The numbers in the column labeled [31] are the lower bounds of nonequivalent d.s.r.g.’s known to the authors of that paper. The word “none” means
that the fact of nonexistence was proved in the corresponding paper. The number in the next-to-last column denotes the total number of nonequivalent d.s.r.g.’s, as established in this paper.

**Remark.** (a) In case #9 there exists an example of a d.s.r.g. which was constructed by Duval. It was shown in [31] that this example by Duval has an intransitive automorphism group. Our computations show that in this case a vertex-transitive d.s.r.g. does not exist.

(b) L. Jørgensen recently constructed intransitive examples in case #10, see Section 8.

(c) The proof of the non-existence of a d.s.r.g. in case #11 is presented in Section 7.

(d) The two nonisomorphic examples in case #21, which were submitted in [31], are in fact equivalent via the transposition of their adjacency matrices. Thus in this case, we get only one class of equivalence (in terms accepted in this paper).

More detailed information about the results of the enumeration is summarized in Table 2. Here only those feasible sets of parameters are listed, for which at least one vertex-transitive d.s.r.g. exists. Each graph is identified by a number that refers to Duval’s list and a letter that distinguishes it from nonequivalent graphs with identical parameters.

Each line of the table together with the identification number of a d.s.r.g. includes: the parameters, rank of the automorphism group Aut(\( \Gamma \)) (which is equal for all our graphs to the rank of \( \mathfrak{M}(A), A = A(\Gamma) \)), name of Aut(\( \Gamma \)), the order of Aut(\( \Gamma \)), references to Section 6 where the description of \( \Gamma \) appears, whether \( \Gamma \) and \( \Gamma^{\top} \) are isomorphic, whether \( \Gamma \) is a Cayley graph, and remarks. We use rather standard names for the groups (except for the wreath products, in which case the notation coincides with [15,31,32]).

The representatives of all the vertex-transitive d.s.r.g.’s with \( n \leq 20 \) are collected in the next section. Altogether we get 35 equivalence classes of d.s.r.g.’s.

**6. Catalogue of d.s.r.g.’s**

In this section we briefly describe representatives of each of the 35 equivalence classes of vertex-transitive d.s.r.g.’s with \( n \leq 20 \).

The representation of each graph is usually given in group-theoretical terms. Most of the graphs are Cayley graphs over a suitable group \( H \). To represent a graph in such a case we describe its connection set \( X \) via certain generators of \( H \). If \( H \) is a dihedral group, then we use a notation for generators of \( D_n \) as in Section 2.3. In a few cases, when a d.s.r.g. \( \Gamma \) is not a Cayley graph, certain ad hoc notation is introduced. We include a few pictorial descriptions of graphs, when a picture provides the reader some additional essential information.

Some extra comments, if necessary, are presented. In particular, we show all the groups over which a given graph can be represented as a Cayley graph.

The sequence in which the graphs are considered in this section does not coincide with the lexicographical order of the identification numbers of graphs.
Table 2
List of feasible parameters and number of nonequivalent realizations

| #  | n  | k | µ | λ | t | Rank | Aut(Γ) | |Aut(Γ)| | Sec. | Γ ≥ Γᵀ | C.G. | Remarks |
|----|----|---|---|---|---|------|--------|--------|------|-------|------|------|---------|
| 1  | 6  | 2 | 1 | 0 | 1 | 6    | D₁     | 6      | 6.2  | Yes   | Yes  | [11] |          |
| 2  | 8  | 3 | 1 | 1 | 2 | 8    | D₄     | 8      | 6.2  | Yes   | Yes  | [11] |          |
| 3a | 10 | 4 | 2 | 1 | 2 | 6    | Z₅ ≻ Z₄ | 20     | 6.1  | No    | No   | [11] |          |
| 3b | 10 | 4 | 2 | 1 | 2 | 10   | D₀     | 10     | 6.2  | Yes   | Yes  | [11] |          |
| 4  | 12 | 3 | 1 | 0 | 1 | 7    | S₀     | 24     | 6.3  | Yes   | Yes  | [11] |          |
| 5  | 12 | 4 | 2 | 0 | 2 | 7    | D₁ ∩ S₁ | 384   | 6.4  | Yes   | Yes  | [11] |          |
| 6a | 12 | 5 | 2 | 2 | 3 | 7    | D₃ ∩ S₂ | 384   | 6.4  | Yes   | Yes  | [11] |          |
| 6b | 12 | 5 | 2 | 2 | 3 | 7    | S₀     | 24     | 6.3  | No    | Yes  | [31] |          |
| 6c | 12 | 5 | 2 | 2 | 3 | 12   | D₀     | 12     | 6.2  | Yes   | Yes  |      |          |
| 8a | 14 | 6 | 3 | 2 | 3 | 6    | Z₇ ≻ Z₆ | 42     | 6.2  | No    | Yes  | [11] |          |
| 8b | 14 | 6 | 3 | 2 | 3 | 14   | D₁     | 14     | 6.2  | Yes   | Yes  | [11] |          |
| 12 | 16 | 7 | 2 | 4 | 5 | 9    | D₄ ∩ S₁ | 2¹¹   | 6.4  | Yes   | Yes  | [11] |          |
| 13a| 16 | 7 | 3 | 3 | 4 | 8    | GL(2,3) | 48     | 6.5  | No    | Yes  |      |          |
| 13b| 16 | 7 | 3 | 3 | 4 | 16   | D₆     | 16     | 6.2  | Yes   | Yes  |      |          |
| 14 | 18 | 4 | 1 | 0 | 3 | 7    | Z₃ ∩ (S₂ ∩ S₁)ₚ₀₅ | 108   | 6.6  | Yes   | Yes  | [11] |          |
| 15 | 18 | 5 | 1 | 2 | 3 | 8    | (S₁ × S₁) × Z₂ | 72    | 6.7  | No    | Yes  |      |          |
| 16 | 18 | 6 | 3 | 0 | 3 | 7    | D₃ ∩ S₁ | 6⁷    | 6.4  | Yes   | Yes  | [11] |          |
| 17a| 18 | 7 | 3 | 2 | 5 | 10   | S₂²   | 36     | 6.5  | No    | Yes  |      |          |
| 17b| 18 | 7 | 3 | 2 | 5 | 10   | E₀ ≻ Z₄ | 36     | 6.5  | No    | No   |      |          |
| 18 | 18 | 8 | 3 | 4 | 5 | 7    | D₁ ∩ S₁ | 6⁷    | 6.4  | Yes   | Yes  | [11] |          |
| 19a| 18 | 8 | 4 | 3 | 4 | 6    | (E₀ ≻ Z₅) ≻ Z₂ | 144   | 6.1  | No    | No   | [11] |          |
| 19b| 18 | 8 | 4 | 3 | 4 | 10   | (Z₁ ∩ Z₃) × Z₂ | 162   | 6.2  | No    | Yes  | [11] |          |
| 19c| 18 | 8 | 4 | 3 | 4 | 10   | E₀ ≻ Z₄ | 36     | 6.1  | Yes   | No   | [11] |          |
| 19d| 18 | 8 | 4 | 3 | 4 | 18   | D₀     | 18     | 6.2  | Yes   | Yes  | [11] |          |
| 19e| 18 | 8 | 4 | 3 | 4 | 18   | D₀     | 18     | 6.2  | No    | Yes  | [11] |          |
| 20 | 20 | 4 | 1 | 0 | 1 | 7    | S₀     | 120   | 6.3  | Yes   | Yes  | [11] |          |
| 21 | 20 | 7 | 2 | 3 | 4 | 7    | S₂      | 120   | 6.3  | No    | Yes  | [31] |          |
| 22a| 20 | 8 | 4 | 2 | 4 | 7    | F₄⁴ ∩ S₂ | 5 × 2¹² | 6.4  | No    | Yes  | [11] |          |
| 22b| 20 | 8 | 4 | 2 | 4 | 11   | D₁₀ ∩ S₂ | 5 × 2¹¹ | 6.4  | Yes   | Yes  | [11] |          |
| 22c| 20 | 8 | 4 | 2 | 4 | 20   | F₄⁴    | 20     | 6.7  | No    | Yes  |      |          |
| 23a| 20 | 9 | 4 | 4 | 5 | 7    | F₄⁴ ∩ S₂ | 5 × 2¹² | 6.4  | No    | Yes  | [11] |          |
| 23b| 20 | 9 | 4 | 4 | 5 | 11   | D₁₀ ∩ S₂ | 5 × 2¹¹ | 6.4  | Yes   | Yes  | [11] |          |
| 23c| 20 | 9 | 4 | 4 | 5 | 12   | Z₁₀ ≻ Z₄ | 40     | 6.7  | No    | No   |      |          |
| 23d| 20 | 9 | 4 | 4 | 5 | 20   | D₁₀    | 20     | 6.2  | Yes   | Yes  |      |          |
| 23e| 20 | 9 | 4 | 4 | 5 | 20   | F₃²    | 20     | 6.7  | No    | Yes  |      |          |

We tried to concentrate in this section only on objective mathematical information. A number of extra remarks and comments on some graphs are postponed to Section 8.

6.1. d.s.r.g.'s constructed from Paley graphs

Graph #3a was given in [11, Section 5]. It is constructed from the Paley graph \(P(5)\). Graph #19a is constructed from \(P(3²)\). Graph #19c also consists of two copies of \(P(3²)\).
6.2. Cayley graphs over dihedral groups

The small Cayley graphs over $D_n$, $n$ odd, can be also constructed following [11, Section 6]. Hence, they belong to a subclass of d.s.r.g.’s which are equivalent to a graph $\Gamma$ with adjacency matrix

$$A(\Gamma) = \begin{pmatrix} T & PT \\ (PT)\top & T \end{pmatrix},$$

where $T$ is the adjacency matrix of a regular tournament and $P$ is a permutation matrix of a permutation of order 2.

Graph #1 was the first example of a d.s.r.g. Duval submitted in his paper. It is the first representative of the series of d.s.r.g.’s over dihedral groups from Lemma 3.1.3. It is a Cayley graph over $D_3$ with connection set $\gamma = X + Xd$, $X = \{c\}$.

Graph #2 is the first representative of the series of d.s.r.g.’s over dihedral groups due to Lemma 3.2.1 with $\gamma = X + d + Xd$, $X = \{c\}$. Duval cites [4] as a source of this example, here it appears in a new interpretation. The graph is depicted in Fig. 2.

Graph #3b has been constructed in [31] and its connection set is $\gamma = X + Xd$, $X = \{c, c^2\}$.

The connection set of graph #6c is $\gamma = X + d + Xd$, $X = \{c, c^2\}$. #6c can also be constructed from #1 with Lemma 3.1.6.

Graph #8a is a Cayley graph over the dihedral group of order 14 with $\gamma = X + Xd$, $X = \{c, c^2, c^4\}$. It is described in [31, Example 6.5]. The graph can also be constructed from the double Paley design on 7 points, see Section 3.2.2.

Graph #8b is a Cayley graph over $D_7$ with $\gamma = X + d + Xd$, $X = \{c, c^2, c^3\}$.

Graph #13b can be obtained with $\gamma = X + d + Xd$, $X = \{c, c^2, c^3\}$.

The connection set of graph #19b is $\gamma = X + d + Xd$, $X = \{c, c^3, c^4, c^7\}$. Besides its representation due to Lemma 3.1.3 this graph also is a Cayley graph over $E_9 \cong \mathbb{Z}_2$. 
Graph #19d is a Cayley graph over the dihedral group of order 18 and \( \gamma = X + X^{-1}d \) where \( X = \{ c, c^2, c^3, c^4 \} \). The dihedral group of order 18 is the first example (and up to 20 vertices the only one) which yields more than one d.s.r.g. (up to isomorphism and transposed graph) with automorphism group \( D_n \) (see #19e).

Graph #19e also is a Cayley graph over \( D_9 \) due to Lemma 3. Here \( \gamma = X + X^{-1}d \) and \( X = \{ c, c^2, c^3, c^5 \} \).

#23d is a Cayley graph over \( D_{10} \) with \( \gamma = X + X^{-1}d \), \( X = \{ c, c^2, c^3, c^4 \} \). This graph can also be constructed from #3b according to Lemma 3.1.6.

6.3. Graphs defined on the flags of incidence structures

Graph #4 is the first in a series due to [31, Proposition 8.2(a)]. It is constructed from a BIBD that corresponds to the incidence structure of vertices and arcs of the complete graph \( K_4 \). The graph can be represented as a Cayley graph over \( A_4 \). Duval described the graph as the first example in a series of d.s.r.g.’s on \( k(k - 1) \) vertices.

Graph #6b is similar to #4, that is, it also is constructed from a BIBD corresponding to the vertex/line incidence relation of \( K_4 \). The method is described in [31, Proposition 8.2(b)]. Similarly to #4 this graph can be represented as a Cayley graph over \( A_4 \).

Graph #20 is constructed from a BIBD corresponding to the incidence relation of vertices and lines of \( K_5 \), cf. [31, Proposition 8.2(a)]. The graph is a Cayley graph over \( F_5^4 \).

Graph #21 is constructed due to [31, Proposition 8.2(b)], cf. parameter set #6b. Like #20 it is based on the incidence relation of vertices and lines of the \( K_5 \). The graph is a Cayley graph over \( F_5^4 \).

6.4. Wreath product constructions

#5 is the first representative for constructing a d.s.r.g. from smaller examples using wreath product, in this case graph #1. The method is described in [11,31, Proposition 9.1(a)]. It is depicted in Fig. 3. Here a “⇒” between two subgraphs indicates that each vertex of the first subgraph (an empty graph with 2 vertices) is adjacent to each vertex of the second one. It can be represented as a Cayley graph over both \( D_6 \) and \( \mathbb{Z}_6, \mathbb{Z}_2 \).

As a Cayley graph over \( D_6 \) it has a connection set \( \gamma = X + Xd, X = \{ c, c^4 \} \).

Graph #6a also originates from #1. The construction is explained in [11,31, Proposition 9.1(b)]. The graph only differs from #5 in that the subgraphs consisting of two vertices are \( K_2 \), see Fig. 4. Analogously to #5 the graph can be represented as a Cayley graph over \( D_6 \) or \( \mathbb{Z}_6, \mathbb{Z}_2 \). As a Cayley graph over \( D_6 \) it has connection set \( \gamma = X + c^3 + Xd, X = \{ c, c^4 \} \).

Graph #12 is derived from graph #2 due to [11, Theorem 7.2], see also [31, Proposition 9.1(b)]. The graph can be also obtained from #2 using Lemma 3.1.6. This graph can be represented as a Cayley graph over the groups \( D_4 \times \mathbb{Z}_2, QD_{16}, t16n10 \) (this name refers to the GAP-library of small transitive groups), \( (\mathbb{Z}_2 \times \mathbb{Z}_4), \mathbb{Z}_2, D_8, \) and \( Q_{16} \). As a Cayley graph over \( D_8 \) the connection set is \( \gamma = X + d + Xd, X = \{ c, c^4, c^5 \} \).
#16 originates from #1 where each vertex is to be replaced by an empty 3-vertex graph. The graph is a Cayley graph over $E_9 \bowtie \mathbb{Z}_2$, $D_9$, and $S_3 \times \mathbb{Z}_3$. As a Cayley graph over $D_9$ this graph has a connection set $\gamma = X + Xd$, $X = \{c, c^4, c^7\}$.

#18 is constructed from #1. Each vertex in graph #1 has to be replaced with a $K_3$. #18 is a Cayley graph over $E_9 \bowtie \mathbb{Z}_2$, $D_9$, and $S_3 \times \mathbb{Z}_3$. As a Cayley graph over $D_9$ the graph has a connection set $\gamma = X + Xd$, $X = \{c, c^4, c^7\}$.

Graph #22a originates from #3a. Each vertex in #3a has to be replaced by an empty 2-vertex subgraph. This graph is a Cayley graph over $F_{5}^4 = \mathbb{Z}_5 \bowtie \mathbb{Z}_4$.

#22b is constructed due to [31, Proposition 9.2(a)]. Hence, each vertex in #3b has to be replaced by an empty 2-vertex subgraph. It has representations as a Cayley graph over $D_{10}$ and $\mathbb{Z}_{10}$, $\mathbb{Z}_2$. As a Cayley graph over $D_{10}$ it has a connection set $\gamma = X + Xd$, $X = \{c, c^2, c^6, c^7\}$. 

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Fig. 3. Graph #5.

Fig. 4. Graph #6a.
Graph #23a is derived from #3a. Each vertex in #3a has to be replaced by a complete 2-vertex subgraph. This graph is a Cayley graph over $\mathbb{Z}_5 \times \mathbb{Z}_4$, cf. #22a.

Graph #23b is constructed from #3b. Each vertex in #3b has to be replaced by a complete 2-vertex subgraph. It has representations as a Cayley graph over $D_{10}$ and $\mathbb{Z}_{10} \times \mathbb{Z}_2$. As a Cayley graph over $D_{10}$ it has a connection set $\gamma = X + Xd + c^5$, $X = \{c, c^2, c^6, c^7\}$.

6.5. New graphs derived from new general constructions

#13a is a Cayley graph over $QD_{16}$. This graph is constructed from antiflags as in Section 3.2.3 with $q = 3$.

Graph #17a is a Cayley graph over $D_3 \times \mathbb{Z}_3$. Graph #17b is one of the few graphs in our list whose automorphism group does not contain a regular subgroup. Both #17a and #17b are constructed over two copies of the two-dimensional vector space over $GF(3)$, see Proposition 3.2.4.

6.6. Known sporadic graph

#14 was already described in [11, Section 4]. Its automorphism group was constructed in [31, Example 9.2]. This graph can be interpreted as a Cayley graph over two groups, $E_9 \rtimes \mathbb{Z}_2$ and $S_2 \times \mathbb{Z}_3$. It is depicted in Fig. 5.

6.7. New sporadic graphs

Graph #15 is a Cayley graph over $S_3 \times \mathbb{Z}_3$. Let $s_1, s_2$ denote the generators of order 3, 2 respectively, of $S_3$, $z$ be a generator of $\mathbb{Z}_3$, $e_z$ be the identity in $\mathbb{Z}_3$. Then $X = \{s_1 s_2 \times e_z, s_2 \times z, s_1 \times z^2, s_1^2 \times z, s_1 s_2^2 \times z^2\}$ defines a connection set $\gamma = X$ for #15.
Graph #22c is a Cayley graph over $F_4^5$, the Frobenius group of order 20. This group is isomorphic to $AGL(1,5)$. With the notation from Section 3.2.2 a connection set of #22c is $\gamma = X$ with $X = \{\mu_1,1, \mu_1,2, \mu_2,2, \mu_2,4, \mu_3,1, \mu_3,3, \mu_3,4, \mu_4,2\}$.

Graph #23c has automorphism group $\mathbb{Z}_{10} \rtimes \mathbb{Z}_4$. It is not a Cayley graph since $\mathbb{Z}_{10} \rtimes \mathbb{Z}_4$ is a minimal transitive permutation group. The minimal transitive groups of small degree (31 so far, revised up to 24) can be found in [26] or its corresponding GAP-library. #23c can also be constructed from #3a using Lemma 3:

7. Nonexistence of a $\mathbb{K}(16,6,3,1,3)$-graph

Consider the parameter set #11 from Table 1; $n = 16$, $k = 6$, $\mu = 3$, $\lambda = 1$, and $t = 3$. Then

$$A^2 = -2A + 3J.$$ Using the formulas for the spectrum of a d.s.r.g. (see [11]), we get that $A$ has eigenvalue 0 with the multiplicity 12. Therefore $\text{rank}(A) = 4$. Since the rank is 4 there are exactly 4 columns in matrix $A$ such that all others are their linear combinations. Each column is a characteristic vector $\chi(A_i)$ of a subset $A_i \subseteq V(I)$ and $|A_i| = 6$. Let us denote by $A_1$, $A_2$, $A_3$, and $A_4$ the subsets whose characteristic vectors span the space of columns. Let $\bar{a}_i = \chi(A_i)$, $1 \leq i \leq 4$. Let $\bar{\mathbf{I}}$ be a vector-column, all 16 components of which are equal to 1.

Since $\bar{\mathbf{I}}$ is a linear combination of columns of $A$, there exist $x_1, x_2, x_3, x_4 \in \mathbb{R}$ such that

$$x_1\bar{a}_1 + x_2\bar{a}_2 + x_3\bar{a}_3 + x_4\bar{a}_4 = \bar{\mathbf{I}}.$$ (7.1)

Taking the scalar product on both sides with $\bar{\mathbf{I}}$ we obtain

$$x_1 + x_2 + x_3 + x_4 = \frac{8}{7}.$$ (7.2)

In a few steps let us show that (7.1) is not possible.

- $A_1 \cap A_2 \cap A_3 \cap A_4 = \emptyset$. Otherwise let $v \in A_1 \cap A_2 \cap A_3 \cap A_4$, then $\bar{\mathbf{I}}_v = 1$ and

$$(x_1\bar{a}_1 + x_2\bar{a}_2 + x_3\bar{a}_3 + x_4\bar{a}_4)_v = x_1 + x_2 + x_3 + x_4,$$

hence $x_1 + x_2 + x_3 + x_4 = 1$ in contradiction to (7.2).
- For each $i$, $1 \leq i \leq 4$, we set

$$\tilde{A}_i = A_i \setminus \left( \bigcup_{j \neq i} A_j \right).$$

Suppose that $\tilde{A}_i \neq \emptyset$ for some $1 \leq i \leq 4$. This means that there exists $v \in V(I)$ such that $v \in A_i$, but $v \notin A_j$ if $j \neq i$. This implies that $x_i = 1$. Due to equality (7.2) there
exists at least one $i$ with $x_i \neq 1$. Without loss of generality, let $x_4 \neq 1$. So we may assume that $A_4 = \emptyset$ or, equivalently, $A_4 \subseteq A_1 \cup A_2 \cup A_3$. (7.1) implies that $A_1 \cup A_2 \cup A_3 \cup A_4 = V(G)$. Therefore $A_1 \cup A_2 \cup A_3 = V(G)$. Since $|A_i| = 6$ and $|V(G)| = 16$ there are only a few possible cases (up to renumbering).

(a) $|A_1 \cap A_2| = |A_2 \cap A_3| = |A_3 \cap A_1| = |A_1 \cap A_2 \cap A_3| = 1$;
(b) $|A_1 \cap A_2 \cap A_3| = 0$, $|A_1 \cap A_2| = 2$, $|A_1 \cap A_3| = 0$, and $|A_2 \cap A_3| = 0$;
(c) $|A_1 \cap A_2 \cap A_3| = 0$, $|A_1 \cap A_2| = 1$, $|A_1 \cap A_3| = 0$, and $|A_2 \cap A_3| = 1$.

- Now we get that the property $\tilde{A}_i \neq \emptyset$ is satisfied for at least two sets of $A_1, A_2, A_3$. To prove this, consider the sets $B_1 = A_1 \setminus (A_2 \cup A_3)$, $B_2 = A_2 \setminus (A_1 \cup A_3)$, and $B_3 = A_3 \setminus (A_1 \cup A_2)$. It is evident that for $i \in \{1, 2, 3\}$ we have
  $$\tilde{A}_i \neq \emptyset \iff B_i \setminus A_4 \neq \emptyset.$$  

The sets $B_1, B_2, B_3$, are pairwise disjoint. It is easy to see that in each case (a)–(c) any two of these sets have together at least 8 elements. Thus $A_4$ may cover at most one of these sets, while for two other sets we get $B_i \setminus A_4 \neq \emptyset$. Therefore, at least two of the coefficients from $x_1, x_2, x_3$ are equal to 1.

- Let us now consider case (a). Without loss of generality, we may assume that $x_1 = x_2 = 1$. Clearly, $A_3 \neq A_4$; therefore there exists $v \in A_4 \setminus A_3$. Then we get
  $$\tilde{I}_c = 1 = (x_1 a_1 + x_2 a_2 + x_3 a_3 + x_4 a_4)_c = (1 + x_4),$$  

hence $x_4 = 0$.

Take now $u \in A_1 \cap A_2 \cap A_3$. Then we get
  $$\tilde{I}_u = 1 = (x_1 a_1 + x_2 a_2 + x_3 a_3 + x_4 a_4)_u = (2 + x_3)$$  

and $x_3 = -1$.

So we get $x_1 + x_2 + x_3 + x_4 = 1$, contrary to Eq. (7.2). Thus case (a) is impossible.

- In a similar manner we get contradictions in cases (b) and (c).

8. Discussion

The main subject of this paper is the constructive enumeration of small vertex-transitive d.s.r.g.’s with the aid of a computer. According to the experience of some of us and of our colleagues in constructive enumeration of other objects, the possible size of resulting catalogues varies in a very wide range. For example, each graph in [27] was a subject of careful group-theoretical investigations. On the other hand, the sizes of catalogues of designs with certain parameters in [2,3] were measured in the thousands and even millions, so that only a small amount of “outstanding” objects was really accurately considered and interpreted.

From this point of view, the use of a computer for the enumeration of small d.s.r.g.’s with a vertex-transitive automorphism group proves its value:

- the catalogues are not too small;
- representation of d.s.r.g.’s requires a lot of routine computations;
• description of the automorphism groups, using GAP, in most cases allows a reasonably convenient identification of the group;
• even on the final step of the computer depiction of the diagrams of the graphs the computer graphics facilities saved a lot of our forces.

At the same time we were able to proceed carefully once we obtained each d.s.r.g. All our results in Section 3 were initiated by thorough observations of certain graphs and a subsequent lucky insight in each case. This allowed us to get concrete theoretical generalizations.

Recently, Jørgensen informed us that the idea of the proof of the nonexistence of a d.s.r.g. with parameter set #11 can be generalized in order to eliminate an infinite set of feasible parameters.

In August 1997 Akihiro Munemasa, at our request, kindly announced to the recipients of [31] a short summary of our results. In particular, it was mentioned that case #10 in Duval’s list still remained open. This challenged Jørgensen to use a computer for the solution of this problem. Finally, he found many d.s.r.g.’s which correspond to this case, in particular with the identical automorphism group. It seems that on the modern level of the development of computer technique the complete enumeration of all d.s.r.g.’s in Duval’s list is a worthwhile task. Significant steps in this direction were recently announced by Jørgensen [28].

The rank $r$ of $\mathcal{M}(A)$, where $A$ is the adjacency matrix of a d.s.r.g. $\Gamma$, is in our opinion a very essential invariant of $\Gamma$. The smallest possible values of $r$ are 6 and 7. A few series of d.s.r.g.’s with these values of $r$ were discovered in [31]. In this paper we present a few other graphs with such small values of $r$, as well as shed a little more light on some known examples. Complete classification of all d.s.r.g.’s with $r = 6, 7$ seems to be an interesting problem in the area of algebraic combinatorics.

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