Networked state estimation over a Gilbert-Elliot type channel

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Abstract—We characterize the stability and achievable performance of networked estimation under correlated packet losses described by the Gilbert-Elliot model. For scalar continuous-time linear systems, we derive closed-form expressions for the mean-square distortion of the optimal estimator. The conditions for stable mean square estimation error are equivalent to those obtained previously for stability of ‘peak distortions’ [3]. We study how the estimator performance depends on loss probability and loss burst length, and show that the mean-square distortion remains bounded if the average burst length does not exceed a calculated bound. The main new finding is that, once we fix the mean length of loss bursts, the average packet loss rate influences the estimator’s performance but not its stability.

I. INTRODUCTION

State estimation is a critical component of many automation systems, with applications in monitoring, detection and control. Recently, there has been a strong interest in wireless automation systems where measurements are sent over a wireless network and hence can be lost or delivered after a time-varying delay. In this context, the issue of estimation under packet losses has been investigated in many different settings. For the discrete-time case with Bernoulli-distributed packet losses, [8] investigated Kalman filtering and identified a threshold condition for the loss probability to guarantee stability of the estimator. However, in reality packet losses tend to be correlated and the Bernoulli loss process is not representative for wireless communication in industrial environments (see, for example, [9]). One simple model for correlated losses is the two-state Markov model due to Gilbert and Elliot [2], [1]. Despite its simplicity, the model has been reported capable of reproducing actual measured loss traces with good accuracy [9].

The optimal state estimator for a linear system under observation losses that are not IID but correlated is well-known (it is the time-varying Kalman filter), but few results exist that characterize the achievable estimator performance and how it depends on critical loss parameters such as overall loss probability and average length of loss bursts. Such results are important, as they give insight and design rules for when it is feasible to use wireless sensing for estimation. Recently, [3] studied peak covariance for discrete-time systems with Markovian packet losses described by the Gilbert-Elliot model. Conditions for when the peak covariance process is stable were derived but no results on estimator performance in the mean square sense were presented. For continuous-time performance, that accounts also for the inter-sample evolution of the estimator error variance, only a handful of results exist (e.g. [5], which studied estimation under sampling according to a Poisson process).

This note considers the stability and achievable continuous-time performance of estimation under packet losses described by the Gilbert-Elliot model. A scalar continuous-time system is considered and closed-form expressions for the mean-square distortion of the optimal estimator are given. This allows us to study how the estimator performance depends on loss probability and loss burst length, and to derive conditions on these parameters that guarantee that the mean-square distortion remains bounded. An important insight of our study is that the average loss probability plays a subordinate role when the packet losses are correlated: for any given average loss rate, it is possible to find parameters for the Gilbert-Elliot model which cause the mean-square distortion to grow unboundedly. Our calculation generalizes our previous studies [6], [7].

II. MODEL AND PROBLEM FORMULATION

Consider a scalar continuous-time process that obeys the stochastic differential equation (SDE):

$$dx_t = ax_t dt + dW_t$$  (1)

with $W_t$ being a standard one dimensional Wiener process independent of $x_0$. The process state is assumed to be measured exactly every $h$ seconds, and the samples are transmitted to an estimator node over an unreliable communication channel.

Packet losses on the channel are modelled using the so-called Gilbert-Elliot model [2], [1]. There are several parameterizations of this model, but we will use the specific Markov chain shown in Figure 1. The transition from the good state $G$ to the bad state $B$ is assumed to take place whenever a packet is lost but the previous one was not. Similarly, the transition in the opposite direction is assumed to take place whenever a packet is successfully transmitted but the previous one was lost. The loss probability in the good state (i.e. the conditional probability of packet loss given that the previous transmission was successful) is denoted $p$ and the loss probability in the bad state (i.e. the conditional
probability of packet loss given that also the previous packet was also lost) is denoted \( q \). Throughout the paper it is assumed that \( p \in (0, 1) \) and \( q \in (0, 1) \). With these definitions, the average sojourn times for the two states are given by:

\[
T_G = \frac{1}{p}, \\
T_B = \frac{1}{1-q}
\]

and the stationary distribution for the Markov chain is:

\[
\pi_G = \frac{T_G}{T_G + T_B} = \frac{1}{1 + \frac{p}{1-q}}, \\
\pi_B = \frac{T_B}{T_G + T_B} = \frac{1}{1 + \frac{q}{1-p}}.
\]

These relationships will be useful for translating results derived in terms of conditional loss probabilities \( p \) and \( q \) into physically relevant quantities such as average loss probability and the expected length of loss-bursts. Notice that the average loss burst length \( T_B \) is determined only by \( q \). On the other hand the average packet loss rate \( \pi_B \) can be varied independently of \( T_B \) over the interval \((0, \infty)\) by choosing \( T_G \) or equivalently \( p \) appropriately.

Denote the MMSE estimate of the state signal given the received samples by \( \hat{x}_t \). The goal of this paper is to determine the minimum achievable mean-square distortion

\[
J = \limsup_{M \to \infty} \frac{1}{M} \int_0^M \mathbb{E} \left[ (x_t - \hat{x}_t)^2 \right] dt
\]

and quantify how \( J \) depends on the system dynamics (1), the sampling interval \( h \) and the characteristics of the channel (e.g., \( p \) and \( q \)).

In section III we first find an expression for the mean square estimation distortion in terms of the mean length of loss bursts and the mean integral quadratic error collected over loss bursts. Then we explicitly evaluate the estimation distortion in terms of the parameters of the Gilbert-Elliot model. We also find that the criterion of stable estimation performance depends only on the mean length of bursts. In section IV, we further focus on the effect of the mean burst length. We characterize the estimation performance as a function of the burst length. Finally, in section V, we provide some numerical computations of the dependence of stability and estimation performance with respect to the burst length and the packet loss rate.

\[ \text{III. Achievable mean-square distortion} \]

Let \( \{R_0, R_1, R_2, \ldots \} \) denote the sequence of times at which successful receptions occur, with \( R_0 = 0 \). Let \( l_i = \sup \{R_i | R_i \leq t \} \) be the process that denotes the time of the last successful reception of a packet. Because of linearity, we get an explicit expression for the least-squares causal estimator of the \( x \)-process based on 1) sequence of received packets, and, 2) the sequence of their reception times:

\[
\hat{x}_t = \mathbb{E} \left[ x_t \big| \{ (R_i, x_{R_i}) | 1 \leq i \leq l_t \} \right], \\
= x_t e^{a(t-t_0)}.
\]

This way of propagating the latest sample to compute the (unobserved)-mean of the \( x \)-process is optimal because there is no correlation between the \( x \)-process and the random sequence of reception times.

In order to evaluate this performance metric, both the average interval between successful transmissions and the average of the squared estimation error in that interval must first be determined.

\[ \text{A. Average time between successful transmissions} \]

The probability mass function for the packet inter-reception time is derived by studying the Markov chain for the channel state and is determined to be:

\[
\mathbb{P} [R_{i+1} - R_i = nh] = \begin{cases} 
1 - p, & n = 1 \\
\frac{pq^{n-2}(1-q)}{n}, & n \geq 2
\end{cases}
\]

The average time interval between consecutive successful transmission is thus given by:

\[
\bar{R} = \mathbb{E} [R_{i+1} - R_i] = (1-p)h + \sum_{n=2}^{\infty} \frac{pq^{n-2}(1-q)}{n}nh
\]

\[
= (1-p)h + (1-q)h \sum_{j=0}^{\infty} (j+2)q^j
\]

\[
= (1-p)h + (1-q)h \left( \frac{q}{(1-q)^2} + \frac{2}{1-q} \right)
\]

\[
= \frac{h(1+p-q)}{1-q}
\]

The expression for \( \bar{R} \) reduces to the one in [6] when packet losses are assumed to be IID. To get the special case of IID packet losses, let \( q = p \) which means that the probability for packet loss is the same in both network condition states.

\[ \text{B. Average estimation error} \]

The estimation distortion of equation(2) is the average power of the estimation error process which is given by:

\[
\xi_t = x_t - \hat{x}_t,
\]

and this is governed by:

\[
d\xi_t = dx_t - d\hat{x}_t = a\xi_t dt + dW_t.
\]
Notice that the error process is reset to zero whenever a transmission is successful. The sequence of intervals between successive receptions is IID. The statistics of the error signal are the same over the different inter-reception intervals - the error signal is governed by the same time-homogeneous SDE. This enables us to express the long-run average of equation (2) as a ratio of two averages.

**Proposition 3.1:** With probability one, the average mean square distortion of equation (2) can be computed as:

\[ J = \frac{\mathbb{E}\left[ \int_{R_{i-1}}^{R_i} \xi_t^2 dt \right]}{\mathbb{E}\left[ R_i - R_{i-1} \right]}, \]  

(6)

where, the index \( i \) does not matter because the sequence \( \{R_i - R_{i-1}\} \) is IID.

**Proof:** This result is obtained by a straightforward calculation of limits using elementary analysis and the law of large numbers. We have:

\[
J = \limsup_{M \to \infty} \frac{1}{M} \mathbb{E}\left[ \int_{0}^{M} \xi_t^2 dt \right],
\]

\[
= \limsup_{M \to \infty} \frac{1}{M} \mathbb{E}\left[ \sum_{t=1}^{M} \left( \int_{R_{i-1}}^{R_i} \xi_t^2 dt + \int_{R_i}^{R_{i+1}} \xi_t^2 dt \right) \right],
\]

\[
= \limsup_{M \to \infty} \frac{1}{M} \mathbb{E}\left[ \sum_{t=1}^{M} \left( (R_i - R_{i-1}) + (M - R_{i+1}) \right) \right],
\]

\[
= \limsup_{M \to \infty} \frac{1}{M} \mathbb{E}\left[ \sum_{t=1}^{M} (R_i - R_{i-1}) + (M - R_{i+1}) \right],
\]

\[
= \limsup_{M \to \infty} \frac{1}{M} \sum_{t=1}^{M} \left( (R_i - R_{i-1}) + (M - R_{i+1}) \right),
\]

\[
= \limsup_{M \to \infty} \frac{1}{M} \sum_{t=1}^{M} (R_i - R_{i-1}) + (M - R_{i+1}),
\]

\[
= \limsup_{M \to \infty} \frac{1}{M} \sum_{t=1}^{M} (R_i - R_{i-1}) + (M - R_{i+1}),
\]

because of 1) the inequality: \( 0 < \lim_{M \to \infty} \frac{M}{M} < \infty \), and, 2) the fact that, whenever the limits in the numerator and in the denominator exist so does their ratio. Hence, the average distortion takes the form:

\[
\mathbb{E}\left[ \int_{R_{i-1}}^{R_i} \xi_t^2 dt \right] = \frac{\mathbb{E}\left[ R_i - R_{i-1} \right]}{\mathbb{E}\left[ R_i - R_{i-1} \right]}. \]

Of course, this expression is valid only when the numerator and the denominator of the above fraction are finite. \( \blacksquare \)

Let \( Y_t = \xi_t^2 \), and use Itô’s formula [4] to determine:

\[ dY_t = d\xi_t^2 = 2\xi_t d\xi_t + dt = (2aY_t + 1) dt + 2Y_t^{1/2} dW_t \]

This equation is solved to be:

\[ Y_t = \int_0^t (2aY_s + 1) ds + \int_0^t 2Y_s^{1/2} dW_s \]

Take the expectation, and use the fact that the expected value of an Itô integral is zero, to get:

\[ \mathbb{E}\left[ Y_t \right] = \int_0^t (2a\mathbb{E}\left[ Y_s \right] + 1) ds \]

Which can be rewritten as:

\[ d\mathbb{E}\left[ Y_t \right] = 2a\mathbb{E}\left[ Y_s \right] dt + dt \]  

(7)

The solution to Equation (7) gives the expected value of the squared error process as:

\[ \mathbb{E}\left[ \xi_t^2 \right] = \mathbb{E}\left[ Y_t \right] = \frac{e^{2at} - 1}{2a} - e^{2at}\mathbb{E}\left[ \xi_0^2 \right] \]

Because the expected error at time \( t = 0 \) is zero, we have:

\[ \mathbb{E}\left[ \xi_0^2 \right] = 0 \]

and hence:

\[ \mathbb{E}\left[ \xi_t^2 \right] = \frac{e^{2at} - 1}{2a} \]

which gives the integral of the squared estimation error between two successful transmissions as:

\[ \int_0^{R_{i+1} - R_i} \mathbb{E}\left[ \xi_t^2 \right] dt = \frac{e^{2a(R_{i+1} - R_i)} - 1}{4a^2} - \frac{R_{i+1} - R_i}{2a} \]

(8)

Now, we have what we need to derive the expression for the mean square distortion.

**Proposition 3.2:** Under the model assumptions in the previous sections, if

\[ qe^{2ah} < 1 \]

then the mean square distortion for the estimator is finite and is given by:

\[ J = \frac{1}{4a^2h(1 + p - q)} \left( \frac{1 - q}{1 - qe^{2ah}} - 1 \right) - \frac{1}{2a} \]

(9)

**Proof:** The proposition can be proved through straightforward calculations. The mean square distortion for the estimator is given by:

\[ J = \limsup_{M \to \infty} \frac{1}{M} \mathbb{E}\left[ \int_0^{R_{i+1} - R_i} (x_t - \hat{x}_t)^2 dt \right] \]

\[ = \frac{1 - q}{h(1 + p - q)} \left( \frac{e^{2a(R_{i+1} - R_i)} - 1}{4a^2} - \frac{R_{i+1} - R_i}{2a} \right) \]

\[ = \frac{1 - q}{h(1 + p - q)} \left( (1 - p)e^{2ah} - 1 \right) + \frac{1}{2a} \]

\[ + \sum_{n=2}^{\infty} pq^{n-2}(1 - q)e^{2an} - \frac{1}{4a^2} - \frac{1}{2a} \]

\[ = \frac{1 - q}{4a^2h(1 + p - q)} \left( (1 - p)(e^{2ah} - 1) + p(1 - q) \sum_{j=0}^{\infty} q^j(e^{2ah(j+2)} - 1) \right) - \frac{1}{2a} \]
If $qe^{2ah} < 1$ and $q < 1$, the sum converges and we get:

$$J = \frac{1 - q}{4a^2 h(1 + p - q)} \left( (1 - p)(e^{2ah} - 1) + p(1 - q) \left( \frac{e^{4ah}}{1 - qe^{2ah}} - \frac{1}{1 - q} \right) \right) - \frac{1}{2a}$$

$$= \frac{1 - q}{4a^2 h(1 + p - q)} \times \left( \frac{(1 - p)e^{2ah} + (p - q)e^{4ah}}{1 - qe^{2ah}} - 1 \right) - \frac{1}{2a}$$

The expression for $J$ in Equation (9) also reduces to the one in [6] when $q = p$. The convergence criterion $qe^{2ah} < 1 \iff 2ah < \ln(1/q)$ is equivalent to the result in Theorem 8 in [3], which was expected. An interesting aspect is that there is no condition on $p$, the failure probability in the good network condition state, for the distortion $J$ to be finite. This is because the average time spent in the bad state, which determines convergence, is independent of $p$. However, even though convergence is independent of $p$, the performance of the estimator, measured as the mean square distortion, is not.

Remark 3.3: For stable systems ($a < 0$) we have $e^{2ah} < 1$ for any finite step size $h > 0$, and hence the distortion $J$ is finite for any loss probability $q < 1$.

IV. THE INFLUENCE OF THE LOSS BURST LENGTH

In many settings it might not be the packet drop probability as such but rather the risk of long loss bursts that is important to consider. Therefore, the effect on the distortion $J$ of the average loss burst length, $T_B$, is investigated, keeping the average packet loss probability, $p_{loss}$, constant. These two parameters can be varied independently of each other, and thus it is possible in this setting to study the effect jitter has on the performance of the estimator. The loss probability is computed to be:

$$p_{loss} = \pi_a p + \pi_b q = \frac{1 - q}{1 - q + p} p + \frac{p}{1 - q + p} q$$

$$= \frac{p}{1 - q + p} = \pi_b$$

and the average loss burst length is:

$$T_B = \frac{1}{1 - q}$$

However, to analyse the impact from varying average loss burst length, we let $p_{loss}$ be constant and rewrite the expressions as:

$$p_{loss} = \frac{p}{1 - q + p} = \frac{1}{\frac{1 - q}{p} + 1} = \frac{1}{\alpha + 1}$$

$$T_B = \frac{1}{1 - q} = \frac{1}{\alpha p}$$

for some chosen constant $\alpha \in (0, \infty)$. We have seen in proposition 3.2 that the parameter $q$ determines the boundedness of the estimation distortion. The parameter $q$ is equivalent to the mean burst length $T_B$. So, if the mean burst length satisfies the stability condition, the packet loss rate does not affect stability. For example, even if $T_B$ is relatively small, one can have a large loss rate $p_{loss}$ if the mean length $T_G$ is small relative to $T_B$. This does not affect the stability of the estimation distortion because the threat to stable distortion comes from the growth of the error variance during the outage durations. The mean length $T_B$ does not affect the statistics of the outage durations.

Proposition 4.1: Under the model assumptions defined in the previous sections and $\alpha \in (0, \infty)$, if

$$T_B < \frac{1}{1 - e^{-2ah}}$$

then the mean square distortion for the estimator is finite and is given by:

$$J = \frac{\alpha}{4a^2 h(1 + \alpha)}$$

$$\times \left( \frac{1}{\alpha} \frac{e^{2ah}}{1 - e^{-2ah} - T_B} + e^{2ah} - 1 \right) - \frac{1}{2a}$$

Proof: The expression (9) for the mean square distortion can be rewritten as:

$$J = \frac{1 - q}{4a^2 h(1 + p - q)}$$

$$\times \left( \frac{pe^{2ah}(e^{2ah} - 1)}{1 - qe^{2ah}} + e^{2ah} - 1 \right) - \frac{1}{2a}$$

$$= \frac{1 - q}{4a^2 h(1 + p - q)}$$

$$\times \left( \frac{e^{2ah}}{\alpha} \frac{1}{1 - e^{-2ah} - \frac{1}{\alpha} T_B} + e^{2ah} - 1 \right) - \frac{1}{2a}$$

$J$ is now a function of only one variable, $T_B$. Because $q = 1 - \frac{1}{T_B}$, the condition for convergence stating that $qe^{2ah} < 1$ can be expressed in $T_B$ as:

$$T_B < \frac{1}{1 - e^{-2ah}}$$

Since we already know that $J$ converges for all $q < 1$ and $h > 0$ if $a < 0$, it will also converge for all $T_B > 1$.

Corollary 4.2: $J$ in Equation (10) is increasing and convex in $T_B$ if $a > 0$, and increasing and concave if $a < 0$.

Proof: The expression for $J$ in Equation (10) can be rewritten in two different ways depending on whether $a > 0$ or $a < 0$.

Case $a > 0$: For this case, $\frac{1}{1 - e^{-2ah}} > 0$, and hence $J(T_B)$ can be written as:

$$J(T_B) = A_1 + \frac{B_1}{C_1 - T_B}, \quad 1 \leq T_B < C_1$$

(11)
where,

\[ A_1 = \frac{\alpha}{4a^2h(1 + \alpha)} \left( e^{2ah} - 1 \right) - \frac{1}{2a}, \]

\[ B_1 = \frac{e^{2ah}}{4a^2h(1 + \alpha)} > 0, \quad \text{and}, \]

\[ C_1 = \frac{1}{1 - e^{-2ah}} > 0. \]

**Case a < 0:** For this case, \( \frac{1}{1 - e^{-2ah}} < 0 \), and hence \( J(T_B) \) can be written as:

\[ J(T_B) = A_2 - \frac{B_2}{C_2 + T_B}, \quad T_B \geq 1, \quad (12) \]

where,

\[ A_2 = \frac{\alpha}{4a^2h(1 + \alpha)} \left( e^{2ah} - 1 \right) - \frac{1}{2a}, \]

\[ B_2 = \frac{e^{2ah}}{4a^2h(1 + \alpha)} > 0, \quad \text{and}, \]

\[ C_2 = \frac{1}{e^{-2ah} - 1} > 0. \]

Obviously, \( J \) is increasing and convex in \( T_B \) in Equation (11), and increasing and concave in \( T_B \) in Equation (12).

For the case \( a < 0 \), \( \lim_{T_B \to \infty} J(T_B) = A_2 \) and thus we can identify the maximum mean square distortion for a stable process to be:

\[ A_2 = \frac{\alpha}{4a^2h(1 + \alpha)} \left( e^{2ah} - 1 \right) - \frac{1}{2a} \quad (13) \]

**V. Numerical Computations**

**A. Freely varying average sojourn times**

Simulations to see how the mean square distortion varies with the average sojourn times of the two Markov states were carried out. The result from computations for an unstable system \( (a = 1) \) with a step size of \( h = 0.1 \) is shown in figure 2. As can be seen in the figure, it is when the average sojourn time for the good state is short and the average sojourn time for bad state is long that the performance starts to deteriorate significantly. Translated into parameters of the Gilbert-Elliot model this means that \( p \) and \( q \) are both large. It is also obvious that distortion becomes unbounded for small \( T_G \) when \( T_B \) grows beyond 5.5, which corresponds to the critical value for convergence that can be derived analytically. For this simulation \( p \) was varied between 0.01 and 0.5 and \( q \) between 0 and 0.81.

The same plot as in figure 2, but for a stable system \( (a = -1) \) is shown in figure 4. For these computations \( p \) was varied between 0.01 and 0.5 and \( q \) between 0 and 0.95. As can be seen in the figure, \( J \) is never unbounded for the stable system in contrast to for the unstable system.

**B. The influence of the loss burst length**

To illustrate the conclusions drawn in section IV, computations with an unstable system \( (a = 1) \) and a stable one \( (a = -1) \) were carried out. In the computations, the loss probability was set constant to \( p_{\text{loss}} = 1/10 \), ie \( \alpha = 9 \), while \( T_B \) was varied. The results are shown in figure 6.
As with the preceding figures, the mean square distortion becomes unbounded for $T_B > 5.5$ for the unstable case with $a = 1$ and is bounded above by the constant $A_2$ from Equation (13) in the stable case.

VI. Conclusions

We have characterized the stability and achievable performance of estimation under correlated packet losses described by the Gilbert-Elliot model. For scalar continuous-time linear systems, we derived closed-form expressions for the mean-square distortion of the optimal estimator. We studied how the estimator performance depends on loss probability and loss burst length. We showed that the mean-square distortion remains bounded if the average burst length does not exceed a calculated bound. The average packet loss rate, as long as it is less than 1, does not determine stability. We characterized the dependence of the estimation performance on the mean burst length. Thus we have carried out an useful generalization of earlier works on the mean square estimation under IID packet losses.

It would be interesting to generalize these results to linear systems of higher order. One is tempted to conjecture that the burst-length then needs to be bounded by a function of the maximum eigenvalue of the system matrix, but leave such investigations to our future work.

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REFERENCES