ON PARIKH MATRICES

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Mateescu et al. (2000) introduced an interesting new tool, called Parikh matrix, to study in terms of subwords, the numerical properties of words over an alphabet. The Parikh matrix gives more information than the well-known Parikh vector of a word which counts only occurrences of symbols in a word. In this note a property of two words $u, v$, called “ratio property”, is introduced. This property is a sufficient condition for the words $uv$ and $vu$ to have the same Parikh matrix. Thus the ratio property gives information on the $M$-ambiguity of certain words and certain sets of words. In fact certain regular, context-free and context-sensitive languages that have the same set of Parikh matrices are exhibited. In the study of fair words, Černý (2006) introduced another kind of matrix, called the $p$-matrix of a word. Here a “weak-ratio property” of two words $u, v$ is introduced. This property is a sufficient condition for the words $uv$ and $vu$ to have the same $p$-matrix. Also the words $uv$ and $vu$ are fair whenever $u, v$ are fair and have the weak ratio property.

1. Introduction

The Parikh mapping, also called Parikh vector, introduced in [12] is a significant contribution in the theory of formal languages as this notion has led to certain important results. One such result [13] is that the image by the Parikh mapping of a context-free language is a semilinear set. The Parikh vector of a word $w$ over an alphabet $\Sigma$ counts the number of occurrences of the symbols of $\Sigma$ in $w$ and thus expresses a property of a word as a numerical property in terms of a vector. But the Parikh mapping is not injective as many words over an alphabet have the same Parikh vector and thus much information is lost in the transition from words to vectors.
An extension of the Parikh mapping, called the Parikh matrix mapping was introduced in [11] based on a certain type of matrices. This new mapping involves an apparently simple but an ingenious technique of associating with every symbol $a_q$ in the alphabet, a triangle matrix with 1s on the main diagonal, the entry 1 in the $q$th row, $(q + 1)$st column (above the main diagonal) and all other entries being zero. When extended to a word $w$ using multiplication of matrices, this mapping associates with every word a matrix, called a Parikh matrix, which is again a triangular matrix, with 1’s on the main diagonal and 0’s below it but the entries above the main diagonal provide information on the number of subwords (also called scattered subwords in the literature) in $w$. The interesting aspect of the Parikh matrix is that it has the classical Parikh vector as the second diagonal. Although the Parikh matrix mapping is still not injective, two words with the same Parikh vector have in many cases different Parikh matrices and thus the Parikh matrix gives more information about a word than a Parikh vector does.

Since the introduction of this interesting notion of a Parikh matrix, a series of papers investigating these matrices has appeared studying various problems related to subwords. See for example, [1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 14, 15, 16], to cite a few. In this note we introduce a property called “ratio property” of two words $w_1, w_2$ which provides a sufficient condition for the words $w_1w_2$ and $w_2w_1$ to have the same Parikh matrix and thus to be $M$-ambiguous [16]. As a consequence we exhibit certain regular, context-free and context-sensitive languages that have the same set of $M$-vectors or equivalently the same set of Parikh matrices.

A different kind of a matrix called precedence matrix or $p$-matrix was introduced in [5] in the study of “fair” words. Here we introduce the notion of a “weak ratio property” of two words $w_1$ and $w_2$. This property provides a sufficient condition for the words $w_1w_2$ and $w_2w_1$ to have the same $p$-matrix. Also the words $w_1w_2$ and $w_2w_1$ are fair whenever $w_1, w_2$ are fair and have the weak ratio property.

2. Preliminaries

Let $\Sigma$ be an alphabet. The set of all words over $\Sigma$ is $\Sigma^*$ and the empty word is $\lambda$. For a word $w \in \Sigma^*$, $|w|$ denotes the length of $w$. A word $u$ is a subword of a word $w$, if there exist words $x_1, \ldots, x_n$ and $y_0, \ldots, y_n$, (some of them possibly empty), such that $u = x_1 \cdots x_n$ and $w = y_0 x_1 y_1 \cdots x_n y_n$. For example if $w = acbbaabecab$ is a word over the alphabet $\{a, b, c\}$, then cbabcb is a subword of $w$. In the literature subwords are also called “scattered subwords”. The number of occurrences of the word $u$ as a subword of the word $w$ is denoted by $|w|_u$. Two occurrences of a subword are considered different if they differ by at least one position of some letter. An ordered alphabet $\Sigma = \{a_1, \ldots, a_k\}$ is an alphabet $\Sigma = \{a_1, \ldots, a_k\}$ with the ordering $a_1 < a_2 < \cdots < a_k$. We denote the subword $a_ia_{i+1} \cdots a_j$ for $1 \leq i \leq j \leq k$ by the notation $a_{i,j}$.

We recall the definition of a Parikh matrix mapping [11], which is a generalization of the Parikh mapping or Parikh vector [12].
A triangle matrix is a square matrix $M = (m_{i,j})_{1 \leq i, j \leq k}$, such that $m_{i,j}$ are non-negative integers for all $1 \leq i, j \leq k$, $m_{i,j} = 0$, for all $1 \leq j < i \leq k$, and, moreover, $m_{i,i} = 1$, for all $1 \leq i \leq k$. The set $M_k$ of all triangle matrices of dimension $k \geq 1$ is a monoid with respect to multiplication of matrices.

**Definition 1.** Let $\Sigma = \{a_1 < a_2 < \cdots < a_k\}$ be an ordered alphabet. The Parikh matrix mapping is the morphism $\Psi_k : \Sigma^* \to M_{k+1}$, defined by the condition: $\Psi_k(\lambda) = I_{k+1}$ and if $\Psi_k(a_q) = (m_{i,j})_{1 \leq i, j \leq (k+1)}$, then for each $1 \leq i \leq (k+1), m_{i,i} = 1$, $m_{i,q+1} = 1$, all other elements of the matrix $\Psi_k(a_q)$ are 0.

For a word $w = a_{i_1}a_{i_2} \cdots a_{i_m}, a_{i_j} \in \Sigma$ for $1 \leq j \leq m$, we have

$$\Psi_k(w) = \Psi_k(a_{i_1})\Psi_k(a_{i_2}) \cdots \Psi_k(a_{i_m}).$$

In other words $\Psi_k(w)$ is computed by multiplication of matrices and the triangle matrix $\Psi_k(w)$ is called the Parikh matrix of $w$.

Let $\Sigma = \{a < b < c\}$ and $w = acbcb$. In fact

$$\Psi_3(a) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \Psi_3(b) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \Psi_3(c) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\Psi_3(acbcb) = \Psi_3(a)\Psi_3(c)\Psi_3(b)\Psi_3(c)\Psi_3(b)$$

$$= \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Note that $\Psi_3(w)$ is a $4 \times 4$ triangle matrix.


**Theorem 1.** Let $\Sigma = \{a_1 < a_2 < \cdots < a_k\}$ and $w \in \Sigma^*$. The Parikh matrix $\Psi_k(w) = (m_{i,j})_{1 \leq i, j \leq (k+1)}$, has the following properties:

i) $m_{i,j} = 0$, for all $1 \leq j < i \leq (k+1)$,

ii) $m_{i,i} = 1$, for all $1 \leq i \leq (k+1)$,

iii) $m_{i,j+1} = |w|_{a_{i,j}}$, for all $1 \leq i \leq j \leq k$. 

We now recall an important fact about Parikh matrices. The Parikh matrix mapping is not injective. For example, the words \( w_1 = abccabbac, w_2 = cabbacabc \) have the same Parikh matrix. The words \( w_1, w_2 \) are then called \( M \)-ambiguous or simply ambiguous \([6, 14]\) or amiable \([1, 2, 4]\). Many of the studies (see for example, \([2, 4, 10, 15]\)) in this area pertain to this problem.

Another generalization of the Parikh mapping was considered in \([5]\) by introducing the notion of a precedence matrix or \( p \)-matrix based on an operation \( \circ \) defined on matrices. We recall these notions.

**Definition 2.** For two \( k \times k \) matrices, \( A, B \) with integer entries, the matrix \( A \circ B \) is defined as follows: The \((i, j)\)th entry of \( A \circ B \) is given by

\[
(A \circ B)_{i,j} = A_{i,j} + B_{i,j} + \text{sign}(|i-j|)A_{i,i}B_{j,j}
\]

where \( A_{i,j}, B_{i,j} \) are the \((i, j)\)th entries of \( A, B \) respectively.

**Definition 3.** Let \( \Sigma = \{a_1 < a_2 \cdots < a_k\} \) be an ordered alphabet. For a symbol \( a_s \in \Sigma, 1 \leq s \leq k \), let \( E_{a_s} \) be the \( k \times k \) matrix defined as \((E_{a_s})_{i,j} = 1\) if \( i = j = s \) and \((E_{a_s})_{i,j} = 0\), otherwise. The precedence morphism or \( p \)-morphism on \( \Sigma \) is the morphism \( \Phi \) given by \( \Phi(a_s) = E_{a_s} \). For a word \( w = a_1a_2 \cdots a_m, a_i \in \Sigma \) for \( 1 \leq j \leq m \) we have \( \Phi(w) = \Phi(a_{i_1}) \circ \Phi(a_{i_2}) \circ \cdots \circ \Phi(a_{i_m}) \). In other words \( \Phi(w) \) is computed by the operation \( \circ \) on matrices as defined in Definition 2. The resulting matrix \( \Phi(w) \) is called the precedence matrix or \( p \)-matrix of \( w \).

Let \( \Sigma = \{a < b < c\} \) and \( w = acbcb \). Then

\[
\Phi(w) = \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) \circ \left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array} \right) \circ \left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right) \circ \left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array} \right) = \left( \begin{array}{ccc} 1 & 2 & 2 \\ 0 & 2 & 1 \\ 0 & 3 & 2 \end{array} \right)
\]

We now state a basic property \([5]\) of \( p \)-matrices.

**Theorem 2.** Let \( \Sigma = \{a_1 < a_2 \cdots < a_k\} \) be an ordered alphabet. For each word \( w \) over \( \Sigma \) with \( p \)-matrix \( \Phi(w) \), we have

\[
\Phi(w)_{i,j} = |w|_{a_i, i = j};
\]

and

\[
\Phi(w)_{i,j} = |w|_{a_i a_j, i \neq j}
\]

where \( \Phi(w)_{i,j} \) is the \((i, j)\)th entry of the matrix \( \Phi(w) \).

### 3. A Ratio Property of Words

In this section, we deal with the alphabet \( \Sigma = \{a < b < c\} \), in order to simplify the presentation. We now introduce a “ratio property” of words. The important problem of injectivity of Parikh matrices has been addressed in several studies \([1, 2, 3, 4, 10, 11, 15]\). Here the ratio property gives a sufficient condition for...
iii) The proof is similar to the proof of ii) noting that repeatedly using property i), we obtain

These give equality of the Parikh matrices. Thus the ratio property gives information on the

Definition 4. Two words \( w_1, w_2 \) over \( \Sigma = \{a < b < c\} \) are said to satisfy the ratio property, written \( w_1 \sim_r w_2 \), if

\[
\Psi_3(w_1) = \begin{pmatrix}
1 & p_1 & p_{1,2} & p_{1,3} \\
0 & 1 & p_2 & p_{2,3} \\
0 & 0 & 1 & p_3 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

and

\[
\Psi_3(w_2) = \begin{pmatrix}
1 & q_1 & q_{1,2} & q_{1,3} \\
0 & 1 & q_2 & q_{2,3} \\
0 & 0 & 1 & q_3 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

satisfy the condition

\[ p_i = s \cdot q_i \quad (i = 1, 2, 3), \quad p_{i,i+1} = s \cdot q_{i,i+1}, \quad (i = 1, 2), \]

where \( s \) is a constant.

The relation \( \sim_r \) is an equivalence relation.

Lemma 1. For any three words \( w_1, w_2, w_3 \) over \( \Sigma = \{a < b < c\} \) having the ratio property so that \( w_1 \sim_r w_2, w_2 \sim_r w_3 \), we have for \( n \geq 0 \),

i) \( \Psi_3(w_1w_2) = \Psi_3(w_2w_1) \)

ii) \( \Psi_3(w_1^n w_2^n) = \Psi_3((w_1w_2)^n) \)

iii) \( \Psi_3(w_1^n w_2^n w_3^n) = \Psi_3((w_1w_2w_3)^n) \)

Proof. i) If \( \Psi_3(w_1) \) and \( \Psi_3(w_2) \) are as in Definition 4, then

\[
\Psi_3(w_1w_2) = \begin{pmatrix}
1 & q_1 + p_1 & q_{1,2} + q_1p_1 + p_1q_2 & q_{1,3} + q_1p_1 + p_1q_2 + p_1q_3 \\
0 & 1 & q_2 + p_2 & q_{2,3} + q_2p_3 + p_2q_3 \\
0 & 0 & 1 & q_3 + p_3 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

\[
\Psi_3(w_2w_1) = \begin{pmatrix}
1 & p_1 + q_1 & p_{1,2} + q_1p_1 + q_1p_2 & p_{1,3} + q_1p_1 + q_1p_2 + q_1p_3 \\
0 & 1 & p_2 + q_2 & p_{2,3} + q_2p_3 + q_2p_3 \\
0 & 0 & 1 & p_3 + q_3 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

If the ratio property is satisfied for \( w_1 \) and \( w_2 \), then we have

\[
\frac{p_1}{q_1} = \frac{p_2}{q_2} = \frac{p_3}{q_3} = \frac{p_{1,2}}{q_{1,2}} = \frac{p_{2,3}}{q_{2,3}} = s
\]

These give \( p_1q_2 = q_1p_2, p_2q_3 = q_2p_3, p_1q_3 = q_1p_3, p_1q_2 + q_1p_2 = q_1p_2q_3 + q_1p_2q_3 \) and thus \( \Psi_3(w_1w_2) = \Psi_3(w_2w_1) \).

ii) On using the property i) of this Lemma and Definition 1, we have

\[
\Psi_3(w_1^n w_2^n) = \Psi_3((w_1w_2)^n)
\]

Repeatedly using property i), we obtain \( \Psi_3(w_1^n w_2^n) = \Psi_3((w_1w_2)^n) \).

iii) The proof is similar to the proof of ii) noting that \( w_1 \sim_r w_3 \). \( \square \)
Remark 1. The ratio property of two words \( w_1, w_2 \) is only a sufficient condition to ensure equality of the Parikh matrices of \( w_1 w_2, w_2 w_1 \).

Lemma 2. Let \( w_1, w_2, w_3 \) be three nonempty words with \( w_1 \sim_r w_2, w_2 \sim_r w_3 \) and \( x = w_1 w_2 \). Then

\( i) \) \( x \) is not related to \( w_3 \) under \( \sim \),

\( ii) \) \( \Psi_3(x^n w_3^m) = \Psi((xw_3)^n) \).

The proof is based on Lemma 1 and it is immediate.

We now exhibit context-free and context-sensitive languages having the same set of Parikh matrices, as those of certain regular languages.

Theorem 3. Let \( w_1, w_2, w_3 \) be words over an ordered alphabet \( \Sigma = \{a < b < c\} \), such that \( w_1 \sim_r w_2 \) and \( w_2 \sim_r w_3 \), and

\[
L_1 = \{(w_1 w_2)^n / n \geq 0\}, L_2 = \{(w_1 w_2 w_3)^n / n \geq 0\},
L_3 = \{w_1^n w_2^n / n \geq 0\}, L_4 = \{w_1^n w_2^n w_3^n / n \geq 0\}.
\]

If \( \Psi_3(L) = \{\Psi_3(w) / w \in L\} \), then \( i) \) \( \Psi_3(L_3) = \Psi_3(L_1) \) and \( ii) \) \( \Psi_3(L_4) = \Psi_3(L_2) \).

That is, the strictly context-free language \( L_3 \) and the regular language \( L_1 \) have the same set of Parikh matrices. A similar statement holds for the strictly context-sensitive language \( L_4 \) and the regular language \( L_2 \).

The result follows from Lemma 1 and Definition 1.

Theorem 4. Let \( \Sigma = \{a < b < c\} \). If \( w_1, w_2 \) are any two words over \( \Sigma \) such that \( w_1 \sim_r w_2 \), and \( x = \alpha w_1 \beta w_2 \gamma, y = \alpha w_2 \beta w_1 \gamma \), then \( \Psi_3(x) = \Psi_3(y) \).

Proof. The result follows on using Lemma 1. In fact we have

\[
\Psi_3(x) = \Psi_3(\alpha w_1 \beta w_2 \gamma) = \Psi_3(\alpha) \Psi_3(w_1 w_2) \Psi_3(\beta) \Psi_3(w_2 w_1) \Psi_3(\gamma) = \Psi_3(y).
\]

In the case of a binary alphabet \( \Sigma = \{a < b\} \), characterizations of Parikh matrices are known [9]. We recall one such characterization here.

For words \( u, v \in \Sigma = \{a < b\} \), define \( u \equiv v \) if there exist words \( x, y, z \) such that \( u = xabyaz, v = xbayabz \). The relation \( \equiv \) is an equivalence relation [9].

Theorem 5. [9] Let \( \Sigma = \{a < b\} \). For words \( u, v \in \Sigma \), the Parikh matrices \( \Psi_2(u) \) and \( \Psi_2(v) \) are equal if and only if \( u \equiv v \).

In the case of a binary alphabet \( \Sigma = \{a < b\} \), for two words \( w_1, w_2 \) over \( \Sigma \) to satisfy the ratio property (i.e. \( w_1 \sim w_2 \)), the condition is simply \( p_i = s \cdot q_i, (i = 1, 2) \) where

\[
\Psi_2(w_1) = \begin{pmatrix}
1 & p_1 & p_{1,2} \\
0 & 1 & p_2 \\
0 & 0 & 1
\end{pmatrix}
\]

and
We now introduce a \weak-ratio property of words. We again take the alphabet \( \Sigma = \{a < b\} \) and define two words \( w_1, w_2 \) to satisfy the condition if

\[
\Phi(w_1) = \begin{pmatrix}
p_1 & p_{1.2} & p_{1.3} \\
p_{2.1} & p_2 & p_{2.3} \\
p_{3.1} & p_{3.2} & p_3
\end{pmatrix}
\]

and

\[
\Phi(w_2) = \begin{pmatrix}
q_1 & q_{1.2} & q_{1.3} \\
q_{2.1} & q_2 & q_{2.3} \\
q_{3.1} & q_{3.2} & q_3
\end{pmatrix}
\]

satisfy the condition

\[ p_i = s \cdot q_i \quad (i = 1, 2, 3) \]

where \( s \) (\( s > 0 \)) is a constant.

Remark 2. Note that when \( \Sigma = \{a < b\} \), the weak ratio property and the ratio property are the same, namely, \( p_i = s \cdot q_i \) (\( i = 1, 2 \)).

Lemma 3. For any three words \( w_1, w_2, w_3 \) over \( \Sigma = \{a < b\} \) having the \weak-ratio property so that \( w_1 \sim_{\text{wr}} w_2, w_2 \sim_{\text{wr}} w_3 \), we have for \( n \geq 0 \),

i) \( \Phi(w_1w_2) = \Phi(w_2w_1) \)

ii) \( \Phi(w_1^n w_2^n) = \Phi((w_1w_2)^n) \)

iii) \( \Phi(w_1^n w_2^n w_3^n) = \Phi((w_1 w_2 w_3)^n) \)

Proof. i) If \( \Phi(w_1) \) and \( \Phi(w_2) \) are as in definition 5, then

\[
\Phi(w_1w_2) = \begin{pmatrix}
p_1 + q_1 & p_{1.2} + q_{1.2} + p_1q_2 & p_{1.3} + q_{1.3} + p_1q_3 \\
p_{2.1} + q_{2.1} + p_2q_1 & p_2 + q_2 & p_{2.3} + q_{2.3} + p_2q_3 \\
p_{3.1} + q_{3.1} + p_3q_1 + q_{3.2} & p_{3.2} + q_{3.2} + p_3q_2 & p_3 + q_3
\end{pmatrix}
\]

and

\[
\Phi(w_2w_1) = \begin{pmatrix}
q_1 + p_1 & q_{1.2} + p_{1.2} + p_2q_1 & q_{1.3} + p_{1.3} + p_3q_1 \\
q_{2.1} + p_{2.1} + q_2p_1 & q_2 + p_2 & p_{2.3} + q_{2.3} + p_3q_2 \\
q_{3.1} + p_{3.1} + q_3p_1 & q_{3.2} + p_{3.2} + q_3p_2 & q_3 + p_3
\end{pmatrix}
\]
If the weak ratio property is satisfied for \( w_1 \) and \( w_2 \), then, clearly, \( \Phi(w_1w_2) = \Phi(w_2w_1) \). The statements ii) and iii) can be proved analogous to Lemma 1.

The notion of fair words and their properties have been studied in [5]. Here in the context of the weak-ratio property, we examine this notion.

**Definition 6.** [5] Let \( \Sigma = \{a < b < c\} \). A word over \( \Sigma \) is said to be fair if \( \|w\|_{xy} = \|w\|_{yx} \) for all \( x, y \in \Sigma \).

**Theorem 7.** For any two fair words \( w_1, w_2 \) over \( \Sigma = \{a < b < c\} \) having the weak-ratio property, the words \( w_1w_2, w_2w_1 \), are also fair.

**Proof.** If \( \Phi(w_1) \) and \( \Phi(w_2) \) are as in definition 5, then we have \( \|w_1w_2\|_{ab} = p_{1,2} + q_{1,2} + p_1q_2 \). Using the well-known property [14] that for any word \( u \) over \( \Sigma \), \( \|u\|_a \|u\|_b = \|u\|_{ab} + \|u\|_{ba} \), we have

\[
\|w_1w_2\|_{ba} = (p_1 + q_1)(p_2 + q_2) - (p_{1,2} + q_{1,2} + p_1q_2) \\
= (p_1p_2 - p_{1,2}) + (q_1q_2 - q_{1,2}) + q_1p_2 = p_{2,1} + q_{1,2} + q_1p_2.
\]

But \( w_1, w_2 \) are fair words and satisfy the weak ratio property so that \( p_{1,2} = p_{2,1}, q_{1,2} = q_{2,1}, p_1q_2 = q_1p_2 \). Hence \( \|w_1w_2\|_{ba} = p_{1,2} + q_{1,2} + p_1q_2 = \|w_1w_2\|_{ab} \). Likewise \( \|w_1w_2\|_{ac} = \|w_1w_2\|_{ca} \) and \( \|w_1w_2\|_{bc} = \|w_1w_2\|_{cb} \). This shows that \( w_1w_2 \) is fair. Similarly, \( w_2w_1 \) is also fair.

**5. Conclusion**

We have stated here the results relating to ratio properties with the alphabet \( \{a, b, c\} \) for simplicity. The results can be extended to a general alphabet but the ratio property has to be suitably extended.

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