EXPERIMENTAL EVIDENCE FOR VIBRATIONAL RESONANCE AND ENHANCED SIGNAL TRANSMISSION IN CHUA’S CIRCUIT

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We consider a single Chua’s circuit and a system of a unidirectionally coupled n-Chua’s circuits driven by a biharmonic signal with two widely different frequencies \( \omega \) and \( \Omega \), where \( \Omega \gg \omega \). We show experimental evidence for vibrational resonance in the single Chua’s circuit and undamped signal propagation of a low-frequency signal in the system of \( n \)-coupled Chua’s circuits where only the first circuit is driven by the biharmonic signal. In the single circuit, we illustrate the mechanism of vibrational resonance and the influence of the biharmonic signal parameters on the resonance. In the \( n (= 75) \)-coupled Chua’s circuits enhanced propagation of low-frequency signal is found to occur for a wide range of values of the amplitude of the high-frequency input signal and coupling parameter. The response amplitude of the \( i \)th circuit increases with \( i \) and attains a saturation. Moreover, the unidirectional coupling is found to act as a low-pass filter.

Keywords: Chua’s circuit; unidirectionally coupled Chua’s circuits; vibrational resonance; enhanced signal propagation.

1. Introduction

A nonlinear system driven by a biharmonic force with two widely different frequencies say, \( \omega \) and \( \Omega \) with \( \Omega \gg \omega \), can exhibit resonance at the low-frequency \( \omega \) when the amplitude \( g \) or frequency \( \Omega \) of the high-frequency component is varied. This high-frequency driving force induced resonance is termed as vibrational resonance [Landa & Mc Clintock, 2000; Blekhman & Landa, 2004]. This phenomenon can be used to identify a weak signal as well as to enhance the response of a system. Vibrational resonance has been analyzed in different
2011] and the chaotic synchronization for data parameters and external perturbations [Koofigar chaotic systems with unknown time-varying parameters and external perturbations [Chen & Duan, 2008], adaptive synchronization developed for a general class of synchronization [Gomes et al., 2012].

Applicability of a graph-theoretical approach on disturbance [Wai et al., 2011] are investigated. Experimental study of vibrational resonance in a single Chua’s circuit and in a PSpice (Personal Simulation Program with Integrated Circuit Emphasis) circuit simulation of n-coupled Chua’s circuits. Constructing a large size circuit on a circuit board and analyzing its performance have limitations due to the physical effects such as parasitic capacitance effects, internal noise, and mismatch in the circuit components. Further, the tolerance effects and circuit loading can also affect the behavior of the circuit. Moreover, parametric identification during the initial stage of circuit design is difficult because for each parametric values one has to make a search of availability of the off-shelf components. In recent years, circuit simulators based on SPICE, for example PSpice, have been widely used for investigating the dynamics of circuits [Tuinenga, 1995; Roberts & Sedra, 1995; Ozden et al., 2004; Breen et al., 2011; Heo et al., 2012; Zhou & Song, 2012]. Evaluation of circuit functions and performance through PSpice is more productive than on a breadboard. With PSpice one can quickly check a circuit idea and perform simulated test measurements and analysis which are difficult, inconvenient and unwise for the circuits built on a breadboard. Therefore, we preferred PSpice circuit simulation of a system of n-coupled Chua’s circuits instead of its real hardware construction.

The single circuit system is driven by the periodic force $f \sin \omega_1 t$ and $g \sin \Omega t$. The response of the circuit displays resonance when the parameter $g$ or $\Omega$ of the high-frequency force is varied. To characterize the vibrational resonance, we use the response amplitude $Q$, the ratio of amplitude $A_\omega$ of the output signal at the frequency $\omega$ of the input signal and the amplitude $f$ of the input periodic signal $f \sin \omega_1 t$. $A_\omega$ can be measured from the power spectrum of the output of the circuit. The signature of the resonance is clearly seen in the quantity $Q$ and time series plot. We are able to identify the influences of the parameters $\omega$, $\Omega$ and $f$ on the vibrational resonance. The critical value of $g$ at which the resonance
2. Single Chua’s Circuit

In order to investigate the vibrational resonance, we drive the circuit by a biharmonic force of the form $F(t) = F_1(t) + F_2(t) = f \sin \omega t + g \sin 2\omega t$ with $\Omega \gg \omega$. Figure 1(a) depicts the resultant Chua’s circuit. The practical realization of the Chua’s diode $N_R$ is shown in Fig. 1(b). $N_R$ consists of two operational amplifiers and six linear resistors. The typical voltage-current characteristic curve of the Chua’s diode is shown in Fig. 1(c). It has five-segment piecewise-linear form. Throughout our study we fix the values of the circuit parameters and approaches a limiting value for large values of $i$. For a range of values of $g$ and the coupling constant (resistivity of the coupling resistor), an undamped signal propagation with the limiting value of $Q > Q_c$ is achieved.

In this case for distant circuits the high-frequency component of the output signal is suppressed and the output signal becomes a rectangular pulse-like form.

Where $f(v_1)$ is the mathematical representation of Chua’s diode characteristic curve: $f(v_1) = G_v v_1 + 0.5(G_a - G_b)|v_1 + BP_1| - |v_1 - BP_1|$. The
The dimensionless form of Eq. (1) are

\begin{align*}
\dot{x} &= \alpha(y - x - f(x)), \\
\dot{y} &= x - y + z, \\
\dot{z} &= \beta(-y + f'(x)\sin\omega'\tau + g' \sin\Omega'\tau),
\end{align*}

(2a, 2b, 2c)

where

\[ f(x) = bx + 0.5(a - b)(|x + 1| - |x - 1|), \]

and

\[ v_1 = xB_P, \quad v_2 = yB_P, \quad i_L = B_PG; \quad t = C_2\tau/G, \]

\[ G = 1/R, \quad a = RG_a, \quad b = RG_b, \quad \alpha = C_2/C_1, \quad \beta = C_2R^2/L, \quad f = f'B_P, \quad g = g'B_P, \quad \omega = \omega'C_2/G \quad \text{and} \quad \Omega = \Omega'C_2/G. \]

In the experiment we consider that \( \Omega \gg \omega \).

Because the driving input signal has two widely different frequencies \( \omega \) (low-frequency) and \( \Omega \) (high-frequency) the output signal of the circuit has components at these two frequencies and their linear combinations. Assume that in the absence of high-frequency input signal \( g \sin \Omega t \), the amplitude \( A_\omega \) of the output signal at the low-frequency \( \omega \) is weak.

We are interested in enhancing the amplitude \( A_\omega \) of the output signal at the frequency \( \omega \) by the high-frequency input signal \( g \sin \Omega t \). To measure \( A_\omega \) we consider the fast Fourier transform (FFT) of the output signal obtained using the Agilent (MSO6014A) Mixed Signal Oscilloscope. A small fluctuation of \( A_\omega \) is observed in the FFT displayed in the instrument. In view of this for better accuracy an average value of \( A_\omega \) over 25 measurements is obtained. The value of \( A_\omega \) measured in the FFT is in dBV. It is then converted into the units of V.

To characterize the resonance we compute \( Q \) at the frequencies \( \omega \) and \( \Omega \) of voltages \( v_1 \) and \( v_2 \) and the current \( i_L \) for a range of values of the control parameter \( g \). Figure 3(a) shows the variation of \( Q \) of \( v_1 \)

**Fig. 2.** The power spectrum of the voltage \( v_1 \) for four values of \( g \). (a) \( g = 1 \) V, (b) \( g = 1.3 \) V, (c) \( g = 1.55 \) V and (d) \( g = 2 \) V. The values of other circuit parameters are \( C_1 = 10 \) nF, \( C_2 = 100 \) nF, \( L = 18 \) mH, \( R = 1.98 \) k\Ω, \( A = 0.3 \) V, \( \omega = 50 \) Hz and \( \Omega = 500 \) Hz.
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Fig. 3. (a) Response amplitudes $Q$ at the low-frequency $\omega$ (continuous curve) and high-frequency $\Omega$ (dotted curve) associated with $v_1$ versus the control parameter $g$ and (b) $Q$ at the low-frequency $\omega$ associated with $v_2$ (continuous curve) and $i_L$ (dotted curve) versus $g$.

At $\omega$ and $\Omega$ with $g$. As $g$ increases from a small value, $Q$ at $\omega$ increases slowly, varies sharply over an interval and reaches a maximum value at a critical value of $g$ denoted as $g_{\text{VR}}$. The value of $g_{\text{VR}}$ is found as 1.3 V. When $g$ is increased further from $g_{\text{VR}}$, the response amplitude decreases rapidly to a small value. This type of resonance behavior is not observed with $Q$ at $\Omega$.

Next, we illustrate the mechanism of vibrational resonance using a time series plot. Figure 4 presents $v_1(t)$ versus $t$ for six fixed values of $g$. In the absence of a biharmonic force, the Chua circuit for the chosen parametric values has two stable equilibrium points $X_+ = (v_1, v_2, i_L) = (5.656 \, \text{V}, 0 \, \text{V}, -2.855 \, \text{mA})$ and $X_- = (-5.656 \, \text{V}, 0 \, \text{V}, 2.855 \, \text{mA})$ and one unstable equilibrium point $X_0 = (0, 0, 0)$.

For $g = 0 \, \text{V}$, $f = 0.3 \, \text{V}$ and $\omega = 50 \, \text{Hz}$, two period-$T = 1/\omega$ orbits coexist — one orbit about $X_+$ and another about $X_-$. That is, on either side of $v_1 = 0$. When the system is driven further by the high-frequency force with $\Omega = 500 \, \text{Hz}$ then for small...
values of $g$, the two periodic orbits coexist and $v_1(t)$ is modulated by the high-frequency force. This is shown in Figs. 4(a) and 4(b) for $g = 1$ V. At a certain value of $g$, crossing of $v_1 = 0$ takes place. We denote $\tau^+$ as the time spent by the trajectory in the region $v_1 > 0$ before switching to the region $v_1 < 0$. Similarly, we define $\tau^-$, $\tau^+$ and $\tau^-$ as the residence times of the trajectory in the regions $v_1 > 0$ and $v_1 < 0$, respectively. We can then calculate the mean residence times $\tau^+_{MR}$ and $\tau^-_{MR}$. For $g$ values just above the onset of switching between $v_1 < 0$ and $v_1 > 0$, the residence times $\tau^+_{MR}$ and $\tau^-_{MR}$ are unequal. An example is shown in Fig. 4(c) where $g = 1.15$ V. $\tau^+_{MR}$ and $\tau^-_{MR}$ vary with $g$. At the critical value $g = g_{VR} = 1.3$ V, $\tau^+_{MR} = \tau^-_{MR} = T/2$ [see Fig. 4(d)]. There is a periodic switching between the regions $v_1 < 0$ and $v_1 > 0$ with period equal to half the time of the period of the low-frequency input signal. The response amplitude $Q$ is maximum at this value of $g$. This is the mechanism for vibrational resonance. Note that $Q$ is not maximum, that is resonance does not occur, at the value of $g$ for which the onset of crossing occurs. When $g$ is further increased from $g_{VR}$, the synchronization between $v_1(t)$ and the input signal $f \sin \omega t$ is lost [see Fig. 4(e)]. For sufficiently large values of $g$ a rapid switching between the regions $v_1 < 0$ and $v_1 > 0$ occurs and now the oscillation is centered around the equilibrium point $X_0$. This is evident in Fig. 4(f) where $g = 2$ V.

We experimentally analyze the influence of the parameters $\omega$, $f$ and $\Omega$ on resonance. Figure 5 presents the results. In Fig. 5(a) as $\Omega$ increases $g_{VR}$ also increases but $Q_{max}$ (the value of $Q$ at

![Fig. 5. The dependence of the response amplitude ($Q$) of the voltage $v_1$ versus the parameter $g$ on (a) the high-frequency $\Omega = 0.5$ kHz, 1.5 kHz and 2.5 kHz with $f = 0.3$ V and $\omega = 50$ Hz, (b) the amplitude $f = 0.05$ V, 0.2 V and 0.4 V with $\omega = 50$ Hz and $\Omega = 1000$ Hz, (c) different combinations of $\omega$ and $\Omega$ with $\Omega/\omega = 10$ where for the curves 1, 2 and 3 the values of ($\omega$, $\Omega$) are (50 Hz, 500 Hz), (150 Hz, 1500 Hz) and (250 Hz, 2500 Hz) while $f = 0.5$ V and (d) variation of $Q$ with the parameter $\Omega$ for three fixed values of $\omega$ with $f = 0.3$ V and $g = 1.75$ V.](image-url)
resonance) decreases. The width of the bell shape part of the resonance profile increases when $\Omega$ increases. In Fig. 5(b), $g_{VR}$ decreases while $Q_{\text{max}}$ increases with an increase in $f$. The width of the bell shape part increases for increasing values of $f$. Figure 5(c) shows $Q$ versus $g$ for different set of values ($\omega, \Omega$) keeping the ratio $\Omega/\omega$ as 10. Increase in $\omega$ and $\Omega$ leads to the effect observed in Fig. 5(a). In Fig. 5(d) increase in $\omega$ increases the value of $\Omega_{VR}$ but decreases the corresponding $Q_{\text{max}}$. Furthermore, we experimentally measure $g_{VR}$ and $Q_{\text{max}}$ for a range of values of high-frequency $\Omega$ for three different values of $\omega$. Figure 6 depicts the variation of $g_{VR}$ and $Q_{\text{max}}$ with $\Omega$. For all the fixed values of $\omega$, $g_{VR}$ increases while $Q_{\text{max}}$ decreases almost linearly with $\Omega$.

![Diagram](image.png)

**Fig. 6.** Plot of (a) $g_{VR}$, the critical value of $g$ at which resonance occurs and (b) $Q_{\text{max}}$, the value of $Q$ at $g = g_{VR}$ versus the high-frequency $\Omega$ of the driving force. The values of the parameters are $f = 0.3 \mathrm{~V}$ and $\omega = 50 \mathrm{Hz}$ (for the symbol circle), $\omega = 100 \mathrm{Hz}$ (square) and $\omega = 150 \mathrm{Hz}$ (triangle).

### 3. System of n-Coupled Chua’s Circuits

In the previous section our focus is on the single Chua’s circuit. The present section is devoted to the case of a system of $n$-coupled Chua’s circuits with specific emphasis on signal propagation in the presence of a biharmonic external force. Study of coupled nonlinear systems are of great interest in different fields. It has been shown that the response of a nonlinear system can be improved by coupling it into an array of systems. Among the various types of coupling, the simple one is the unidirectional linear coupling introduced by Visarath In and his collaborators to induce oscillations in undriven, overdamped and bistable systems [In et al., 2003a; In et al., 2003b]. This coupling is found to give rise to synchronization [In et al., 2005], propagation of waves of dislocations [Lindner & Bulsara, 2006], enhanced signal propagation in coupled overdamped bistable oscillators [Yao & Zhan, 2010] and in coupled maps [Rajasekhar et al., 2012] and propagation and annihilation of solitons [Lindner & Bulsara, 2006; Breen et al., 2011]. Further, it is utilized in electronic sensors and microelectronic circuits [Razavi, 2008]. We consider a system of unidirectionally coupled $n$-Chua’s circuits where only the first circuit is driven by the biharmonic force. We perform a PSpice simulation with $n = 75$ units. We show the evidence for improved transmission of low-frequency signal by the combined action of a high-frequency signal and a unidirectional coupling.

The system of $n$-coupled Chua’s circuits is shown in Fig. 7. The coupling between the $i$th and $(i + 1)$th circuits is made by feeding the voltage across the capacitor $C_i$ of the $i$th circuit to the $(i + 1)$th circuit through a buffer as shown in Fig. 7. The state equations for the coupled circuits shown in Fig. 7 are [Kapitaniak et al., 1994]

\[
C_i \frac{dv_i(t)}{dt} = \left( \frac{1}{R} \right) (v_{i-1}(t) - v_i(t)) - f(v_i(t)), \quad (3a)
\]

\[
C_i \frac{dv_{i-1}(t)}{dt} = \left( \frac{1}{R} \right) (v_i(t) - v_{i-1}(t) + v_{i+1}(t))), \quad (3b)
\]

\[
f \frac{dv_i(t)}{dt} = -v_i(t) + \delta_i( f \sin \omega t + g \sin \Omega t) + \epsilon_i (v_{i-1}(t) - v_{i+1}(t)), \quad (3c)
\]

where, $\delta_1 = 1$, $\epsilon_1 = 0$ and $\delta_i = 0$ and $\epsilon_i = 1$ for $i = 2, 3, \ldots, n$ and $v_R = i R C$. 

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The buffer circuit makes the coupling as unidirectional. We note that coupled circuits do not form a ring since the output of the last circuit is not fed to the first circuit. The system is a one-way open coupled Chua’s circuits. The high input and low output impedances of the buffer ensures that the flow of the signal between first and (i + 1)th circuits is in forward direction, that is, from the ith circuit to the (i + 1)th circuit only. The strength of the coupling is characterized by the coupling resistor $R_C$. Ullner et al. [2003] reported propagation of the low-frequency signal in a system of coupled oscillators. They considered excitable oscillators with all the oscillators driven by the high-frequency periodic force. First 100 oscillators alone are driven by the low-frequency force and are uncoupled. The rest of the oscillators are unidirectionally coupled. They performed numerical simulation. In our experimental work, we consider a simple setting of n-coupled circuits. As shown in Fig. 7, the first circuit alone is subject to biharmonic input signal and the coupling is unidirectional. In the systems considered in [Ullner et al., 2003], if the coupling is removed then each oscillator exhibits oscillatory motion. In contrast to this, in the present case, if the coupling is removed then the first circuit alone exhibits oscillatory motion while the trajectory of all other circuits settle to the stable equilibrium point $X_1$ or $X_\infty$ depending upon the initial state of the circuit.

We fix the values of the circuit parameters as $C_1 = 10 \, \text{nF}$, $C_2 = 100 \, \text{nF}$, $L = 18 \, \text{mH}$, $R = 2.15 \, \text{k} \Omega$, $\omega = 100 \, \text{Hz}$, $\Omega = 1 \, \text{k} \Omega$ and $f = 0.3 \, \text{V}$ and treat $R_C$ and $g$ as the control parameters. Figure 8(a) presents the variation of $Q_i$ with $i$ (the number of the Chua circuit) for five values of $g$ where $R_C = 1 \, \text{k} \Omega$. For each fixed value of $g$, $Q_i$ varies with $i$ and approaches a limiting value. When $Q_2 > Q_1(< Q_1)$ then $Q_i$ increases (decreases) with $i$ and reaches a saturation with $Q_{25} > Q_1(< Q_1)$. The signal propagation through the coupled circuits is termed as undamped when $Q_{25} > Q_1$ is otherwise damped.

We measure $Q_i$ for $g = 1.2 \, \text{V}$ and for five different values of $R_C$. The result is presented in Fig. 8(b).

Figure 9 shows $Q_i$ versus $i$ for two values of the coupling resistor. For $R_C = 2.165 \, \text{k} \Omega$, $Q_i$ versus $g$ [Fig. 9(a)] decays to zero as $i$ increases. This is an example of damped signal propagation. In Fig. 9(b) where $R_C = 1 \, \text{k} \Omega$ the signal propagation...
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Fig. 9. \(Q_i\) as a function of \(i\) and \(g\) for two values of \(R_C\): illustrating (a) damped propagation of signal (for \(R_C = 2.165 \, k\Omega\)) and (b) undamped signal propagation (for \(R_C = 1 \, k\Omega\)) through the unidirectionally coupled Chua’s circuits. The thick red curve in (b) represents \(Q_1\). In (a) \(Q_1\) is not shown because \(Q_i\)’s with \(i > 1\) are much lower than \(Q_1\).

is undamped \(Q_i > Q_1\), \(i > 1\) for a range of values of \(g\). For \(R_C \in [1 \, k\Omega, 2.5 \, k\Omega]\) and \(g \in [0.5 \, V, 2.0 \, V]\) we experimentally identify the regions for which undamped signal propagation occurs. Figure 10 displays the result. Only for certain sets of values of \(R_C\) and \(g\) undamped signal propagation occurs. \(Q_{75} = 0\) for (i) all values of \(R_C\) if \(g < 0.68 \, V\) and (ii) all values of \(g\) if \(R_C \geq 2.165 \, k\Omega\).

Fig. 10. Undamped signal propagation (marked by red color) in the \((g-R_C)\) parameters space.

Fig. 11. Evolution of \(v_1(t)\) with time at four different nodes denoted as \(i\) where \(R = 2.15 \, k\Omega\), \(R_C = 1 \, k\Omega\) and \(g = 1.1 \, V\). Notice the suppression of high-frequency oscillations as \(i\) increases. (a) \(i = 1\), (b) \(i = 5\), (c) \(i = 15\) and (d) \(i = 50\).

values of \(g\) if \(R_C \geq 2.165 \, k\Omega\). Figure 11 presents another interesting result. In this figure we have plotted \(v_1\) of the \(i\)th circuit versus \(t\) for four values of \(i\). \(v_1\) is periodic with period \(T = 1/\omega\). The \(v_1\) of the first circuit \((i = 1)\) is modulated by the high-frequency drive. Since \(\Omega/\omega = 10\), \(v_1\) has ten peaks over one period. The high-frequency oscillation weakens as the number of circuit \(i\) increases as seen clearly in Figs. 11(b) and 11(c) for \(i = 5\).
The PSpice simulation study of a system of $n$-coupled Chua’s circuits reveals undamped signal propagation for a range of values of the amplitude $g$ of the high-frequency input signal and the coupling parameter $R_C$. Another interesting result is the suppression of the high-frequency component in distant circuits while maintaining an enhanced signal propagation of low-frequency signal. The $Q_i$ either decays to zero or approaches a nonzero constant value with increase in $i$. For most of the parametric choices considered in the present work, $Q_i$ attained a stationary value for $i > 50$. Therefore, we have considered 75 coupled circuits. Study of stochastic, coherence and vibrational resonances in variants of Chua’s circuits such as switch controlled Chua’s circuit and multiscroll Chua’s circuit would bring out practical applications of these circuits in both weak signal detection (ac as well as dc) and output signal amplification.

A theoretical method has been developed to investigate vibrational resonance in nonlinear oscillators with polynomial potentials [Landa & McClintock, 2000; Blekhman & Landa, 2004; Chizhevsky, 2008]. In this approach one can obtain a linear system for the low-frequency component of the solution. Since, the equation of motion of Chua’s circuit is piecewise-linear it is very difficult to separately write the equations of motion for the slow and fast components (with frequencies $\omega$ and $\Omega$, respectively).

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