On chromaticity of hypergraphs

Mieczysław Borowiecki\textsuperscript{a}, Ewa Łazuka\textsuperscript{b}

\textsuperscript{a}Faculty of Mathematics, Computer Science and Econometrics, University of Zielona Góra, Podgórna 50, 65-246 Zielona Góra, Poland
\textsuperscript{b}Department of Applied Mathematics, Lublin University of Technology, Nadbystrzycka 38D, 20-618 Lublin, Poland

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Abstract

Let $H = (X, \mathcal{E})$ be a simple hypergraph and let $f(H, \lambda)$ denote its chromatic polynomial. Two hypergraphs $H_1$ and $H_2$ are chromatic equivalent if $f(H_1, \lambda) = f(H_2, \lambda)$. The equivalence class of $H$ is denoted by $\langle H \rangle$. Let $\mathcal{H}$ and $\mathcal{K}$ be two classes of hypergraphs. $\mathcal{H}$ is said to be chromatically characterized in $\mathcal{K}$ if for every $H \in \mathcal{H} \cap \mathcal{K}$ we have $\langle H \rangle \cap \mathcal{K} = H \cap \mathcal{K}$.

In this paper we prove that uniform hypertrees and uniform unicyclic hypergraphs are chromatically characterized in the class of linear hypergraphs.

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1. Introduction

A simple hypergraph $H = (X, \mathcal{E})$ consists of a finite non-empty set $V(H) = X$ of vertices and a family $E(H) = \mathcal{E}$ of edges which are distinct non-empty subsets of $X$ of the cardinality at least 2. $H$ is $h$-uniform if $|e| = h$ for each edge $e \in \mathcal{E}$. A hypergraph is linear if any two of its edges do not intersect in more than one vertex. The number of edges containing a vertex $v$ is its degree $d_H(v)$.

A cycle $C$ of length $k$ in $H$ is a subhypergraph comprising $k$ distinct vertices $v_1, v_2, \ldots, v_k$ and $k$ distinct edges $e_1, e_2, \ldots, e_k$ of a hypergraph $H$ such that $v_{i-1}, v_i \in e_i$ for each $i \in \{1, 2, \ldots, k\}$ (indices are taken modulo $k$) (see [1]). A cycle $C$ is elementary if $d_C(v_i) = 2$ for each $i \in \{1, 2, \ldots, k\}$ and $d_C(u) = 1$ for each other vertex $u \in \bigcup_{i=1}^{k} e_i$.

A hypertree is a connected linear hypergraph without cycles. A unicyclic hypergraph is a connected hypergraph containing exactly one cycle.

Let $H$ be a hypergraph and $\lambda$ be a positive integer. A $\lambda$-coloring of $H$ is a function $f : V(H) \to \{1, 2, \ldots, \lambda\}$ such that for each edge $e \in E(H)$ there exist $x, y \in e$ for which $f(x) \neq f(y)$. The number of $\lambda$-colorings of $H$ is given by a polynomial $f(H, \lambda)$ of degree $|V(H)| \lambda$, called the chromatic polynomial of $H$.

Hypergraphs $H_1$ and $H_2$ are chromatic equivalent ($H_1 \sim H_2$) if $f(H_1, \lambda) = f(H_2, \lambda)$. The equivalence class determined by $H$ under $\sim$ is denoted by $\langle H \rangle$. A hypergraph $H$ is chromatically unique if $\langle H \rangle = \{H\}$. A class $\mathcal{H}$ of hypergraphs is said to be chromatically characterized if for any $H \in \mathcal{H}$ the condition $\langle H \rangle = \mathcal{H}$ holds.

E-mail addresses: m.borowiecki@wmie.uz.zgora.pl (M. Borowiecki), e.lazuka@pollub.pl (E. Łazuka).

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These notions were first introduced and studied only for graphs by Chao and Whitehead [3]. Afterwards many scientists, among them Dohmen, Jones and Tomescu [4–6, 8], started to study the chromaticity of hypergraphs. Till now only few chromatically equivalent or chromatically unique hypergraphs are known [2, 8].

Let \( \mathcal{K} \) and \( \mathcal{H} \) be two classes of hypergraphs. \( \mathcal{K} \) is said to be chromatically characterized in the class \( \mathcal{X} \) if for every \( H \in \mathcal{X} \cap \mathcal{H} \) we have \( \langle H \rangle = \mathcal{K} \cap \mathcal{H} \). We prove that uniform hypertrees and uniform unicyclic hypergraphs are chromatically characterized in the class of linear hypergraphs.

2. Some known results

In this section we remind some known results which will be used in the following sections.

**Theorem 1** (Tomescu [8]). Let \( H \) be a hypergraph of order \( n \). Then \( f(H, \lambda) = \lambda^n + a_1 \lambda^{n-1} + \cdots + a_{n-1} \lambda \), where

\[
a_{n-i} = \sum_{j \geq 0} (-1)^j N(i, j) \quad \text{for} \ i \in \{1, 2, \ldots, n-1\},
\]

where \( N(i, j) \) denotes the number of subhypergraphs of \( H \) with \( n \) vertices, \( i \) connected components and \( j \) edges.

**Theorem 2** (Borowiecki and Lazuka [2]). If \( H \) is a hypergraph such that \( H = H_1 \cup \cdots \cup H_k \) for \( k \geq 2 \), where

\[
H_i \cap H_j = K_p \quad \text{for} \ i \neq j \quad \text{and} \quad \bigcap_{i=1}^{k} H_i = K_p,
\]

where \( K_p \) is a complete graph with \( p \geq 1 \) vertices, then

\[
f(H, \lambda) = [f(K_p, \lambda)]^{1-k} f(H_1, \lambda) f(H_2, \lambda) \cdots f(H_k, \lambda).
\]

**Theorem 3** (Dohmen [4]). If \( H \) is a hypergraph such that \( f(H, \lambda) = a_0 \lambda^n + a_1 \lambda^{n-1} + \cdots + a_{n-1} \lambda + a_n \) for \( n \geq 1 \), then \( n = |V(H)| \), \( a_0 = 1 \) and \( a_n = 0 \). Moreover, if for a certain \( h \in \{2, 3, \ldots, n\} \) each edge of \( H \) contains at least \( h \) vertices, then \( a_1 = a_2 = \cdots = a_{h-2} = 0 \) and \( a_{h-1} = -b \), where \( b \) is the number of \( h \)-edges of \( H \).

3. Some lemmas

Lemmas 1–3, which will be proved below, show reverse reasoning to Theorem 3, i.e., knowing some of the coefficients of the chromatic polynomial \( f(H, \lambda) \) we deduce some information about the structure of \( H \).

**Lemma 1.** Let \( H \) be a hypergraph of order \( n \geq 1 \) and \( f(H, \lambda) = \lambda^n + a_1 \lambda^{n-1} + \cdots + a_{n-1} \lambda \). If \( a_{n-1} \neq 0 \) then \( H \) is connected.

**Proof.** By Theorem 1, we have \( a_{n-1} = \sum_{j \geq 0} (-1)^j N(1, j) \). Since \( a_{n-1} \neq 0 \), it follows that there must exist at least one connected spanning subhypergraph of \( H \). Hence \( H \) must be connected. \( \square \)

**Lemma 2.** Let \( H \) be a hypergraph of order \( n \geq 3 \) and \( f(H, \lambda) = \lambda^n + a_1 \lambda^{n-1} + \cdots + a_{n-1} \lambda \). If for a certain \( h \geq 3 \) and for a certain \( b \geq 0 \) the following equalities hold

\[
a_1 = a_2 = \cdots = a_{h-2} = 0 \quad \text{and} \quad a_{h-1} = -b,
\]

then \( H \) has no \( j \)-edges for \( j \in \{2, 3, \ldots, h-1\} \) and has exactly \( b \) \( h \)-edges.

**Proof.** In order to prove that \( H \) has no \( j \)-edges for \( 2 \leq j \leq h-1 \) we use induction.

1. By Theorem 1, \( a_1 = \sum_{j \geq 0} (-1)^j N(n-1, j) \), where \( N(n-1, j) \) is the number of spanning subhypergraphs of \( H \) having \( j \) edges and \( n-1 \) connected components. The subhypergraph of this property must consist of one 2-edge and \( n-2 \) isolated vertices. Since \( a_1 = 0 \), it follows that \( H \) has no 2-edges.
Lemma 3. Let $H$ be a linear hypergraph of order $n \geq 3$ and $f(H, \lambda) = \lambda^n + a_1 \lambda^{n-1} + \cdots + a_n$. If for a certain $l \geq 1$, for a certain $2 \leq z \leq l$ and for a certain $h \geq 3$

$$a_k(h-1) = (-1)^k \binom{l}{k} \quad \text{for} \quad k \in \{0, 1, \ldots, z\},$$

where $k(h-1) < n$, and $a_i = 0$ for each

$$i \in \{i \in \mathcal{N}: i < z(h-1) \land i \neq k(h-1) \quad \text{for} \quad k \in \{0, 1, \ldots, z-1\}\},$$

then $H$ has no $j$-edges for $j \in \{2, 3, \ldots, z(h-1) + 1\}\{h\}$, has $l$ $h$-edges and does not contain $h$-uniform cycles of length $\leq z$.

Proof. We shall analyze the coefficients of $f(H, \lambda)$.

1. $H$ has $n \geq 3$ vertices, $a_1 = a_2 = \cdots = a_{h-2} = 0$ and $a_{h-1} = (-1)^{h-1} \binom{l}{1} = -l$ in its chromatic polynomial. Therefore, by Lemma 2 for $b = l$, we know that $H$ has no $j$-edges for $2 \leq j \leq h-1$ and has $l$ $h$-edges. Let the symbol $M(n-i, 1)$ denotes the number of spanning subhypergraphs of $H$ consisting of one $(i+1)$-edge and $n-(i+1)$ isolated vertices. It is easy to see that the mentioned spanning subhypergraphs are the only subhypergraphs of $H$ with $n$ vertices, $n-i$ connected components and one edge, i.e., $M(n-i, 1) = N(n-i, 1)$. In order to simplify the proof we shall use the symbol $N(n-i, 1)$ instead of $M(n-i, 1)$.

2. Let us notice that $a_1 = a_{h+1} = \cdots = a_2^{(h-1)} = 0$.

(a) By Theorem 1, $a_h = \sum_{j=0}^{h}(-1)^j M(n-h, j)$. Since $H$ is linear, it does not exist a spanning subhypergraph having only $h$-edges and $n-h$ connected components. Hence $a_h = (-1)^{h} M(n-h, 1)$. Since $a_h = 0$, $H$ has no $(h+1)$-edges.

(b) Let us assume that $H$ has no $(h+s)$-edges for $s \in \{1, 2, \ldots, r\}$ for a certain $1 \leq r \leq h-3$. By Theorem 1, we have $a_{h+s} = \sum_{j=0}^{h+s}(-1)^j M(n-(h+s), j)$. Similarly as above, we deduce that $a_{h+s} = (-1)^{h+s} M(n-(h+s), 1)$. Since $a_{h+s} = 0$, $H$ has no $(h+s+1)$-edges.

By induction, $H$ has no $(h+s)$-edges for $s \in \{1, 2, \ldots, h-2\}$.

3. We have $a_{2(h-1)} = (-1)^2 \binom{l}{2}$, whereas, by Theorem 1,

$$a_{2(h-1)} = \sum_{j=0}^{h-2}(-1)^j M(n-(2h-2), j)$$

$$= (-1)^{h-1} M(n-(2h-2), 1) + (-1)^2 M(n-(2h-2), 2),$$

where $M(n-(2h-2), 2)$ is the number of spanning subhypergraphs consisting of two $h$-edges (disjoint or having exactly one vertex in common) and the corresponding number of isolated vertices. The number of $h$-edges equals $l$, so $M(n-(2h-2), 2) = \binom{l}{2}$. Hence

$$(-1)^2 \binom{l}{2} = (-1)^{h-1} M(n-(2h-2), 1) + (-1)^2 \binom{l}{2},$$

which implies $M(n-(2h-2), 1) = 0$. It means that $H$ has no $(2h-1)$-edges.
(4) Let us notice that \(a_{2(h-1)+1} = a_{2(h-1)+2} = \cdots = a_{2(h-1)+h-3} = 0\).

(a) By Theorem 1 and the previous results, we have

\[
a_{2(h-1)+1} = \sum_{j \geq 0} (-1)^j N(n - (2(h - 1) + 1), j)
= (-1)^1 N(n - (2(h - 1) + 1), 1).
\]

Since \(a_{2(h-1)+1} = 0\), \(H\) has no \(2h\)-edges.

(b) Let us assume that \(H\) has no \((2h + s)\)-edges for \(s \in \{0, 1, \ldots, r\}\) for a certain \(0 \leq r \leq h - 5\). By Theorem 1 and the previous results, we have

\[
a_{2h+r} = \sum_{j \geq 0} (-1)^j N(n - (2h + r), j)
= (-1)^1 N(n - (2h + r), 1).
\]

But \(a_{2h+r} = 0\), so \(H\) has no \((2h + r + 1)\)-edges.

By induction, \(H\) has no \((2h + s)\)-edges for \(s \in \{0, 1, \ldots, h - 4\}\).

(5) We have \(a_{2(h-1)+h-2} = a_{3h-4} = 0\), whereas, by Theorem 1,

\[
a_{3h-4} = (-1)^1 N(n - (3h - 4), 1) + (-1)^3 N(n - (3h - 4), 3),
\]

where \(N(n - (3h - 4), 3)\) is the number of spanning subhypergraphs consisting of an \(h\)-uniform 3-cycle and \(n - (3h - 3)\) isolated vertices. Both numbers are non-negative, what implies

\[
N(n - (3h - 4), 1) = N(n - (3h - 4), 3) = 0.
\]

It means that \(H\) has no \((3h - 3)\)-edges and no three of its \(h\)-edges form a cycle.

(6) We have \(a_{3h-3} = (-1)^3 \binom{1}{3}\), whereas, by Theorem 1,

\[
a_{3h-3} = (-1)^1 N(n - (3h - 3), 1) + (-1)^3 N(n - (3h - 3), 3),
\]

where \(N(n - (3h - 3), 3)\) is the number of spanning subhypergraphs consisting of exactly three \(h\)-edges not forming a cycle and the corresponding number of isolated vertices. By the previous results, \(H\) does not contain an \(h\)-uniform 3-cycle. Therefore,

\[
N(n - (3h - 3), 3) = \binom{l}{3} \quad \text{and} \quad N(n - (3h - 3), 1) = 0.
\]

It means that \(H\) has no \((3h - 2)\)-edges. In the next part of the proof we use induction on \(r\) and \(t\). We treat the parts 4, 5 and 6 as the first step.

(7) For \(k \in \{3, 4, \ldots, z - 1\}\) we have \(a_{k(h-1)+1} = \cdots = a_{k(h-1)+h-2} = 0\) and \(a_{(k+1)(h-1)} = (-1)^{k+1} \binom{l}{k+1}\). Let us assume that for a certain \(2 \leq r \leq z - 2\), \(H\) has no \((s(h - 1) + t)\)-edges for \(s \in \{2, 3, \ldots, r - 1\}\) and \(t \in \{2, 3, \ldots, h\}\), and no its \(h\)-edges form a cycle of length \(s + 1\) for \(s \in \{2, 3, \ldots, r - 1\}\). Let us consider the coefficients

\[
a_{r(h-1)+1} = \cdots = a_{r(h-1)+h-2} = 0 \quad \text{and} \quad a_{(r+1)(h-1)} = (-1)^{r+1} \binom{l}{r+1}.
\]

(a) By Theorem 1, we have

\[
a_{r(h-1)+1} = (-1)^1 N(n - (r(h - 1) + 1), 1).
\]

Since \(a_{r(h-1)+1} = 0\), \(H\) has no \((r(h - 1) + 2)\)-edges.
(b) Let us now assume that for a certain $2 \leq t \leq h - 3$, $H$ has no $(r(h - 1) + t')$-edges for $t' \in \{2, 3, \ldots, t\}$. By Theorem 1, we have
\[
a_{r(h-1)+t} = (-1)^t N(n - (r(h - 1) + t), 1).
\]
Since $a_{r(h-1)+t} = 0$, $H$ has no $(r(h - 1) + t + 1)$-edges.
By induction, $H$ has no $(r(h - 1) + t')$-edges for $t' \in \{2, 3, \ldots, h - 2\}$. 

(8) By Theorem 1, we have
\[
a_{r(h-1)+h-2} = (-1)^t N(n - (r(h - 1) + h - 2), 1) \\
+ (-1)^{t+1} N(n - (r(h - 1) + h - 2), r + 1),
\]
where $N(n - (r(h - 1) + h - 2), r + 1)$ is the number of spanning subhypergraphs consisting of $h$-edges forming a cycle of length $r + 1$ and the corresponding number of isolated vertices. Let us consider two cases.
(a) If $r$ is even then
\[
-N(n - (r(h - 1) + h - 2), 1) - N(n - (r(h - 1) + h - 2), r + 1) = 0.
\]
Both numbers are non-negative so
\[
N(n - (r(h - 1) + h - 2), 1) = 0
\]
and
\[
N(n - (r(h - 1) + h - 2), r + 1) = 0.
\]
It means that $H$ has no $(r(h - 1) + h - 1)$-edges and does not contain an $h$-uniform cycle of length $r + 1$.
(b) If $r$ is odd then
\[
N(n - (r(h - 1) + h - 2), 1) = N(n - (r(h - 1) + h - 2), r + 1).
\]

(9) Let us consider $a_{(r+1)(h-1)}$. Similarly as above, we obtain
\[
(-1)^t \binom{l}{r+1} = (-1)^t N(n - (r+1)(h - 1), 1) \\
+ (-1)^{t+1} N(n - (r+1)(h - 1), r + 1),
\]
where $N(n - (r+1)(h - 1), r + 1)$ is the number of spanning subhypergraphs consisting of $r + 1$-edges not forming a cycle and the corresponding number of isolated vertices. Let us consider two cases.
(a) For even $r$, by the case (8a), no $r$-edges form $(r+1)$-cycle, so $N(n - (r+1)(h - 1), r + 1) = \binom{l}{r+1}$.
Hence
\[
-N\left(\binom{l}{r+1}\right) = -N(n - (r+1)(h - 1), 1) - \binom{l}{r+1},
\]
and then $N(n - (r+1)(h - 1), 1) = 0$, so $H$ has no edges of the cardinality $(r+1)(h - 1) + 1$.
(b) For odd $r$, we have
\[
N(n - (r+1)(h - 1), r + 1) = \binom{l}{r+1} - N(n - (r(h - 1) + h - 2), r + 1).
\]
We obtain
\[
\binom{l}{r+1} = -N(n - (r+1)(h - 1), 1) \\
+ \binom{l}{r+1} - N(n - (r(h - 1) + h - 2), r + 1).
\]
All the numbers in this equality are non-negative, so
\[ N(n - (r + 1)(h - 1), 1) = 0 \]
and
\[ N(n - (r(h - 1) + h - 2), r + 1) = 0. \]

By the case (8b), we also obtain \( N(n - (r(h - 1) + h - 2), 1) = 0. \) It means that \( H \) has neither \((r + 1)(h - 1)\)-edges nor the edges of the cardinality \((r + 1)(h - 1) + 1\) and does not contain an \( h \)-uniform cycle of length \( r + 1. \)

By induction, \( H \) has no \((s(h - 1) + t)-\text{edges}\) for \( s \in \{2, 3, \ldots, z - 1\} \) and \( t \in \{2, 3, \ldots, h\} \), moreover, \( H \) does not contain \( h \)-uniform cycles of length \( s + 1 \) for \( s \in \{2, 3, \ldots, z - 1\}. \)

4. Chromaticity of uniform hypertrees

A hypertree is a connected linear hypergraph without cycles. Dohmen gave the explicit formula of the chromatic polynomial of an \( h \)-uniform hypertree.

**Theorem 4 (Dohmen [5]).** If \( T_m^h \) is an \( h \)-uniform hypertree with \( m \) edges, where \( h \geq 2 \) and \( m \geq 0 \), then
\[ f(T_m^h, \lambda) = \lambda^{h-1} - 1)^m. \]

Of course hypertrees, even uniform, are not chromatically unique. In the following theorem we prove that uniform hypertrees are chromatically characterized in the class of linear hypergraphs.

**Theorem 5.** If \( H \) is a linear hypergraph and
\[ f(H, \lambda) = \lambda^{(h-1)^m}, \quad \text{where} \ h \geq 3 \text{ and } m \geq 1, \]
then \( H \) is an \( h \)-uniform hypertree with \( m \) edges.

**Proof.** If for \( h \geq 3 \) and \( m \geq 1 \) the chromatic polynomial of \( H \) is of the form (1), then
\[ f(H, \lambda) = \left( \begin{array}{c} m \\ 0 \end{array} \right) \lambda^m(h-1)+1(-1)^0 + \left( \begin{array}{c} m \\ 1 \end{array} \right) \lambda^{(m-1)(h-1)+1}(-1)^1 + \cdots + \left( \begin{array}{c} m \\ k \end{array} \right) \lambda^{(m-k)(h-1)+1}(-1)^k + \cdots + \left( \begin{array}{c} m \\ m \end{array} \right) \lambda^m(-1)^m. \]

Of course the degree of \( f(H, \lambda) \) equals \( m(h - 1) + 1 \). Let \( n = m(h - 1) + 1 \) and \( f(H, \lambda) = \sum_{i=0}^n a_i \lambda^{n-i}. \) There are nonzero coefficients of the polynomial giving by (1) only at \( \lambda^{(m-k)(h-1)+1} \) for \( k \in \{0, 1, \ldots, m\}. \) Let us note that \( (m-k)(h-1)+1 = n-k(h-1), \) and so the mentioned nonzero coefficients of \( f(H, \lambda) \) are of the form
\[ a_k(h-1) = (-1)^k \left( \begin{array}{c} m \\ k \end{array} \right) \quad \text{for } k \in \{0, 1, \ldots, m\}. \]

By studying the coefficients of \( f(H, \lambda) \), we obtain the following results:

- \( a_0 = (-1)^0 \left( \begin{array}{c} m \\ 0 \end{array} \right) = 1, \) so \( H \) is a hypergraph of order \( m(h - 1) + 1; \)
- \( a_{m-1} = a_{m(h-1)} = (-1)^m \left( \begin{array}{c} m \\ m \end{array} \right) \neq 0, \) so, by Lemma 1, the hypergraph \( H \) is connected.

It is easy to see that for \( h \geq 3 \) and \( m \geq 1 \) the condition \( m(h - 1) + 1 \geq 3 \) holds. By Lemma 3, for \( n = m(h - 1) + 1, \)
\( l = m \) and \( z = m, \) it follows that \( H \) does not have \( j \)-edges for \( j \in \{2, 3, \ldots, m(h - 1) + 1\}\backslash\{h\}, \) has \( m \)-edges and does not contain \( h \)-uniform cycles of length \( \leq m. \) Therefore \( H \) is an \( h \)-uniform hypertree.

**Theorem 5** needs the linearity of a hypergraph. There might exist hypergraphs which are not linear but are chromatically equivalent with \( h \)-uniform hypertrees. However, by **Theorem 6**, these hypergraphs could not be \( h \)-uniform.
Theorem 6 (Tomescu [8]). If $h$-uniform hypergraphs $H_1$ and $H_2$ are chromatically equivalent and $H_1$ is linear, then $H_2$ is also linear.

5. Chromaticity of uniform unicyclic hypergraphs

A unicyclic hypergraph is a connected hypergraph containing exactly one cycle. One of such hypergraphs is an elementary cycle. Dohmen [5] gave the explicit formula of an $h$-uniform elementary cycle, while Tomescu [8] proved that such a hypergraph is chromatically unique.

Theorem 7 (Dohmen [5]). If $C^h_m$ is an $h$-uniform elementary cycle with $m$ edges, where $h \geq 2$ and $m \geq 3$, then

$$f(C^h_m, \lambda) = (\lambda^{h-1} - 1)^m + (-1)^m(\lambda - 1).$$

We prove that uniform unicyclic hypergraphs are chromatically characterized in the class of linear hypergraphs.

Theorem 8. Let $H$ be a linear hypergraph. $H$ is an $h$-uniform unicyclic hypergraph with $m + p$ edges and a cycle of length $p$ if and only if

$$f(H, \lambda) = (\lambda^{h-1} - 1)^{m+p} + (-1)^p(\lambda - 1)(\lambda^{h-1} - 1)^m,$$

where $h \geq 3$, $m \geq 0$ and $p \geq 3$.

Proof. Necessity. Without loss of generality, by Theorem 2, we can assume that a linear hypergraph $H$ consists of an $h$-uniform elementary $p$-cycle $C^h_p$ and a hypertree $T^h_m$ with $m$ edges in such a way that $C^h_p$ and $T^h_m$ have exactly one vertex in common. By Theorems 4, 7 and 2, we have

$$f(H, \lambda) = \frac{f(C^h_p, \lambda) \cdot f(T^h_m, \lambda)}{f(K_1, \lambda)} = (\lambda^{h-1} - 1)^{m+p} + (-1)^p(\lambda - 1)(\lambda^{h-1} - 1)^m.$$

Sufficiency. Let $H$ be a linear hypergraph with $f(H, \lambda)$ of the form (2). Then

$$f(H, \lambda) = \binom{m + p}{0} \lambda^{(h-1)(m+p)}(-1)^0 + \cdots + \binom{m + p}{m + p} \lambda^0(-1)^{m+p}$$

$$+ (-1)^p(\lambda - 1) \left\{ \binom{m}{0} \lambda^{(h-1)m}(-1)^0 + \cdots + \binom{m}{m} \lambda^0(-1)^m \right\}.$$

Thus, $f(H, \lambda) = f_1(H, \lambda) + f_2(H, \lambda)$, where

$$f_1(H, \lambda) = \binom{m + p}{0} \lambda^{(h-1)(m+p)}(-1)^0 + \cdots + \binom{m + p}{m + p} \lambda^0(-1)^{m+p}$$

and

$$f_2(H, \lambda) = (-1)^p(\lambda - 1) \left\{ \binom{m}{0} \lambda^{(h-1)m}(-1)^0 + \cdots + \binom{m}{m} \lambda^0(-1)^m \right\}.$$

The polynomial $f_1(H, \lambda)$ is of degree $(m + p)(h - 1)$ and $f_2(H, \lambda)$ is of degree $m(h - 1) + 1$. The degree $n$ of $f(H, \lambda)$ equals $(m + p)(h - 1)$ because for $p \geq 3$ and $h \geq 3$ we have $(m + p)(h - 1) > m(h - 1) + 1$.

We study which exponents of $\lambda$ occur in $f(H, \lambda)$. There are only the exponents $(h - 1)(m + p - k)$ for $k \in \{0, 1, \ldots, m + p\}$ in $f_1(H, \lambda)$ and the exponents $(h - 1)(m - j) + 1$ and $(h - 1)(m - j)$ for $j \in \{0, 1, \ldots, m\}$ in $f_2(H, \lambda)$.

(i) It is not possible that $(h - 1)(m + p - k) = (h - 1)(m - j) + 1$ and $(h - 1)(m - j) + 1 = (h - 1)(m - i)$ for $k \in \{0, 1, \ldots, m + p\}$ and $i, j \in \{0, 1, \ldots, m\}$.

(ii) The equality $(h - 1)(m + p - k) = (h - 1)(m - j)$, for $k \in \{0, 1, \ldots, m + p\}$ and $j \in \{0, 1, \ldots, m\}$, implies that $k = p + j$, what is possible only for $k \in \{p, p + 1, \ldots, p + m\}$. 


Finally, if \( f(H, \lambda) = \sum_{i=0}^{n} a_i \lambda^{n-i} \) then we have the following nonzero coefficients of \( f(H, \lambda) \):

- at \( \lambda^{(h-1)(m+p-k)} = \lambda^{h-1(m+p-k)(h-1)} = \lambda^{n-k(h-1)} \):
  \[
a_k(h-1) = (-1)^k \binom{m+p}{k} \quad \text{for } k \in \{0, 1, \ldots, p-1\};
\]

- at \( \lambda^{(h-1)(m+p-k)} = \lambda^{n-k(h-1)} = \lambda^{n-(p+j)(h-1)} \) for \( k = p + j \):
  \[
a_{(p+j)}(h-1) = (-1)^{p+j} \left( \binom{m+p}{j+p} + (-1)^{p+1} \binom{m}{j} \right)
  = (-1)^{p+j} \left[ \binom{m+p}{j+p} - \binom{m}{j} \right] \quad \text{for } j \in \{0, 1, \ldots, m\};
\]

- at \( \lambda^{(h-1)(m-j)+1} = \lambda^{h-1(m+p)-(p+j)+1} = \lambda^{n-(p+j)(h-1)-1} \):
  \[
a_{(p+j)}(h-1)+1 = (-1)^{p+j} \binom{m}{j} \quad \text{for } j \in \{0, 1, \ldots, m\}.
\]

And now we are still studying the coefficients of \( f(H, \lambda) \).

- \( a_0 = (-1)^0 \left( \binom{m+p}{0} \right) = 1 \), the degree of \( f(H, \lambda) \) is equal to \((m+p)(h-1)\), so \( H \) is of order \((m+p)(h-1)\).

- \( a_{n-1} = a_{(m+p)+(h-1)-1} = (-1)^{0+m} \left( \binom{m}{m} \right) \neq 0 \), so, by Lemma 1, \( H \) is connected.

If we assume \( n = (m+p)(h-1), l = m+p \) and \( z = p-1 \) then, by Lemma 3, \( H \) does not have \( j \)-edges for \( j \in \{2, 3, \ldots, (p-1)(h-1) + 1\} \). \( H \) has \( m+p \) \( h \)-edges and does not contain \( h \)-uniform cycles of length \( \leq p-1 \).

Now we shall analyse the coefficients \( a_i \) of \( f(H, \lambda) \) for \( i > (p-1)(h-1) \). In the next part of this proof by the symbol \( N(n-i, 1) \) we shall denote the number of spanning subhypergraphs of \( H \) consisting of one \((i+1)\)-edge and \( n-(i+1) \) isolated vertices. Let us notice that since \( H \) is a linear hypergraph, the spanning subhypergraphs of \( H \) containing \( h \)-uniform cycles will be counted only by the coefficients \( a_{(p+j)(h-1)-1} \) for \( j \in \{0, 1, \ldots, m\} \).

(1) We use induction in this part of the proof. We have
\[
a_{(p-1)(h-1)+1} = a_{(p-1)(h-1)+2} = \cdots = a_{(p-1)(h-1)+h-3} = 0.
\]

(a) By Theorem 1 and the properties of \( H \) which are already known,
\[
a_{(p-1)(h-1)+1} = \sum_{j \geq 0} (-1)^j N(n - ((p-1)(h-1) + 1), j)
  = (-1)^1 N(n - ((p-1)(h-1) + 1), 1).
\]
Since \( a_{(p-1)(h-1)+1} = 0 \), \( N(n - ((p-1)(h-1) + 1), 1) = 0 \). It means that \( H \) does not have \(((p-1)(h-1)+2)\)-edges.

(b) Let us now assume that for a certain \( 2 \leq r \leq h-3 \), \( H \) does not have \(((p-1)(h-1)+s)\)-edges for \( s \in \{2, 3, \ldots, r\} \). By Theorem 1 and the induction hypothesis,
\[
a_{(p-1)(h-1)+r} = \sum_{j \geq 0} (-1)^j N(n - ((p-1)(h-1) + r), j)
  = (-1)^1 N(n - ((p-1)(h-1) + r), 1).
\]
Since \( a_{(p-1)(h-1)+r} = 0 \), \( N(n - ((p-1)(h-1) + r), 1) = 0 \). It means that \( H \) does not have \(((p-1)(h-1)+r+1)\)-edges.

By induction, \( H \) has no edges of the cardinality \((p-1)(h-1)+s\) for \( s \in \{2, 3, \ldots, h-2\} \).
(2) We have \( a_{(p-1)(h-1)+h-2} = a_{p(h-1)-1} = (-1)^p \binom{m}{0} = (-1)^p \). By Theorem 1,
\[
a_{p(h-1)-1} = (-1)^1 N(n - (p(h-1) - 1), 1) \\
+ (-1)^p N(n - (p(h-1) - 1), p),
\]
where \( N(n - (p(h-1) - 1), p) \) is the number of spanning subhypergraphs consisting only of \( h \)-edges forming a \( p \)-cycle. Let us consider two cases.
(a) For even \( p \) we have
\[
1 = -N(n - (p(h-1) - 1), 1) + N(n - (p(h-1) - 1), p),
\]
which implies \( N(n - (p(h-1) - 1), p) \geq 1. \)
(b) For odd \( p \) we have
\[
-1 = -N(n - (p(h-1) - 1), 1) - N(n - (p(h-1) - 1), p),
\]
what implies \( N(n - (p(h-1) - 1), p) \in \{0, 1\}. \)
(3) Now we consider the next coefficient of \( f(H, \lambda) \):
\[
a_{p(h-1)} = (-1)^p \left[ \binom{m+p}{p} - \binom{m}{p} \right] = (-1)^p \left[ \binom{m+p}{p} - 1 \right].
\]
As it was previously
\[
a_{p(h-1)} = (-1)^1 N(n - p(h-1), 1) + (-1)^p N(n - p(h-1), p),
\]
where \( N(n - p(h-1), p) \) is the number of spanning subhypergraphs which are acyclic and each of them has only \( p \) \( h \)-edges. Let us consider two cases.
(a) For even \( p \)
\[
\binom{m+p}{p} - 1 = -N(n - p(h-1), 1) + N(n - p(h-1), p).
\]
Since the number of all \( h \)-edges of \( H \) is equal to \( m+p \) and \( H \) contains \( N(n - (p(h-1) - 1), p) \) \( h \)-uniform cycles of length \( p \), it follows that
\[
N(n - p(h-1), p) = \binom{m+p}{p} - N(n - (p(h-1) - 1), p).
\]
Therefore \( N(n - p(h-1), 1) + N(n - (p(h-1) - 1), p) = 1. \) By the case (2a), \( N(n - (p(h-1) - 1), p) = 1 \) and consequently
\[
N(n - p(h-1), 1) = 0, \quad N(n - p(h-1), p) = \binom{m+p}{p} - 1
\]
and
\[
N(n - (p(h-1) - 1), 1) = 0.
\]
It means that \( H \) has neither \( p(h-1) \)-edges nor \( (p(h-1) + 1) \)-edges and contains exactly one \( h \)-uniform \( p \)-cycle.
(b) For odd \( p \)
\[
-\binom{m+p}{p} + 1 = -N(n - p(h-1), 1) - N(n - p(h-1), p).
\]
By the case (2b), we obtain \( N(n - (p(h-1) - 1), p) \in \{0, 1\}. \) For \( N(n - (p(h-1) - 1), p) = 0 \) it would be \( N(n - (p(h-1) - 1), 1) = 1 \) and \( N(n - p(h-1), p) = \binom{m+p}{p}. \) And then
\[
-\binom{m+p}{p} + 1 = -N(n - p(h-1), 1) - \binom{m+p}{p},
\]
Let us notice that

\[ N(n - (p(h - 1) - 1), 1) = 0, \quad N(n - p(h - 1), p) = \binom{m + p}{p} - 1 \]

and consequently

\[ N(n - p(h - 1) - 1, 1) = 0. \]

Now, as it was in (3a), we deduce that \( H \) has neither \( p(h - 1) \)-edges nor \((p(h - 1) + 1)\)-edges and contains exactly one \( h \)-uniform \( p \)-cycle.

In the next part of the proof we use induction on \( r \) and \( t \). We treat the parts 1, 2 and 3 as the first step.

(4) Let us notice that \( a(p+k-1)(h-1)+1 = \cdots = a(p+k-1)(h-1)+h-3 = 0, \)

\[ a(p+k)(h-1)-1 = (-1)^{p+k} \binom{m}{k} \]

and

\[ a(p+k)(h-1) = (-1)^{p+k} \left[ \binom{m + p}{k + p} - \binom{m}{k} \right] \quad \text{for} \quad k \in \{1, 2, \ldots, m\}. \]

Let us assume that for a certain \( 1 \leq r \leq m \), a hypergraph \( H \) does not have \((p+s-1)(h-1)+t\)-edges and its \( h \)-edges do not form a \((p+s)\)-cycle for \( s \in \{1, 2, \ldots, r - 1\} \) and \( t \in \{2, 3, \ldots, h\} \). Consider the coefficients

\[ a(p+r-1)(h-1)+1 = \cdots = a(p+r-1)(h-1)+h-3 = 0. \]

(a) By the properties of \( H \) which are already known, the induction hypothesis and Theorem 1, we have

\[ a(p+r-1)(h-1)+1 = (-1)^1 N(n - ((p + r - 1)(h - 1) + 1), 1). \]

Since \( a(p+r-1)(h-1)+1 = 0, N(n - ((p + r - 1)(h - 1) + 1), 1) = 0. \) It means that \( H \) does not have \((p + r - 1)(h - 1) + 2\)-edges.

(b) Let us now assume that for a certain \( 2 \leq t \leq h - 3 \), \( H \) does not have \(((p + r - 1)(h - 1) + t')\)-edges for all \( t' \in \{2, 3, \ldots, t\} \). By the previous conclusions and by Theorem 1, we have

\[ a(p+r-1)(h-1)+t = (-1)^1 N(n - ((p + r - 1)(h - 1) + t), 1). \]

Since \( a(p+r-1)(h-1)+t = 0, H \) has no \(((p + r - 1)(h - 1) + t + 1)\)-edges.

By induction, \( H \) does not have \(((p + r - 1)(h - 1) + t')\)-edges for \( t' \in \{2, 3, \ldots, h - 2\} \).

(5) We have \( a(p+r)(h-1)-1 = (-1)^{p+r} \binom{m}{r} \) and by Theorem 1,

\[ a(p+r)(h-1)-1 = (-1)^1 N(n - ((p + r)(h - 1) - 1), 1) \]

\[ + (-1)^{p+r} N(n - ((p + r)(h - 1) - 1), p + r), \]

where \( N(n - ((p + r)(h - 1) - 1), p + r) \) denotes the number of spanning subhypergraphs of \( H \), which are not necessarily connected, have only \( p + r \) \( h \)-edges and contain exactly one cycle that is either a \( p \)-cycle or a \((p + r)\)-cycle. Assume that

\[ N(n - ((p + r)(h - 1) - 1), p + r) = N_c(p) + N_c(p + r), \]

where \( N_c(p) \) denotes the number of the subhypergraphs containing a \( p \)-cycle, whereas \( N_c(p + r) \) denotes the number of the subhypergraphs with a \((p + r)\)-cycle. We know that \( H \) has exactly one \( p \)-cycle, so \( N_c(p) = \binom{m}{r} \) because one has to choose \( r \) from among \((m + p) - p h\)-edges of \( H \). Therefore

\[ (-1)^{p+r} \binom{m}{r} = - N(n - ((p + r)(h - 1) - 1), 1) \]

\[ + (-1)^{p+r} \binom{m}{r} + (-1)^{p+r} N_c(p + r) \]

and then \( N(n - ((p + r)(h - 1) - 1), 1) = (-1)^{p+r} N_c(p + r) \). Let us consider two cases.
(a) If $p + r$ is even then $N(n - ((p + r)(h - 1) - 1), 1) = N_c(p + r)$.
(b) If $p + r$ is odd then $N(n - ((p + r)(h - 1) - 1), 1) = N_c(p + r) = 0$. It means that $H$ has no $(p + r)(h - 1)$-edges and does not contain an $h$-uniform cycle of length $p + r$.

(6) We have $a_{(p+r)(h-1)} = (-1)^{p+r} \left[ \binom{m+p}{r+p} - \binom{m}{r} \right]$, whereas, by Theorem 1 and by the induction hypothesis,
$$a_{(p+r)(h-1)} = (-1)^{1} N(n - (p + r)(h - 1), 1)$$
$$+ (-1)^{p+r} N(n - (p + r)(h - 1), p + r),$$
where $N(n - (p + r)(h - 1), p + r)$ denotes the number of acyclic spanning subhypergraphs of $H$ which have only $p + r$ $h$-edges. Let us notice that
$$N(n - (p + r)(h - 1), p + r) = \binom{m+p}{r+p} - N_c(p) - N_c(p + r)$$
$$= \binom{m+p}{r+p} - \binom{m}{r} - N_c(p + r).$$

Hence
$$(-1)^{p+r} \left[ \binom{m+p}{r+p} - \binom{m}{r} \right]$$
$$= -N(n - (p + r)(h - 1), 1) + (-1)^{p+r} \left[ \binom{m+p}{r+p} - \binom{m}{r} - N_c(p + r) \right],$$
and then $N(n - (p + r)(h - 1), 1) = (-1)^{p+r+1} N_c(p + r)$. Let us consider two cases.

(a) If $p + r$ is even then $N(n - (p + r)(h - 1), 1) = N_c(p + r) = 0$. By the case (5a), we obtain
$$N(n - ((p + r)(h - 1) - 1), 1) = 0,$$
and so, $H$ has neither $(p + r)(h - 1)$-edges nor $(p + r)(h - 1) + 1$-edges and does not contain an $h$-uniform cycle of length $p + r$.

(b) If $p + r$ is odd then $N_c(p + r) = N(n - (p + r)(h - 1), 1) = 0$, by the case (5b). It means that $H$ has neither $(p + r)(h - 1)$-edges nor $(p + r)(h - 1) + 1$-edges and does not contain an $h$-uniform cycle of length $p + r$.

By induction, $H$ has no $((p + s - 1)(h - 1) + r)$-edges and does not contain $h$-uniform cycles of length $p + s$ for $s \in \{1, 2, \ldots, m\}$ and $t \in \{2, 3, \ldots, h\}$. We proved that a linear hypergraph $H$ with the chromatic polynomial given by (2) is connected, $h$-uniform and unicyclic with the cycle of length $p$, what completes the proof. □

Theorem 8 is also true for $h = 2$, i.e., for graphs. This case was presented in [7]. However the proofs of the cases $h = 2$ and $h \geq 3$ require quite different methods and use different theorems.

Similarly as it was for $h$-uniform hypertrees, there might exist hypergraphs which are not linear and not $h$-uniform but are chromatically equivalent with $h$-uniform unicyclic hypergraphs.

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References