On graphs with a local hereditary property

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Abstract

Let $P$ be an induced hereditary property and $L(P)$ denote the class of all graphs that satisfy the property $P$ locally. The purpose of the present paper is to describe the minimal forbidden subgraphs of $L(P)$ and the structure of local properties. Moreover, we prove that $L(P)$ is irreducible for any hereditary property $P$. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

We consider finite undirected graphs without loops or multiple edges. A graph $G$ has vertex-set $V(G)$ and edge-set $E(G)$, and $H \preceq G$ means that $H$ is an induced subgraph of $G$. We say that $G$ contains $H$ whenever $G$ contains an induced subgraph isomorphic to $H$.

In general, we follow the notation and terminology of [3].

Let $I$ denote the class of all graphs. If $P$ is a proper nonempty isomorphism closed subclass of $I$, then $P$ will also denote the property of being in $P$. We shall use the terms class of graphs and property of graphs interchangeably.

A property $P$ is called (induced) hereditary, if every (induced) subgraph of any graph with property $P$ also has property $P$ and additive, if the disjoint union $H \cup G \in P$ whenever $G \in P$ and $H \in P$. Obviously, any hereditary property is induced hereditary, too.

Let us denote by $L$ ($M$, resp.) the set of all hereditary (induced hereditary, resp.) properties of graphs. Corresponding sets of additive properties are denoted by $L^a$ and $M^a$, respectively. The sets $L$, $L^a$, $M$ and $M^a$, partially ordered by the set inclusion, form

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complete distributive lattices with the set intersection as the meet operation. Obviously, \((L, \subseteq)\) is a proper sublattice of \((M, \subseteq)\), for more details see [1,2].

We list some properties to introduce the necessary notions which will be used in the paper. Let \(k\) be a nonnegative integer.

\[
\mathcal{O} = \{ G \in \mathcal{I} : G \text{ is totally disconnected} \},
\]
\[
\mathcal{O}_k = \{ G \in \mathcal{I} : \text{each component of } G \text{ has at most } k + 1 \text{ vertices} \},
\]
\[
\mathcal{I}_k = \{ G \in \mathcal{I} : \text{G contains no subgraph isomorphic to } K_{k+2} \}.
\]

Let \(P_1, P_2, \ldots, P_n\) be properties of graphs. We say that a graph \(G\) has property \(P_1 \circ \cdots \circ P_n\), if the vertex set of \(G\) can be partitioned into \(n\) sets \(V_1, \ldots, V_n\) such that \(G[V_i]\), the subgraph of \(G\) induced by \(V_i\), has the property \(P_i\) for \(i = 1, \ldots, n\). It is easy to see that \(P_1 \circ \cdots \circ P_n\) is (induced) hereditary and additive whenever \(P_1, P_2, \ldots, P_n\) are (induced) hereditary and additive, respectively. An (induced) hereditary property \(R\) is said to be reducible if there exist two (induced) hereditary properties \(P_1\) and \(P_2\) such that \(R = P_1 \circ P_2\) and irreducible, otherwise.

For a given irreducible property \(P \in \mathcal{M}\), a reducible property \(R \in \mathcal{M}\) is called a minimal reducible bound for \(P\) if \(P \subset R\) and for each reducible property \(R' \subset R\), \(P \not\subseteq R'\). The family of all minimal reducible bounds for \(P\) will be denoted by \(\mathbf{B}(P)\).

Let \(\mathcal{P}\) be a class of graphs. A graph \(G = (V, E)\) is said to satisfy a property \(P\) locally if \(G[N(v)] \in \mathcal{P}\) for every \(v \in V(G)\). The class of graphs that satisfy the property \(P\) locally will be denoted by \(L(P)\) and we shall call such a class a local property.

Early investigations dealt mostly with the case \(|\mathcal{P}| = 1\); i.e., when all neighborhoods are isomorphic. Summaries of results of this type can be found in the survey papers of Hell [5] and Sedláček [7]. The major question is the existence of an appropriate \(G\).

More recently, the cases when \(\mathcal{P}\) consists of all cycles, all paths, all matchings, or all forests were investigated. Also, results concerning some extremal problems on such classes of graphs have been obtained, see [4,8].

Assume that \(\mathcal{P}\) is an induced hereditary property. The purpose of the present paper is to describe the minimal-forbidden subgraphs of \(L(\mathcal{P})\) and the structure of local properties. Moreover, we prove that \(L(\mathcal{P})\) is irreducible for any \(\mathcal{P} \in \mathcal{L}\).

2. Forbidden subgraphs

Any induced hereditary property \(\mathcal{P}\) can be characterized by the set of minimal-forbidden-induced subgraphs:

\[
\mathcal{C}(\mathcal{P}) = \{ H \in \mathcal{I} : H \not\in \mathcal{P} \text{ but } (H - v) \in \mathcal{P} \text{ for any } v \in V(H) \}.
\]

It is easy to prove that a property \(\mathcal{P} \in \mathcal{M}\) is additive if and only if all minimal-forbidden-induced subgraphs of \(\mathcal{P}\) are connected.

**Lemma 1.** If \(H \in \mathcal{C}(L(\mathcal{P}))\), then \(H\) has a universal vertex.
Proof. Let $F \in \mathcal{C}(L(\mathcal{P}))$ and suppose $F$ has no universal vertex. Let $v \in V(F)$. Then, there is a vertex $x \in V(F)$ such that $x \not\in N_F(v)$. Since $F - x \in L(\mathcal{P})$, it follows that $F[N_{F-x}(v)] \in \mathcal{P}$. But $N_{F-x}(v) = N_F(v)$, hence $F[N_F(v)] \in \mathcal{P}$. This proves that $F \in L(\mathcal{P})$, a contradiction. $\square$

Lemma 2. $\mathcal{C}(L(\mathcal{P})) = \{K_1 + H: H \in \mathcal{C}(\mathcal{P})\}$.

Proof. Let $F \in \mathcal{C}(L(\mathcal{P}))$. By Lemma 1, $F$ has a universal vertex $v$. Let $H = F - v$. We shall prove that $H \in \mathcal{C}(\mathcal{P})$, i.e., $H \not\in \mathcal{P}$ and $H - u \not\in \mathcal{P}$ for every $u \in V(H)$.

Suppose that $H \in \mathcal{P}$. This implies that $H[N_H(u)] \in \mathcal{P}$ for every $u \in V(H)$. Moreover, $F[N_F(u)] \in \mathcal{P}$, because of

(1) If $u$ is a universal vertex of $F$, too, then $F - u$ is isomorphic to $F - v$ which is equal to $H \in \mathcal{P}$.

(2) If $u$ is not a universal vertex of $F$, let $w$ be a vertex in $F$ that is nonadjacent to $u$. Then $F[N_F(u)] = F[N_{F-w}(u)] \in \mathcal{P}$, since $F - w \in \mathcal{P}$.

Thus, by (1) and (2) we have $F[N(x)] \in \mathcal{P}$ for every $x \in V(F)$, i.e., $F \in L(\mathcal{P})$, a contradiction. Thus, $H \not\in \mathcal{P}$.

Now suppose that $H$ is not a minimal-forbidden subgraph of $\mathcal{P}$, i.e., there is a vertex $u \in V(H)$ such that $H - u \not\in \mathcal{P}$. Hence, $H' = (H - u) + K_1 \not\in L(\mathcal{P})$. But $H'$ is isomorphic to $F - u \in L(\mathcal{P})$, a contradiction. Thus, $H \not\in \mathcal{C}(\mathcal{P})$.

Conversely, let $H \in \mathcal{C}(\mathcal{P})$ and $F = K_1 + H, V(K_1) = w$. Obviously $F \not\in \mathcal{C}(\mathcal{P})$ since $H \not\in \mathcal{P}$. Suppose that there is a proper induced subgraph $H' < F, H' \in \mathcal{C}(L(\mathcal{P}))$. Since the graph $H \in L(\mathcal{P}), w \in V(H')$ and the graph $H'$ has a universal vertex $u \neq w$ such that $H' - u \in \mathcal{C}(\mathcal{P})$. However $H' - u$ is isomorphic to $H' - w < H$, a contradiction. Thus, $F \not\in \mathcal{C}(L(\mathcal{P}))$. $\square$

Lemmas 1 and 2 imply the following theorems.

Theorem 1. A property $\mathfrak{L} = L(\mathcal{P})$ for some $\mathcal{P} \in \mathcal{M}$ if and only if every $H \in \mathcal{C}(\mathfrak{L})$ has a universal vertex.

Theorem 2. Let $\mathcal{P}, \mathfrak{L} \in \mathcal{M}$. Then,

(a) $L(\mathcal{P}) \in \mathcal{M}^a$,
(b) $\mathcal{P} \subseteq L(\mathcal{P})$,
(c) If $\mathcal{P} \subseteq \mathfrak{L}$, then $L(\mathcal{P}) \subseteq L(\mathfrak{L})$,
(d) $L(\mathcal{P}) = L(\mathfrak{L})$ if and only if $P = Q$.
(e) If $G \in \mathcal{P}$, then $G + K_1 \in L(\mathcal{P})$.

Proof. Conditions (a)–(c) follow immediately from the corresponding definitions. To prove (d), suppose that $\mathcal{P} \neq \mathfrak{L}$. Then there is a graph $H$ such that, say, $H \not\in \mathcal{P}$ and $H \in \mathfrak{L}$. From this it follows that: $K_1 + H \not\in L(\mathcal{P})$ and $K_1 + H \in L(\mathfrak{L})$. Thus, $L(\mathcal{P}) \neq L(\mathfrak{L})$, a contradiction. Condition (e) follows from the proof of Lemma 2. $\square$
Let \( \mathcal{P} \in \mathcal{M} \). Then

\[ c(\mathcal{P}) = \sup\{k: \mathcal{K}_{k+1} \in \mathcal{P}\} \]

is called the completeness of \( \mathcal{P} \).

From the definition of completeness, Lemma 2 and Theorem 2, we have the following:

**Theorem 3.** Let \( \mathcal{P} \in \mathcal{M} \). Then \( c(L(\mathcal{P})) = c(\mathcal{P}) + 1 \).

Let \( \mathcal{L}_k = \{\mathcal{P} \in \mathcal{L}: c(\mathcal{P}) = k\} \). The set \( \mathcal{M}_k \) is defined analogously. \((\mathcal{M}, \subseteq)\) and \((\mathcal{L}, \subseteq)\) are distributive sublattices of \((\mathcal{M}, \subseteq)\) and \((\mathcal{L}, \subseteq)\), respectively (see [2]).

By the previous results we have immediately

**Theorem 4.** Let \( L(\mathcal{L}_k) = \{L(\mathcal{P}): \mathcal{P} \in \mathcal{L}_k\} \). Then,

(a) \( L(\mathcal{L}_k) \subset \mathcal{L}_{k+1} \),
(b) \( L(\mathcal{L}_k) \) is isomorphic to \( \mathcal{L}_k \),
(c) \( L(\mathcal{L}) \) is isomorphic to \( \mathcal{L} \).

3. Irreducibility

There are different approaches to show that an additive hereditary property \( \mathcal{P} \) is irreducible in \( \mathcal{L}^a \) (see [1], Chapter 3). The following deep Theorem of Nešetřil and Rödl implies that some properties have exactly one-trivial minimal reducible bound.

**Theorem 5** (Nešetřil and Rödl [6]). Let \( \mathcal{G}(\mathcal{P}) \) be a finite set of 2-connected graphs. Then for every graph \( G \) of property \( \mathcal{P} \) there exists a graph \( H \) of property \( \mathcal{P} \) such that for any partition \( \{V_1, V_2\} \) of \( V(H) \) there is an \( i = 1 \) or \( 2 \) for which the subgraph \( H[V_i] \) induced by \( V_i \) in \( H \) contains \( G \).

**Corollary 1** (Borowiecki et al. [1]). Let \( \mathcal{G}(\mathcal{P}) \) be a finite set of 2-connected graphs, then the property \( \mathcal{P} \) has exactly one minimal-reducible bound \( \mathcal{R} = \mathcal{C} \circ \mathcal{P} \).

Particularly for the property \( L(\mathcal{C}_k) \) it follows:

**Lemma 3.** \( \mathcal{B}(L(\mathcal{C}_k)) = \{\mathcal{C} \circ L(\mathcal{C}_k)\} \) for \( \mathcal{C}_k \in \mathcal{L}^a \).

**Proof.** The set of minimal-forbidden subgraphs for \( \mathcal{C}_k \in \mathcal{L} \) is given by

\[ \mathcal{G}(\mathcal{C}_k) = \{G \in \mathcal{G}: |V(G)| = k + 2 \text{ and } G \text{ is connected}\} \]

Thus, if \( H \in \mathcal{G}(L(\mathcal{C}_k)) \), then \( H \) is 2-connected. Since \( \mathcal{G}(L(\mathcal{C}_k)) \) is finite, then each minimal reducible bound \( \mathcal{C} \circ \mathcal{C}' \) for \( L(\mathcal{C}_k) \), by Corollary 1, has the form \( \mathcal{C} \circ L(\mathcal{C}_k) \), i.e., one of factors has to be \( L(\mathcal{C}_k) \). By the minimality we have \( \mathcal{C} = \mathcal{C}' \) which proves the lemma. \( \square \)
From above the following lemma follows.

**Lemma 4.** The property $L(O_k)$ is irreducible for $O_k \in \mathbb{L}$, $k \geq 1$.

**Theorem 6.** If $\mathcal{P} \in \mathbb{L}^a$, then $L(\mathcal{P})$ is irreducible.

**Proof.** Suppose that for some property $\mathcal{P} \in \mathbb{L}^a$, $L(\mathcal{P})$ is reducible, i.e., $L(\mathcal{P}) = \mathcal{Q}_1 \circ \mathcal{Q}_2$. Then by Theorem 3 $c(\mathcal{Q}_1 \circ \mathcal{Q}_2) = k + 1$, $k \geq 1$ and $L(O_k) \subset \mathcal{Q}_1 \circ \mathcal{Q}_2$. By Lemma 3, $L(O_k) \subset \mathcal{Q}_1 \circ L(O_k) \subset \mathcal{Q}_1 \circ \mathcal{Q}_2$. But $c(\mathcal{Q}_1 \circ L(O_k)) = c(\mathcal{Q}_1) + c(L(O_k)) + 1 = k + 2 \leq c(\mathcal{Q}_1 \circ \mathcal{Q}_2) = k + 1$, a contradiction. \(\square\)

4. Concluding remarks

Let $L'(\mathcal{P}) = L(L'^{-1}(\mathcal{P}))$ and $L'(\mathbb{L}_k) = L(L'^{-1}(\mathbb{L}_k))$ for $r \geq 2$.

By induction on $r$ we can prove the following statements.

**Theorem 7.** Let $\mathcal{P} \in \mathbb{M}$ and $r \geq 1$. Then,

1. $\mathcal{C}(L'(\mathcal{P})) = \{K_r + H : H \in \mathcal{C}(\mathcal{P})\}$.
2. $c(L'(\mathcal{P})) = c(\mathcal{P}) + r$.
3. $L'(\mathcal{P}) \subseteq L'^{-1}(\mathcal{P})$.
4. $L'(\mathcal{P}) \in \mathbb{M}^a$.
5. $L'(\mathbb{L}_k) \subseteq \mathbb{L}_{k+r}$.
6. $L'(\mathbb{L}_k)$ is isomorphic to $\mathbb{L}_k$.
7. If $r \geq 2$, then $L'(\mathcal{P})$ is irreducible for any $\mathcal{P} \in \mathbb{L}$.

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