A Note on the Computation of the Bit Error Rate for Binary Block Codes*

Michele Elia
Dipartimento di Elettronica
Politecnico di Torino
Corso Duca degli Abruzzi 24
I-10129 Torino, Italy

Submitted by I. F. Blake

ABSTRACT

The bit error rate of binary codes on the binary symmetric channel with raw error probability $p$ is a performance index of primary importance in many applications. It is the probability that decoded information symbols are in error after complete decoding. In this paper the class of linear block codes having a transitive group of automorphisms is considered. It is shown that a linear homogeneous differential transformation of the weight enumerator enables us to obtain a closed expression for the bit error rate in terms of $p$. The application of the method is general enough to allow the computation of bit error rates for all cyclic codes.

I. INTRODUCTION

Linearity pervades nearly all the manifold aspects of coding theory. The most remarkable classes of error control codes are defined as subspaces of finite vector spaces over a Galois field. The dual code $C^\perp$ of a given linear code $C$ is defined as the subspace of vectors orthogonal to all vectors of $C$. The distributions of vector Hamming weights of paired dual codes are connected by a linear transformation. Here linearity will be applied to exploit properties of another important parameter describing code performance, the symbol error probability.

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Let $V_n$ be the $n$-dimensional vector space over the Galois field GF(2). Let $\text{wt}(U)$ denote the Hamming weight of the vector $U \in V_n$, i.e. the number of nonzero components of $U$. The Hamming distance $d(U, V)$ between $U$ and $V$ is defined as the number of components in which the two vectors differ. Equivalently $d(U, V) = \text{wt}(U + V)$. A binary linear code $(n, k, d)$ is a $k$-dimensional vector subspace of the vector space $V_n$ with minimum Hamming distance between vectors equal $d$; thus the code can correct $t = [(d - 1)/2]$ errors. The weight enumerator is defined as the polynomial

$$W(X, Y) = \sum_{i=0}^{n} A_i X^i Y^{n-i},$$

where $A_i$ is the number of vectors of weight $i$.

Let $\mathbb{Z}[[X, Y]]$ be the algebra of polynomials in $X$ and $Y$, and let $H_n[[X, Y]]$ be the $(n + 1)$-dimensional subspace of homogeneous polynomials $q(X,Y)$ of degree $n$. A linear transformation on $X$ and $Y$ induces a linear transformation in $H_n[[X, Y]]$. The transformation, denoted $M_c$ and defined as

$$M_c[q(X, Y)] = q(X + Y, X - Y),$$

was introduced by MacWilliams [3] to show the connection between weight enumerators of dual codes. In fact we have the relation

$$W^\perp(X, Y) = \frac{1}{2^k} M_c[W(X, Y)].$$

The bit error probability, also called bit error rate and henceforth shortened to BER, is the average probability that an information bit is in error after complete decoding. Complete decoding means that every received word is assigned to a definite codeword. In the case of linear codes of block length $n$ complete decoding is realized through a coset decomposition of $V_n$ with respect to the code. Such a decomposition is usually called a standard array, and different aims may be accomplished by giving different rules for selecting the coset leaders.

Slepian [1] has shown that coset leaders of minimum weight yield a maximum likelihood decoding, thus minimizing the probability that the decoded codeword is not the transmitted one, i.e., the word error probability $P_w$ is as small as possible. In [8] it is shown that maximum likelihood decoding, hereafter abbreviated MLD, does not necessarily minimize the BER. In fact,
for many codes a smaller BER is obtained by taking each coset leader to be either the unique element of minimum weight not greater than $t$, or the unique element with all zeros in information positions. We shall call this last rule unique coset leader decoding, and hereafter it will be abbreviated UCLD.

Sporadic expressions for the BER in terms of $p$, the raw error probability of the binary symmetric channel, have been derived [2, 5, 6, 7]; in [8] the performances of MLD and UCLD are compared, asymptotically as $p \to 0$, for the special class of quasiperfect linear codes. However, in general, the decoding rule which gives the minimum BER is difficult to deal with (see [12] for a clever reporting) and the improvement in the code's performance is not always significant, despite the increased complexity. Therefore it is of some interest to have an expression for the bit error probability for block codes over the binary symmetric channels when UCLD is applied.

Here we will show that the BER for block codes having a transitive group of automorphisms $G$ can be obtained from the weight distribution by means of a differential linear transformation. It must be observed that several important classes of codes satisfy this requirement; some of the more common ones are cyclic codes, extended primitive binary BCH codes, uniformly packed linear codes, and shortened codes of cyclic codes which have a 2-transitive group of automorphisms; see [2, 5]. The condition of the automorphism group for the codes will allow a simple counting argument which will be used in the following to evaluate the bit error rate; otherwise, BER computation could require exhaustive analysis.

II. DEFINITIONS

This section is mainly devoted to recalling definitions as well as known results for easy reference. We just report, with few comments, the items useful in the computation of BER from the weight enumerator.

The leader weight enumerator is defined as the polynomial

$$L(X, Y) = \sum_{i=0}^{n} l_i X^i Y^{n-i},$$

where $l_i$ is the number of coset leaders of weight $i$.

For every binary block code, linear or nonlinear, the bit error rate $P_{\text{symb}}$ is defined, [2], as

$$P_{\text{symb}} = \frac{1}{kM} \sum_{i=1}^{k} \sum_{j=1}^{M} \text{Prob}\{x_i^{(j)} \neq \hat{x}_i | X^{(j)} \text{ was sent}\}, \quad (1)$$
where $M$ denotes the cardinality of the code, and the code vectors $X^{(j)}$, $j = 1, \ldots, M$, are equally likely. If attention is restricted to linear codes, then Equation (1) can be written in a simpler and useful form

$$P_{\text{symb}} = \frac{1}{k} \sum_{e} f(e) p^{\text{wt}(e)} (1 - p)^{n - \text{wt}(e)},$$

where $f(e)$ is the number of incorrect information bits after decoding with the assumption that the all zero word is transmitted, and the summation is extended all over the $2^n$ binary $n$-tuples. For computational purposes Equation (2) will be more conveniently rewritten as follows:

$$P_{\text{symb}} = \sum_{i=0}^{n} B_i p^i (1 - p)^{n - i} = \sum_{i=0}^{n} E_i p^i,$$

where

$$B_i = \frac{1}{k} \sum_{\text{wt}(e) = i} f(e)$$

and $\sum_{\text{wt}(e) = i}$ sums $f(e)$ over all error patterns of Hamming weight $i$.

Now let $B(X, Y)$ denote the generating polynomial of the $B_i$'s, that is,

$$B(X, Y) = \sum_{i=0}^{n} B_i X^i Y^{n-i}.$$

Thus it is seen that

$$P_{\text{symb}} = B(p, 1 - p).$$

This last expression suggests the introduction of an operator $L_1$ which replaces $X$ and $Y$ respectively with $p$ and $1 - p$; it is immediately verified that $L_1$ is linear.

By using $L_1$, the expression (5) can be rewritten as

$$P_{\text{symb}} = L_1 [B(X, Y)].$$

It is known that the word error probability $P_w$ may be obtained from
$L(X, Y)$, whereas the probability $P_d$ of detecting some errors at the receiver front end may be computed from $W(X, Y)$. In both cases using $L_1$ we respectively have

$$P_w = 1 - L_1[L(X, Y)]$$

and

$$P_d = 1 - L_1[W(X, Y)].$$

In the next section we will introduce a linear operator $L_2$ that enables us to mechanically derive $B(X, Y)$ from the weight enumerator $W(X, Y)$. We will show that the operator $L_2$ admits an explicit representation as antisymmetric homogeneous differential operator in the algebra $Z[[X, Y]]$.

III. COUNTING

As described in the previous section, the derivation of $B(X, Y)$ for systematic linear block codes is based on the evaluation of

$$B_j = \frac{1}{k} \sum_{\text{wt}(e) - j} f(e),$$

where $f(e)$ is the number of 1’s in information positions after the decoding when the received word is $e$ and the all zero word is transmitted.

The evaluation of $B_j$ is essentially a counting problem. Thus to this purpose we introduce the set of matrices $T^{(j)}$ whose rows are all the $\binom{n}{j}$ binary words of weight $j$, $0 \leq j \leq n$. In each matrix $T^{(j)}$ the rows are ordered so as to define a partition in $s \leq 2t + 2$ submatrices $Q_1, Q_2, \ldots, Q_s$, where each submatrix is made up of words coming from all the codewords of a fixed given weight and modified by correctable suitable error patterns. The rows of each $Q_i$, $1 \leq i \leq s - 1$, are vectors of the form

$$Z = C + E,$$

where $C$ is a codeword and $E$ is a coset leader, i.e. a correctable error pattern. Let $j + h_2 - h_1$ be the weight of $C$; let $h_1 + h_2$ be the weight of $E$, where $h_1$ denotes the number of 1’s falling in the 0’s positions of $C$, and $h_2$ denotes the number of 1’s falling in 1’s positions of $C$, so that the weight of $Z$
will be \( \text{wt}(C) + h_1 - h_2 = j \); and finally let \( h_2 - h_1 \) be fixed and different from zero. The rows of the last submatrix \( Q_s \) are either words coming from codewords of weight \( j \), or words that according to UCLD are decoded without changes in information positions.

For later use it is useful to know the number of rows in each matrix \( Q_i \). Set \( b = \lfloor (t - h) / 2 \rfloor \); if \( h_1 > h_2 \), then the number \( N_{j-h} \) of rows of the submatrix associated to codewords of weight \( j - h \), is given by

\[
N_{j-h} = A_{j-h} \sum_{h_2 = 0}^{b} \binom{n}{j+h_2} \binom{j-h}{h_2}.
\]

Similarly, if \( h_1 < h_2 \), the number \( N_{j+h} \) of rows of the submatrix associated to codewords of weight \( j + h \) is given by

\[
N_{j+h} = A_{j+h} \sum_{h_1 = 0}^{b} \binom{n-j-h_1}{h_1} \binom{j+h}{h_1+h}.
\]

The number of rows of \( Q_s \) is given by the difference

\[
N_s = \binom{n}{j} - \sum_{n=1}^{t} [N_{j-h} + N_{j+h}].
\]

Every matrix defined above has the same number of 1’s in all its columns. This statement can be proved as follows. In \( T^{(j)} \) all the columns have the same number of 1’s, since the interchange of two columns leaves the set of rows invariant. A similar argument also applies to any submatrix \( Q_i \) of \( T^{(j)} \).

Let us consider first the matrices \( Q_i \), \( 1 \leq i \leq s-1 \), where, as said before, every row has the form \( C + E \), in which \( C \) is a codeword of weight \( j - h_2 + h_1 \) and \( E \) is a coset leader of weight \( h_1 + h_2 \). Let \( P(g) \) denote a permutation matrix associated to the permutation \( g \) of \( G \), and let \( ZP(g) \) be the vector that results from applying the permutation \( g \) to the \( n \) components of \( Z \). We have \( (C + E)P(g) = CP(g) + EP(g) \), where \( CP(g) \) is still a codeword and \( EP(g) \) is a vector of weight not greater than \( t \) that modifies \( CP(g) \) in the same way that \( E \) modifies \( C \). Note that if \( E \) is a coset leader, of weight not greater than \( t \), then \( EP(g) \) is a coset leader too. Therefore \( (C + E)P(g) \) is just transformed into another row of \( Q_i \). This implies the existence of a permutation matrix \( R(g) \) such that

\[
R(g)Q_i P(g) = Q_i \quad \text{for every } g \in G \text{ and for every } i \leq s-1.
\]
and we can conclude that every matrix $Q_i$, $1 \leq i \leq s - 1$, has the same number of 1's in every column. Finally, also the matrix $Q_s$ has the same number of 1's in every column, because this number is the difference of equal numbers.

Now we are ready to estimate $B_j$. Owing to previous results, the contribution coming from matrices $Q_i$, $1 \leq i \leq s - 1$, is obtained by summing the numbers

$$N_{j-h}(j-h) + N_{j+h}(j+h)$$

for $h$ from 1 to $t$, then multiplying by the ratio $k/n$ to take account of the $k$ information columns, and finally dividing by $k$:

$$B_{1j} = \frac{1}{n} \sum_{h=1}^{t} \left[ N_{j-h}(j-h) + N_{j+h}(j+h) \right].$$

The contribution coming from the last matrix $Q_s$ is simply obtained multiplying the number $N_s$ by $j/n$:

$$B_{0j} = \frac{j}{n} \left( \binom{n}{j} - \sum_{h=1}^{t} \left[ N_{j-h} + N_{j+h} \right] \right).$$

In conclusion we have

$$B_j = B_{0j} + B_{1j}$$

$$= \frac{1}{n} \left( \binom{n}{j} + \sum_{h=1}^{t} \frac{h}{n} \left[ N_{j+h} - N_{j-h} \right] \right).$$

Multiplying this expression by $X^jY^{n-j}$ and summing over $j$, we get

$$B(X, Y) = \frac{1}{n} \sum_{j=0}^{n} X^jY^{n-j} \left( \binom{n}{j} + \sum_{h=1}^{t} \frac{h}{n} \left[ N_{j+h} - N_{j-h} \right] \right)$$

$$= X(X+Y)^{n-1} + \frac{1}{n} \left\{ \sum_{j=0}^{n} X^jY^{n-j} \sum_{h=1}^{t} \frac{h}{n} \left[ N_{j+h} - N_{j-h} \right] \right\}. \quad (9)$$
The final expression for $B(X, Y)$ is obtained after some algebraic manipulations outlined in the following steps:

**Step 1.** Interchange summations in Equation (9) and use Equations (6) and (7):

$$X(X + Y)^{n-1} + \frac{1}{n} \sum_{h=1}^{t} \sum_{h_2=0}^{b} \sum_{j=0}^{n} hX^jY^{n-j}$$

$$\times \left[ A_{j+h} \left( \begin{array}{c} n-j-h \\ h_2 \\ \end{array} \right) \left( \begin{array}{c} j+h \\ h_2+h \\ \end{array} \right) - A_{j-h} \left( \begin{array}{c} n-j+h \\ h_2+h \\ \end{array} \right) \left( \begin{array}{c} j-h \\ h_2 \\ \end{array} \right) \right].$$

**Step 2.** Perform the substitution $h = u - 2h_2$, define $v = \lfloor u/2 \rfloor$, separate the two summations on $j$, and in them respectively substitute $j = i + 2h_2 - u$ and $j = i - 2h_2 + u$:

$$X(X + Y)^{n-1} + \frac{1}{n} \sum_{u=1}^{t} \sum_{h_2=0}^{v} (u - 2h_2)$$

$$\times \left\{ \sum_{i=0}^{n} X^{i+2h_2-u}Y^{n-i-2h_2+u}A_i \left( \begin{array}{c} n-i \\ h_2 \\ \end{array} \right) \left( \begin{array}{c} i \\ u-h_2 \\ \end{array} \right) \right\}$$

$$- \sum_{i=0}^{n} X^{i-2h_2+u}Y^{n-i+2h_2-u}A_i \left( \begin{array}{c} n-i \\ u-h_2 \\ \end{array} \right) \left( \begin{array}{c} i \\ h_2 \\ \end{array} \right).$$

**Step 3.** Rewrite the expression of step 2 to enhance the action of differential operators as follows:

$$X(X + Y)^{n-1} + \frac{1}{n} \sum_{u=1}^{t} \sum_{h_2=0}^{v} (u - 2h_2)$$

$$\times \left\{ X^{h_2}Y^{u-h_2} \sum_{i=0}^{n} X^{i+h_2-u}Y^{n-i-h_2}A_i \left( \begin{array}{c} n-i \\ h_2 \\ \end{array} \right) \left( \begin{array}{c} i \\ u-h_2 \\ \end{array} \right) \right\}$$

$$- Y^{h_2}X^{u-h_2} \sum_{i=0}^{n} X^{i-h_2}Y^{n-i+h_2-u}A_i \left( \begin{array}{c} n-i \\ u-h_2 \\ \end{array} \right) \left( \begin{array}{c} i \\ h_2 \\ \end{array} \right).$$
Step 4. Finally find

$$B(X, Y) = X(X + Y)^{n-1} + \left[ L_2(t) \left\{ W(X, Y) \right\} \right],$$  \hspace{1cm} (10)

where the linear differential antisymmetric homogeneous operator $L_2(t)$, associated with $t$-error correcting codes, has been introduced:

$$L_2(t) = \frac{1}{n} \sum_{u=1}^{t} D_u$$  \hspace{1cm} (11)

with

$$D_u = \frac{1}{u!} \sum_{h_2=0}^{u} (u - 2h_2) \binom{u}{h_2}$$

$$\times \left( Y^{h_2} X^{u-h_2} \frac{\partial^u}{\partial Y^{u-h_2} \partial X^{h_2}} - X^{h_2} Y^{u-h_2} \frac{\partial^u}{\partial X^{u-h_2} \partial Y^{h_2}} \right)$$

In particular the first three expressions for $L_2(t)$, corresponding to minimum distances from 3 up to 8, are

$$L_2(1) = \frac{1}{n} \left( Y \frac{\partial}{\partial X} - X \frac{\partial}{\partial Y} \right),$$

$$L_2(2) = \frac{1}{n} \left[ \left( Y \frac{\partial}{\partial X} - X \frac{\partial}{\partial Y} \right) + \left( Y^2 \frac{\partial^2}{\partial X^2} - X^2 \frac{\partial^2}{\partial Y^2} \right) \right],$$

$$L_2(3) = \frac{1}{n} \left[ \left( Y \frac{\partial}{\partial X} - X \frac{\partial}{\partial Y} \right) + \left( Y^2 \frac{\partial^2}{\partial X^2} - X^2 \frac{\partial^2}{\partial Y^2} \right) \right.$$ \hspace{1cm} 

$$+ \frac{1}{2} \left( Y^3 \frac{\partial^3}{\partial X^3} - X^3 \frac{\partial^3}{\partial Y^3} \right)$$ \hspace{1cm} 

$$+ \frac{1}{2} \left( X Y^2 \frac{\partial^3}{\partial X^2 \partial Y} - Y X^2 \frac{\partial^3}{\partial Y^2 \partial X} \right) \right].$$

It is interesting to observe that $L_2(t)$ is the sum of $t$ differential operators,
where each operator originates from the correction of all error patterns of a
given weight $i$, for every $i$ in the range between 1 and $t$.

IV. CONCLUSIONS

In this paper we have introduced the operator $L_2(t)$ that enables us to
obtain the bit error probability from the weight enumerator. The existence of
$L_2(t)$ is seen to follow from the uniqueness of coset leaders and the
transitivity of the symmetry permutation group. These requirements are not
mandatory; they are sufficient conditions, and so far easy counterexamples
show that the transformations may either work or not with codes without a
transitive group of symmetry. Hence necessary and sufficient conditions on
the code for Equation (10) to work will be welcome. Moreover, several other
interesting questions still remain open, for example a combinatorial interprer-
tation of the surprising symmetry of the operator $L_2(t)$.

There exists a nice commutation property between $D_1$ and $M_c$, i.e.

$$\left(Y \frac{\partial}{\partial X} - X \frac{\partial}{\partial Y}\right) M_c = -M_c \left(Y \frac{\partial}{\partial X} - X \frac{\partial}{\partial Y}\right),$$

which can be verified by straightforward computation. It can be used to
calculate the BER of one-error-correcting codes from the weight enumerators
of their dual codes. For example, for Hamming codes, the BER, known from
[10], can be simply obtained given the weight enumerator of their dual codes:

$$W(X, Y) = X^n + nX^{(n-1)/2}Y^{(n+1)/2}.$$

The following steps are straightforward. Apply $L_2(1)$ to $W(X, Y)$:

$$L_2(1)W(X, Y) = \frac{1}{n} \left[ nXY^{n-1} + \frac{n(n - 1)}{2}X^{(n-3)/2}Y^{(n+3)/2} - \frac{n(n + 1)}{2}X^{(n+1)/2}Y^{(n-1)/2} \right]. \quad (12)$$
Apply $M_c$ to Equation (12):

$$Q(X, Y) = M_c L_2(1) W(X, Y)^\prime$$

$$= \frac{1}{n} \left[ n(X - Y)(X + Y)^{n - 1} + \frac{n(n - 1)}{2} (X + Y)^{(n - 3)/2} (X - Y)^{(n + 3)/2} - \frac{n(n + 1)}{2} (X + Y)^{(n + 1)/2} (X - Y)^{(n - 1)/2} \right].$$

**TABLE 1**

**COEFFICIENTS FOR BER COMPUTATION OF SOME KNOWN $(n, k, d)$ CODES**

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Use Equation (10):

$$B(Y) = X(X+Y)^{n-1} - \frac{1}{n+1} Q(X,Y).$$

Apply $L_1$:

$$P_{\text{sym}} = \frac{1}{2^m} \left\{ 1 + (2^m - 2)p - (1 - 2p)2^{m-1} - 1 \left[ 1 + 2(2^m - 2)(p - p^2) \right] \right\}.$$

Finally using Equation (10), the BER of several codes has been derived, in particular the known expressions of all perfect codes have been checked. Numerical results obtained using the program mumath are summarized in Table 1, where are reported the coefficients $E_i$ to be used in equation (3) for evaluating the BER.

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REFERENCES


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