Communication

Interval-regularity does not lead to interval monotonicity

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Abstract


We give the construction of an infinite family of interval-regular graphs which are not interval monotone, thus disproving a conjecture of H.M. Mulder.

0. Introduction

A connected graph $G$ is said to be interval-regular if, for any two vertices $u$ and $v$ of $G$, the number of neighbours of $u$ that lie on a shortest $(u,v)$-path is precisely the distance between $u$ and $v$. Foldes [1] proved that an equivalent property is the following.

**Theorem.** Let $G$ be a connected graph then $G$ is interval-regular if and only if for any two vertices $u$ and $v$ of $G$ there is exactly $d(u,v)!$ shortest $(u,v)$-path in $G$.

The hypercube $Q_n$ is an example of such a graph and in the same paper Foldes proved that the bipartite interval-regular graphs are the hypercubes. For any two vertices $u$ and $v$ of $G=(V,E)$ the interval between $u$ and $v$ is the set:

$I(u,v)=\{w \in V/ w$ lies on a shortest $(u,v)$-path$.}$

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A natural idea is to study the convexity of intervals and to introduce the notion of interval monotone graph [2, 3]: A graph $G=(V, E)$ is interval monotone if and only if for any $u$ and $v$,

$$x, y \in I[u, v] \Rightarrow I(x, y) \subseteq I(u, v).$$

Mulder proposed the following conjecture [2, 3] based on the observation of all known interval-regular graphs.

**Conjecture.** An interval-regular graph is interval monotone.

### 1. A family of interval-regular graphs

Let $n = 2m$ be an even integer and $V_n$ be the set of words of length $n$ over $\{0, 1\}$. For $i=1, \ldots, n$ let $f_i$ be the mapping

$$x_1 \cdots x_{i-1} x_i x_{i+1} \cdots x_{2m} \xrightarrow{f_i} x_1 \cdots x_{i-1} \tilde{x}_i x_{i+1} \cdots x_{2m}.$$

Let $\theta$ be a permutation of $\{1 \cdots n\}$ such that $\theta^2 = \text{Id}$ and let $T_\theta$ be the induced mapping from $V_n$ to $V_n$ defined by

$$x_1 \cdots x_i \cdots x_n \xrightarrow{T_\theta} x_{\theta(1)} \cdots x_{\theta(i)} \cdots x_{\theta(n)}.$$

We have $T_\theta^2 = \text{Id}$, $f_i^2 = \text{Id}$, $f_i \circ f_j = f_j \circ f_i$.

**Definition.** $G_{m, \theta}$ is the graph with vertex-set $V_n$ and where two vertices $x$ and $y$ are joined by an edge if and only if

$$\exists i \text{ with } f_i(x) = y \text{ (blue edge) or } x \neq y \text{ and } T_\theta(x) = y \text{ (red edge)}. $$

Let $w$ be the Hamming weight function (number of 1 in the word). If $f_i(u)=v$ then $|w(u) - w(v)| = 1$ and if $T_\theta(u)=v$ then $w(u)=w(v)$ thus an edge cannot be red and blue.

The spanning subgraph of the blue edges of $G_{m, \theta}$ is the hypercube $Q_n$.

**Theorem 1.** $G_{m, \theta}$ is interval-regular.

**Proposition 1.** $T_\theta \circ f_i = f_{\theta(i)} \circ T_\theta$.

This is obvious: on the first hand we have

$$x_1 \cdots x_i \cdots x_n \xrightarrow{f_i} x_1 \cdots \tilde{x}_i \cdots x_n \xrightarrow{T_\theta} x_{\theta(1)} \cdots \tilde{x}_i \cdots x_{\theta(n)}.$$
where $\bar{x}_i$ is in the position $\theta^{-1}(i) = \theta(i)$ and in the second hand

$$
x_1 \cdots x_i \cdots x_n \overset{T_{\theta}}{\longrightarrow} x_{\theta(1)} \cdots x_{\theta(\theta(i))} \cdots x_{\theta(n)} \overset{f_{\theta(i)}}{\longrightarrow} x_{\theta(1)} \cdots \bar{x}_i \cdots x_{\theta(n)}
$$

where $\bar{x}_i$ is also in the position $\theta(i)$.

**Proposition 2.** A shortest path in $G_{m, \theta}$ uses at most one red edge.

Assume

$$(\cdots \circ T_{\theta} \circ f_{i_1} \circ \cdots \circ f_{i_k} \circ T_{\theta} \circ \cdots)(x) = y$$

then by Proposition 1

$$(\cdots \circ f_{\theta(i_1)} \circ \cdots \circ f_{\theta(i_k)} \circ T_{\theta} \circ \cdots)(x) = y$$

and there is a shorter path because $T_{\theta}^{2} = \text{Id}$.

**Proposition 3.** If there is a shortest path in $G_{m, \theta}$ between $x$ and $y$ using a red edge then every geodesic between $x$ and $y$ uses a red edge.

We know that the Hamming weight function verifies: if $f_{i}(u) = v$ then $|w(u) - w(v)| = 1$ and if $T_{\theta}(u) = v$ then $w(u) = w(v)$. Then if there exists a path using a red edge between $x$ and $y$ with length $L$, then the parity of $w(x) - w(y)$ is the parity of $L - 1$ and if there is a path between $x$ and $y$ of length $L$ using only blue edges then $w(x) - w(y)$ and $L$ have the same parity.

**Proposition 4.** If there is a shortest path in $G_{m, \theta}$ between $x$ and $y$ using only blue edges then $x$ and $y$ are joined by exactly $d(x, y)!$ geodesics.

This is an immediate consequence of Proposition 3 because the shortest paths between $x$ and $y$ use only the edges of $Q_n$ and there is exactly $d(x, y)!$ geodesics in $Q_n$.

**Definition.** Assume that $x$ and $y$ are joined by a geodesic using a red edge then we have

$$f_{i_1} \cdots f_{i_k} \circ T_{\theta} \circ f_{\theta(i_k+1)} \cdots f_{\theta(i_p)}(x) = y$$

and thus

$$f_{i_1} \cdots f_{i_k} \circ f_{\theta(i_k+1)} \cdots f_{\theta(i_p)} \circ T_{\theta}(x) = y.$$  

Notice that $i_1, \ldots, i_k, \theta(i_{k+1}), \ldots, \theta(i_p)$ are all distinct otherwise $x$ and $y$ will be joined by a shorter path. We say that the set $\{i_1, \ldots, i_k, \theta(i_{k+1}), \ldots, \theta(i_p)\}$ is the standard set of the geodesic. Clearly if $\{i_1, \ldots, i_p\}$ is the standard set of some geodesic then for all partition of $\{i_1, \ldots, i_p\}$ in $A$ and $B$ we have $A \cap \theta(B) = \emptyset$. 

Proposition 5. All geodesics between $x$ and $y$ have the same standard set.

If

$$f_{i_1} \cdots f_{i_k} \circ T_\theta(x) = f_{j_1} \cdots f_{j_k} \circ T_\theta(x)$$

then for $u = T_\theta(x)$ we have

$$f_{i_1} \cdots f_{i_k}(u) = f_{j_1} \cdots f_{j_k}(u)$$

which implies $\{i_1 \cdots i_k\} = \{j_1 \cdots j_k\}$.

Proposition 6. If there is a shortest path in $G_m$ between $x$ and $y$ using a red edge then $x$ and $y$ are joined by exactly $d(x, y)!$ geodesics.

All geodesics between $x$ and $y$ have the same standard set $\{i_1, \ldots, i_{d(x, y)}\}$ but such a standard set is common to at most $d(x, y)!$ geodesics. Reciprocally for every $k$ in $\{1, \ldots, d(x, y)\}$ and every permutation $\sigma$ of $\{i_1, \ldots, i_{d(x, y)}\}$ we have

$$f_{\sigma(i_1)} \cdots f_{\sigma(i_{d(x, y)})} \circ T_\theta \circ f_{\sigma^{-1}(i_1)}(\sigma(i_k)) \cdots f_{\sigma^{-1}(\sigma(i_{d(x, y)})} = y$$

and all the induced geodesics are distinct.

2. A family of counterexample

Let $\theta$ be the permutation defined by $\theta(i) = i + 1$ if $i$ is odd $i - 1$ if $i$ is even. We have $\theta^2 = 1$ and $T_\theta$ is the mapping

$$x_1 x_2 \cdots x_{i-1} x_i x_{i+1} x_{i+2} \cdots x_n \xrightarrow{T} x_2 x_1 \cdots x_{i-2} x_i x_{i+1} x_{i+3} \cdots x_{n-1}.$$

For every integer $m$ by the above construction we obtain an interval-regular graph $G_m$ of order $2^{2m}$. The first examples are $K4-e$ and the graph $G_2$ shown in Fig. 1.

Theorem 2. $G_m$ is not interval monotone for $m > 1$.

Let $x = (0 \cdots 0)^y = (111 \cdots 0)$. $T$ does not change the Hamming weight thus every geodesic starting from $x$ uses only the edges of $Q_n$ thus the vertices $x$ and $y$ are at distance 3 in $G_m$ and $I(x, y) = \{(abc0 \cdots 0)\}$.

Let $u = (1000 \cdots 0)$ and $v = (0110 \cdots 0)$. We have $u$ and $v$ in $I(x, y)$. $u$ and $v$ are not adjacent and thus are at distance 2 by the geodesics:

$$\begin{align*}
(10000 \cdots 0) & \xrightarrow{T} (01000 \cdots 0) \\ & \xrightarrow{f_3} (01100 \cdots 0)
\end{align*}$$

and

$$\begin{align*}
(10000 \cdots 0) & \xrightarrow{f_4} (10010 \cdots 0) \\ & \xrightarrow{T} (01100 \cdots 0)
\end{align*}$$

But $(10010 \cdots 0)$ is not in $I(x, y)$. 

Fig. 1.

References