Shape and Topology Optimization for Compliant Mechanisms Using Level Set-Based Parameterization Method

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Abstract—A new parameterization approach for optimal synthesis of compliant mechanisms using level set-based shape and topology optimization approach is presented in this work. The structural free design boundary is implicitly represented by embedding into a higher-dimensional scalar function as its zero level set. The compactly supported radial basis function with a favourable smoothness is introduced to construct the trial function. Structural optimization is advanced by iteratively evaluating the optimization problem using a sequential convex programming scheme. The coefficients together with the collection matrix of the radial basis functions are then utilized to renew the level set function by performing the size optimization. Therefore, the presented method is capable of addressing shape fidelity and topology changes simultaneously, especially with the capability of keeping design boundary smooth during the optimization process. Optimal synthesis of compliant mechanisms is applied to demonstrate the potentials, in which the mechanical efficiency is treated as the objective function, while the limitation of the maximal input displacement and the prescribed material usage are treated as two global constraints to further narrow the design domain. A benchmark example is applied to demonstrate the benefits and the advantages of the proposed method.

Keywords: compliant mechanisms, shape optimization, topology optimization, level set method, compactly supported radial basis function, sequential convex programming

I. Introduction

A compliant mechanism [1] is a relatively new breed of mechanical device that achieves at least some of its mobility from the flexibility of its structural members in contrast to a rigid-link mechanism only gains movement through its movable joints, such as hinges, bearings, and slides. Optimal design of compliant mechanisms has been becoming popular in many areas. Advantage of compliant mechanisms is that fewer parts, fewer assembly processes and even no lubrication are needed. The optimal synthesis of compliant mechanisms can be classified into two types generally. The first is the lumped compliant mechanisms produced by the pseudo-rigid-body model based on the method of rigid-body kinematic synthesis [2]. Although the pseudo-rigid-body model can simplify the synthesis of compliant mechanisms by unifying compliant concept and the rigid-body mechanism theories. But, it is difficult to be extended to microsystems, and the lumped compliance only allows local bending around a few flexural hinges so that the mechanism is easily to over stress and overstrain. The other approach is a synthesis way applied to design the distributed compliant mechanisms [3-4], which has been flourishing since the introduction of the work [5] based on topology optimization [6]. The distributed means the jointless mechanism achieve its flexibility throughout the structure by topology and shape of the material continuum. This approach provides a favorite way to design compliant micro-devices in microsystems because its continuity and monolithic can avoid fabrication and assembly in micro-scale. With the popularity of micro-systems, this approach using topology optimization can greatly contribute to the design of distributed micro-compliant mechanisms. Topology optimization has been identified as the most promising structural optimization because it can establish an overall framework at the conceptual stage without any pre-knowledge. A variety of schemes have been established since the identifying works firstly proposed by Bendsøe & Kikuchi [7]. Topology optimization is mainly applied to determine the best arrangement of a given sum of material to iteratively eliminate and redistribute the material in the design space so that a prescribed objective is extremized. Topology optimization, which can be essentially classified as large-scale integer programming with 0 and 1 discrete design variables, is usually ill-posed and those gradient methods with continuous design variable cannot be applied directly. To make the original optimization problem well-posed, homogenization methods [7] and SIMP approaches [8], have been applied to relax the optimization problem. The element-based methods have characterized with some promising properties, and some additional schemes should be employed to smear out the numerical instabilities [6]. Structural boundary should be smoothly reconstructed to eliminate the boundary zigzag features. Hence, topology and shape optimizations have to be performed during two different stages respectively.

Sethian and Wiegmann [9] should be among the first researchers to introduce the boundary level set methods into structural optimization within an framework of an Eulerian approach [10]. Wang and co-workers [11-12] proposed a way of implicitly embedding the free design boundary into a higher-dimensional scalar function as its zero level set to formulate the level set equation as a first-order Hamilton-Jacobi equation. The level set function is iteratively solved by using an appropriate finite difference
scheme to implement the update of the level set function. Allaire and co-works [13] studied the structural shape and topology optimization using a combination of the shape derivative and the level set method for font propagation considering different objective functions. The highlight of these works is to bridge the shape derivative [14] with the powerful level set methods [15].

Currently, the Hamilton-Jacobi PDE schemes are the dominating methods for level set structural optimization. With level set-based implicit representation, the design boundary is updated by updating the level set function in terms of the normal velocity field. The finite difference method based on an Eulerian grid is usually used to solve the partial differential equation numerically. Accordingly, upwind scheme, velocity extension and re-initialization should be properly settled to make the discrete level set equation well-posed. The level set function representing the same design boundary cannot be uniquely determined since any Lipschitz continuous function can be adopted as long as it can keep the zero level set unchangeable. Hence, to stabilize the numerical implementation, the level set function is recommended to be a signed distance function to avoid its drifting away from the initial shape because upwind schemes has a tendency of losing the surface in under-resolved regions or inducing unwanted dissipation of the front during the evolving process [16-17]. The process of re-initialization is often regularly employed to resurrect the level set function by maintaining it to be a signed distance function. An easier scheme [11-12] is to periodically develop a new Hamilton-Jacobi PDE to implement the re-initialization of the level set function to compensate numerical errors. To guarantee the stability of numerical process, the time step, decided by the smallest element size, must be small enough to satisfy Courant- Friedrichs-Lewy (CFL) condition, and thus a large sum of iterations is indispensable to stabilize the numerical solution of the partial differential equation. CFL condition is a necessary condition to provide the explicit time-marching scheme for numerically solving a PDE and the time step is then constrained to provide enough information to fully travel adjacent grids to evaluate the first-order spatial derivatives. The normal velocity field, originally derived from shape derivative, is only meaningful on the free design boundary. However, in the level set-based shape representation, the velocities in the level set equation need to be explicitly evaluated on the Eulerian grids. So, the velocity field must be extended to a set of grids either in a narrow band by using the fast marching-based methods [18-19], or the design domain by incorporating some numerical schemes such as the “ersatz material” approach [13]. A simple way is to make the normal velocity field constant along the normal direction such that a new Hamilton-Jacobi PDE is formulated [11-12]. But the most troublesome thing is the introduction of the discontinuity to the normal velocity field by strain energy densities when the extension crossing the meshes.

Hence, the aim of this work is to develop an efficient parameterization method by considering a more general nonlinear programming derived from optimal synthesis of compliant mechanisms. For the sake of the simplicity, only the linear case is considered but it cannot disable the generality of the proposed method to nonlinear cases. The fixed finite element nodes can also be regarded as the compactly supported radial basis function (CS-RBF) knots. The CS-RBFs [20-21] are used to interpolate the level set function such that the Hamilton-Jacobi PDE optimization problem is converted to a time-separable size optimization problem. The propagation for the free design boundary is advanced by a sequential of solutions by solving the size optimization by using the method of moving asymptotes (MMA) [22-24] because its stability and computational efficiency has been demonstrated. Thus, the presented method can not only inherit the favorable features of the level set-based implicit representation, but also overcome the difficulties of the conventional PDE level set schemes.

II. Implicit Presentation Using Level Set Method

Level set methods is first used for tracking, modeling and simulating the propagation of the moving boundaries with topology changes of merging and breaking naturally in many areas, such as fluid mechanics, computer animation, material science, crack propagation and image processing, etc. It performs in the way of handling interface shape and topology changes by mathematically solving the Hamilton-Jacobi PDE. Thus, the solution of the initial value problem for a partial differential equation with first-order of time derivatives can be exploited to prompt the motion of the interface. A natural way of describing the boundaries is to iteratively solve this equation on a fixed Eulerian grids. In structural optimization, a level set model is developed by implicitly embedding the design boundary into a higher-dimensional level set function as zero level set, and a first-order Hamilton-Jacobi PDE is developed by mathematically solving the Hamilton-Jacobi PDE. Thus, the solution of the initial value problem for a partial differential equation with first-order of time derivatives can be exploited to prompt the motion of the interface. A natural way of describing the boundaries is to iteratively solve this equation on a fixed Eulerian grids. In structural optimization, a level set model is developed by implicitly embedding the design boundary into a higher-dimensional level set function as zero level set, and a first-order Hamilton-Jacobi PDE is developed by differentiating both sides of the level set model with respect to time. Structural optimization is described by advancing the shape fidelity and topology changes by iteratively renewing the level set function that has been governed by a set of initial value PDEs using finite difference schemes on the fixed Eulerian grid. Level set function can be specified in any form provided that it is nonintersecting and Lipschitz continuous. The different parts of the design domain is indicated using the following definition

\[
\begin{align*}
\Phi(x,t) &> 0 \quad \forall x \in \Omega \setminus \partial \Omega \quad \text{(inside)} \\
\Phi(x,t) &= 0 \quad \forall x \in \partial \Omega \cap D \quad \text{(boundary)} \\
\Phi(x,t) &< 0 \quad \forall x \in D \setminus \Omega \quad \text{(outside)}
\end{align*}
\]  

where \( D \subset R^d \) (here \( d = 2 \) ) is regarded as a design domain involved all admissible shapes \( \Omega \) (\( \Omega \subset D \)) depicted as figure 1.
Because the pseudo-time \( t \) has been introduced to indicate the dynamic process of shape deformation, the initially static boundary \( \Gamma = \{ x | \Phi(x) = 0 \} \) can be viewed as a dynamic boundary \( \Gamma(t) = \{ x | \Phi(x,t) = 0 \} \), which can be located by the solution of zero level set \( \Phi(x,t) = 0 \), and the Hamilton-Jacobi equation is given by differencing both sides with respect to \( t \)

\[
\frac{\partial \Phi(x,t)}{\partial t} + \nabla \Phi \cdot v(x) = 0, \quad \Phi(x,0) = \Phi_s(x) \tag{2}
\]

where only the normal velocity field \( v_n \) contributes to the geometrical change of the interface

\[
v_n = v \cdot \left( \frac{\nabla \Phi}{|\nabla \Phi|} \right) \tag{3}
\]

The local unit direction normal to the interface is given as

\[
n = \left( \frac{\nabla \Phi}{|\nabla \Phi|} \right), \quad |\nabla \Phi| = \sqrt{\nabla \Phi \cdot \nabla \Phi} \tag{4}
\]

Then the Hamilton-Jacobi equation is re-written as

\[
\frac{\partial \Phi(x,t)}{\partial t} + v_n |\nabla \Phi| = 0, \quad \Phi(x,0) = \Phi_s(x) \tag{5}
\]

The Hamilton-Jacobi equation, defined on the zero level set, is known as the level set equation. Because the reference domain \( D \) has been introduced as a bounded open set of \( \mathbb{R}^2 \) to include all the admissible shapes of the design domain \( \Omega \) (Figure 1.a), the material and the void areas together with their boundaries can be uniformly discerned by introducing a characteristic function and its related derivative function: Heaviside function \( H \) and Dirac delta function \( \delta \)

\[
H(\Phi) = \begin{cases} 1 & \text{if } \Phi \geq 0 \\ 0 & \text{if } \Phi < 0 \end{cases} \quad \text{and} \quad \delta(\Phi) = \frac{dH(\Phi)}{d\Phi} \tag{6}
\]

\( H(\Phi) = 1 \) is used as a to indicate the solid material and \( H(\Phi) = 0 \) denotes the weak material in the reference domain. \( \Omega \) and \( \partial \Omega \) can be separately expressed as \( \Omega = \{ x : H(\Phi(x)) = 1 \} \) and \( \partial \Omega = \{ x : \delta(\Phi(x)) > 0 \} \). The smeared \( H(\Phi) \) and \( \delta(\Phi) \) are usually given to stabilize the numerical implementation. The boundary and the domain have the following mapping relation

\[
d\Gamma = \delta(x)d\Omega = \delta(\Phi) |\nabla \Phi| d\Omega \tag{7}
\]

where \( \delta(x) = \nabla H \cdot |\nabla \Phi| = \delta(\Phi) |\nabla \Phi| \).

The implicit level set function can address the shape fidelity and topology changes simultaneously in contrast to the classical shape optimization or the homogenization-based methods. The level set function is also characterized with a simple initial topology, but it cannot disable its capability of representing complicated boundary changes by appearing holes, splitting to shape multiple boundaries or forming a single boundary by merging other boundaries. Level set equation is parametric free because the normal component only influences the shape geometry while the tangential part influences the shape parameterization. It has been known that the level set equation is described either as a first-order Hamilton-Jacobi PDE, and therefore the theory of viscosity solutions [25] can be used to ensure a physically meaningful problem. With the unique entry of the Hamilton-Jacobi convection condition, the accuracy, robustness and efficiency can be ensured, even for a nonsmooth or a singular boundary.

### III. Parameterization of Level Set Model

This section introduces the compactly supported radial basis functions into the level set model to develop an implicit boundary parameterization method for structural shape and topology optimization freely from the solution of differential equations (PDEs and ODEs). Radial basis functions have flourished because of its favorable interpolating behaviors which is responsible for its popularity in approximating multivariate scattered data in many engineering areas. The radial basis functions can be classified into two groups: the globally supported radial basis functions (GS-RBF) [26] and the compactly supported radial basis functions (CS-RBF) [20-21].

The conditionally positive definite as well as the positive definite GS-RBFs have been applied to interpolate the embedding level set function in structural shape and topology optimization [27-29]. Although the GS-RBFs possess many attractive features, the selection of shape parameter \( c \) will influence the interpolating accuracy and so far there is no well-established mathematical theory to help select the optimal value. Furthermore, the fully dense matrix will influence the computational efficiency when approximating systems with a large amount of candidate points. However, the CS-RBFs have been introduced as an alternative way to perform multivariate interpolations by taking advantage of their strictly positive definite property, sparse matrix, unnecessary selection of the shape parameter and the approximating completeness. The support of the CS-RBFs, a limited sub-domain around a sample knot, is applied to measure the contribution of a particular node. CS-RBFs can be selected to satisfy the continuity of the resulting interpolant because we can choose the desired basis function among a mass of existed ones to model all the admissible design. Here, the widely investigated CS-RBF
with C4 continuity proposed by Wendland [21] is used to interpolate the level set function. A mathematically simplest size optimization is developed. The shape and derivatives for the popularly used CS-RBFs with C4 continuity is given in figures 2.

**Wendland-C4**

(1) Shapes of the CS-RBF with four order smoothness

\[
\max \left\{ 0, (1-r)^4 \right\} \left( 35 r^2 + 18 r + 3 \right) \quad (8)
\]

(2) Derivatives of the CS-RBF with four order smoothness

\[
\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial r} \frac{\partial r}{\partial x} = \max \left\{ 0, (1-r)^4 \right\} \left( -245 r^2 - 91 r \right) \frac{\partial r}{\partial x} \quad (9)
\]

\[
\frac{\partial \phi}{\partial y} = \frac{\partial \phi}{\partial r} \frac{\partial r}{\partial y} = \max \left\{ 0, (1-r)^4 \right\} \left( -245 r^2 - 91 r \right) \frac{\partial r}{\partial y} \quad (11)
\]

The radius of support is given as following

\[
r = \frac{d_s}{d_{\text{min}}} = \frac{\sqrt{(x-x_i)^2 + (y-y_i)^2}}{d_{\text{min}}} \quad (13)
\]

The size of the domain at a knot is \( d_{\text{min}} = d_{\text{min}} \cdot C_i \), where \( d_{\text{min}} \) can be regarded as a scaling factor, its value for a static analysis is typically located in the range of 2.0-4.0 [30]. \( C \) is determined by searching enough nodes in the neighborhood of the current sample knot. The derivatives of \( r \) in different directions is defined as

\[
\frac{\partial r}{\partial x} = \frac{1}{d_{\text{min}}} \frac{(x-x_i)}{\sqrt{(x-x_i)^2 + (y-y_i)^2}} \quad (14)
\]

\[
\frac{\partial r}{\partial y} = \frac{1}{d_{\text{min}}} \frac{(y-y_i)}{\sqrt{(x-x_i)^2 + (y-y_i)^2}} \quad (15)
\]

We address the significance of the support radius because the errors of the matrix and computational efficiency are tightly related to bandwidth of the matrix. A small support cannot guarantee the basis functions to span the inner-constraint gaps, and a too large support would disable the matrix sparseness. A support that can ensure the invertibility and the sparseness of the matrix should be favorable. Thus, the trial function for the level set function can be obtained by centrally positioning CS-RBFs at those knots over the whole design domain

\[
\Phi(x) = \phi(x)^T \alpha = \sum_{i=1}^{N} \phi_i(x) \alpha_i \quad (16)
\]

where \( \phi(x) \) is the shape function that can be uniquely determined by the positions of the scattered knots.

\[
\phi(x) = \left[ \phi_1(x), \phi_2(x), ..., \phi_N(x) \right]^T \in \mathbb{R}^N \quad (17)
\]

and \( \alpha \) is the expansion coefficient vector given as

\[
\alpha = [\alpha_1, \alpha_2, ..., \alpha_N]^T \in \mathbb{R}^N 
\]

The level set function is parameterized by employing the locally supported CS-RBF. The interpolant for the level set function can be formulated by finding the solution of the following linear system with the given interpolating data located at these knots. The matrix form is given as

\[
B \alpha = h \quad (19)
\]

where the pre-prescribed interpolating values is defined as

\[
h = (h_1, h_2, ..., h_N)^T \in \mathbb{R}^N 
\]

The collection matrix is then constructed as

\[
B = \left[ \begin{array}{cccc} 
\phi_1(x_1) & \phi_1(x_2) & \cdots & \phi_1(x_N) \\
\phi_2(x_1) & \phi_2(x_2) & \cdots & \phi_2(x_N) \\
\vdots & \vdots & \ddots & \vdots \\
\phi_N(x_1) & \phi_N(x_2) & \cdots & \phi_N(x_N) 
\end{array} \right] \in \mathbb{R}^{N \times N} 
\]

The vector of the coefficient of the CS-RBF is given

\[
\alpha = B^{-1} h 
\]

The present interpolating scheme is performed under the assumption that all the CS-RBF knots are fixed in the design domain. Thus the space and time can be thoroughly separated. Thus, the time dependence is only embodied by the expansion coefficients and the renewal of the design boundary is equivalent to sequentially update the expansion coefficients. The implicit level set function is further rewritten as

\[
\Phi(x,t) = \phi(x)^T \alpha(t) 
\]

The level set model is reshaped as

\[
\phi(x)^T \dot{\alpha}(t) - \nabla \phi(x)^T \alpha(t) = 0 
\]

Then \( \nabla \phi(x) \) is now directly expressed as

\[
\nabla \phi(x) = \left[ \begin{array}{c} \phi(x)^T \end{array} \right] \frac{\dot{\alpha}(t)}{dt}, \text{ where } \dot{\alpha}(t) = \frac{d\alpha(t)}{dt} 
\]

The original time-dependent PDE is transformed to an easier size optimization problem in a parametric design domain where some efficient and optimization-orientated schemes can be employed. The movement of the design boundary can be implemented by employing mathematical programming schemes to calculate the coefficients and substituting them into the interpolant to update the shape and topology, and this procedure will be repeated until the optimal configuration is obtained. The CFL condition to stabilize the numerical process can be eliminated because the initially Hamilton-Jacobi PDE is then transformed to a size optimization problem. The move limit in mathematical programming is different from the time step size of CFL condition when an explicit time-marching scheme is used to numerically solve the Hamilton-Jacobi PDE since the latter is determined by the smallest grid size to ensure an
accurate solution of the first-order spatial derivatives and the former doesn’t have any consequence with the grid size.

The normal velocity field is determined by the radial basis function and the expansion coefficients. The explicit calculation of normal velocities and velocity extension to all the knots is unnecessary because the velocity field will be substituted into the shape derivatives as an alternative way to evaluate design sensitivities. The process of the structural optimization is thus implicitly governed by the moving coefficients that are periodically calculated in terms of the design sensitivities of all knots.

**IV. Optimal Synthesis of Compliant Mechanisms**

In this section, a favorable formulation for the optimal synthesis of compliant mechanism design is established by employing an objective to evaluate the mechanical performance, while the prescribed material usage and the maximal input displacement are imposed as additional constraints to further narrow the searching domain so as to engender a feasible resulting mechanism. A variety of methods can be applied to construct the objective function of the optimization problem. In this work, the mechanical efficiency is used to get a balanced trade-off between the desired force and the displacement outputs in a prescribed direction [31] rather than using the strain energy and the geometrical advantage, the latter can ensure a desired displacement output while the former is used to embody the structural stiffness.

The design domain for compliant mechanisms is illustrated in figure 3, where a linear spring model is attached at the output port to simulate the reaction force of the workpiece. Artificial spring model [31] is applied to simulate the mutual reaction between the work-piece and the workpiece. Artificial spring model [31] is applied to simulate the mutual reaction between the work-piece and the workpiece. Artificial spring model [31] is applied to simulate the mutual reaction between the work-piece and the workpiece.

The displacements in the design domain for compliant mechanisms is shown in figure 3, where a linear spring model is attached at the output port to simulate the reaction force of the workpiece. Artificial spring model [31] is applied to simulate the mutual reaction between the work-piece and the workpiece. Artificial spring model [31] is applied to simulate the mutual reaction between the work-piece and the workpiece.

The mechanical efficiency is measured by the ratio of output work to input work involving the contributions of MA and GA to the compliant mechanism design, in which signGA indicates the direction of the output displacement.

\[
ME = \text{sign}(GA) (MA \times GA)
\]

The geometrical advantage GA is re-written as

\[
GA = \frac{\Delta_{\text{out}}}{\Delta_{\text{in}}}
\]

The mathematical formulation is established as

\[
\begin{cases}
\text{Minimize} : J(u_1, u_2, \Phi) = -ME(u_1, u_2, \Phi) \\
G_1 = \int_D g_1(u_1, u_2) H(\Phi) d\Omega \leq 0, \\
G_2 = \int_D g_2(u_1, u_2) H(\Phi) d\Omega \leq 0, \\
\end{cases}
\]

where \( G_1 \) is introduced to limit the input displacement to control the maximum stress level in resulting mechanism [31], and \( G_2 \) is used to restrict material usage. Where

\[
a_1(u_1, u_2, \Phi) = \int_D C_{ijkl} \epsilon_{ij}(u_1) \epsilon_{kl}(u_1) H(\Phi) d\Omega
\]
The shape functional and the design variables using the sensitivity analysis methods. In this work, the concept of shape derivative [11] is recalled with the aim of using a gradient method for the design sensitivity analysis with a smooth bounded open set. One way to find the shape derivatives is the Fréchet derivatives [11] following the way of the shape diffeomorphism suggested by Murat and Simon. The loading functionals are respectively specified as

\[ \left\{ \begin{array}{l}
\mathcal{J}(u, \Phi) = \int_\Omega \phi(u) H(\Phi) d\Omega \\
\mathcal{G}_i(u, \Phi) = \int_{\Gamma_i} \gamma d\Gamma
\end{array} \right. \]

The state equilibrium equation is expressed as

\[ \frac{d}{dt} \rho_\alpha (u, w) = -\frac{\partial}{\partial w} C_{\alpha \beta} e_\alpha (u) e_\beta (w) \]

\[ - \left( - \nabla (\omega \cdot \nabla) \Phi - \nabla \cdot (\nabla \Phi \cdot \nabla \omega) \right) \]

where, the normal velocity operator is satisfy

\[ \partial \Phi |_{\Gamma} = 0 \]

The shape derivative for the objective function can also be expressed in another way using the chain rule

\[ \frac{d \mathcal{J}(u, \Phi)}{d\alpha} = \int_\Omega \frac{\partial \phi(u, w)}{\partial \alpha} H(\Phi) d\Omega \]

The sensitivity of the volume constraint is obtained as

\[ \frac{d \mathcal{G}_i(u, \Phi)}{d\alpha} = \int_{\Gamma_i} \frac{\partial \gamma}{\partial \alpha} d\Gamma \]

The sensitivity of the displacement constraint is given as

\[ \frac{d \mathcal{G}_i(u, \Phi)}{d\alpha} = \int_{\Gamma_i} \frac{\partial \zeta(u, w)}{\partial \alpha} \frac{\partial H(\Phi)}{\partial \alpha} d\Gamma \]

where, \( i = 1, 2, \ldots, N \) indicate the number of the radial basis function knots. The coupled shape density function of the displacement constraint is written as

\[ \xi(u, w) = \begin{cases}
M A \cdot C_{\alpha \beta} e_\alpha (u) e_\beta (w) \\
+ [M (B + D) + G A \cdot E] C_{\alpha \beta} e_\alpha (u) e_\beta (w) \\
+ [M A \cdot C + G A \cdot F] C_{\alpha \beta} e_\alpha (u) e_\beta (w)
\end{cases} \]

The six coefficients are respectively given as

\[ A = \frac{u_{in} - k u_{in} u_{out}}{(u_{in} - k u_{in} u_{out})^2} \]

\[ B = \frac{k u_{in}}{(u_{in} - k u_{in} u_{out})^2} \]

\[ C = \frac{k u_{in} u_{out}}{(u_{in} - k u_{in} u_{out})^2} \]

\[ E = \frac{k^2 u_{out}}{1 - k u_{out}} \]

\[ F = \frac{k^2 u_{out}}{(1 - k u_{out})^2} \]

The popular methods of solving topology optimization of continuum structures can be roughly classified into two categories, such as optimality criteria methods [33] and mathematical programming schemes. Optimality criteria methods have been successfully applied to solve those problems with a large sum of design variables while a few constraints. However, it is not always available to develop an updating scheme if the objective function is not convex and for problems with a complicated objective function or multiple constraints. This work employs the method of moving asymptotes (MMA) [23-24], belonging to the sequential convex programming, to solve the design of the compliant mechanisms since it has been substantiated to be more flexible and mathematically well-founded for
continuum topology optimization. MMA methods operate by solving a sequence of linearized, convex, and separable sub-problems and the approximation of a design function are computed based on the function value and the first or second derivatives. In each step, dual methods or interior-point approaches can be applied to solve the sub-problems to renew the design variables.

VII. Numerical example

Compliant micro-inverter is used to demonstrate the design of compliant mechanisms using structural topology optimization schemes because it has been widely studied as one benchmark example. The material properties are used such as the elastic modulus for the solid material is \( E = 1 \text{GPa} \) and for the weak material is \( E = 0.0001 \text{GPa} \) and Poisson’s ratio is \( \nu = 0.3 \). The design domain of a micro-inverter is shown in figure 5, in which the left-upper and left-lower parts are fixed as the Dirichlet boundary, a force is horizontally loaded at the center point of the right side as non-homogenous Neumann boundary. The objective is to maximize the mechanical efficiency while the prescribed material usage is limited as a volume fraction of 25%. Only the lower half of the design domain is discretized by taking the symmetry into account.

The design domain is discretized with 5000 elements, and the contours of the zero level set in different stages are given in figure 6, and the related high-dimensional level set surfaces are given in figure 7. It can be seen that the designs are all characterized with a smooth boundary, which means the proposed algorithm has the capacity of addressing the shape fidelity and topology changes at the same time compared to conventional shape optimization and the element-based topology optimization that cannot perform shape and topology optimization simultaneously. It is well-known a smooth design boundary is meaningful for the boundary representation based shape optimization. The curves of the mechanical efficiency and the volume constraint over the iterations are shown in figure 8. From their convergent histories, we can find the mechanical efficiency rapidly increases from 0.0216 to 0.568 at the initial 38 iteration, and structural topology optimization is also almost completed before 38th iteration. After that, it takes 274 iterations to make the mechanical efficiency gradually increase from 0.5685 to 0.6851, and these iterations are used to perform the shape optimization to achieve a uniform distribution of the coupled-strain energy density. The changing process of the zero-level set contours is given in figure 9.
References

V. Conclusion
This paper presents a novel parameterization method employing CS-RBFs to interpolate the level set function with favorable smoothness and accuracy. It can address structural shape fidelity and topology changes with a smooth boundary during the optimization process, which is important for the boundary shape-based optimization. The mathematically difficult PDE is transformed into an easiest size optimization. MMA method is directly applied to solve the parameterization optimization problem. The free design boundary propagation is iteratively advanced in terms of solutions of the size optimization problem. The efficiency has been demonstrated by employing an widely studied numerical application.