Low Energy Construction of Fault-Tolerant Topologies in Wireless Networks

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ABSTRACT
This paper studies asymmetric power assignments for various network topologies under the $k$-resilience criterion. We aim to minimize the total energy consumption, which is usually NP-Hard for a desired link topology. We develop a general approximation framework for various topology problems under the $k$-fault resilience criterion in the plane. We use it to obtain an $O(k^5)$ approximation ratio for three $k$-fault tolerant topology problems: multicast, broadcast and convergecast. In addition, we consider the problem of $k$-fault tolerant bounded-hop broadcast. We derive an $O(k^4)$ approximation factor for this problem. To the best of our knowledge, these are the first non-trivial results for these problems.

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C.2.1 [Computer-Communication Networks]: Network Architecture and Design—Wireless communications; G.2.2 [Discrete Mathematics]: Graph Theory—Network Problems

General Terms
Algorithms, Design, Reliability, Theory

Keywords

1. INTRODUCTION
A wireless ad-hoc network consists of several transceivers, communicating by radio. Each transceiver $t$ is assigned a transmission power $p(t)$, which results in some transmission range, denoted by $r_t$. This is customary to assume that the minimal transmission power required to transmit to a distance $d$ is $d^\alpha$, where the distance-power gradient $\alpha$ is usually taken to be in the interval $[2, 4]$ (see [32]). Thus, a transceiver $t$ receives transmissions from $s$ if $p(s) \geq d(s, t)^\alpha$, where $d(s, t)$ is the Euclidean distance between $s$ and $t$. The transmission possibilities resulting from a power assignment induce a communication graph. Research efforts have focused on finding power assignments, for which the induced communication graph satisfies a certain topology property, while minimizing the total cost. The strong connectivity (all-to-all) property has been the first studied problem in this area. The importance of this property is due to the fact that in a wireless network, it can be useful that every station can communicate with all the other ones. Other key properties are broadcast (one-to-all) and multicast (one-to-many), which are very close from the designer point of view, while broadcast being a special case of multicast. In some cases it is essential to minimize the number of hops from the root transceiver to other nodes in the network. This is called the bounded-hop broadcasting. An additional property, convergecast (many-to-one), is critical for data gathering.

This paper is organized as follows. In the rest of this section we present the model, previous work and briefly describe our results. In Section 2 we present our framework for planar $k$-fault resistance and present approximation algorithms for the $k$-broadcast, $k$-multicast and $k$-convergecast problems. In Section 3 we address the $k$-fault tolerant bounded-hop topology property and derive an approximation algorithm for this problem. To the best of our knowledge, these are the first non-trivial results for these problems.

1.1 The Model
We are given a set $T$ of $n$ transceivers $t_1, t_2, \ldots, t_n$, positioned in $\mathbb{R}^d$, $d \geq 1$. We define the cost of an undirected graph $G(T) = (T, E)$, with edges $E$ to be:

$$C_G = \sum_{(s,t) \in E} c(s, t),$$

where the cost of an edge $(s, t) \in E$ is $c(s, t) = d(s, t)^\alpha$. Once each transceiver is assigned a transmission power $p(t) = r_t^\alpha$, an ad-hoc network is created. A power assignment for $T$ is a vector of transmission powers $\{p(t) \mid t \in T\}$ and is denoted by $A(T)$ (usually abbreviated to $A$). The resulting directed communication graph is denoted by $H_A = (T, E_A)$, where $E_A$ is the set of directed edges created as a result from the power assignment $A(T)$:

$$E_A = \{(t, s) \mid p(t) \geq d(t,s)^\alpha\}.$$
That is, there is a directed edge from \( t \) to \( s \) if \( t \) has sufficient transmission power to reach \( s \). Throughout this paper we refer to transceivers as nodes. The cost of the power assignment is defined as the sum of all transmission powers:

\[
C_A = \sum_{i\in T} p(i).
\]

A square of graph \( G = (V, E) \) is defined as \( G^2 = (V, E^2) \), where \((u,v) \in E^2 \) if \((u,v) \in E \) or \( \exists w : (u,v), (v,w) \in E \). A Hamiltonian cycle, is a graph cycle that visits each node exactly once. A graph possessing a Hamiltonian cycle is said to be a Hamiltonian graph or simply Hamiltonian. The cost of a Hamiltonian cycle \( h = (t_1, t_2, \ldots, t_n, t_{n+1} = t_1) \) is:

\[
C_h = \sum_{i=1}^{n} c(t_i, t_{i+1}) = \sum_{i=1}^{n} d(t_i, t_{i+1})^\alpha
\]

Given a graph \( G \) we denote by \( P(G) \) the topological properties it holds. In this paper we address the following properties:

- **Strong Connectivity** — A graph \( G = (V, E) \) is strongly connected if for any two nodes \( u, v \in V \) there exists a path from \( u \) to \( v \).
- **Multicast** — A graph \( G = (V, E) \) is multicasting from a root node \( r \in V \) to a subset of nodes \( M \subseteq V \) if for any node \( v \in M \) there exists a path from \( r \) to \( v \).
- **Broadcast** — A graph \( G = (V, E) \) is broadcasting from a root node \( r \in V \), if for any \( v \neq r \in V \) there exists a path from \( r \) to \( v \). This is a special case of multicast with \( M = V \).
- **Bounded-Hop Broadcast** — A graph \( G = (V, E) \) is bounded-hop broadcasting from a root node \( r \in V \) in at most \( h \) hops, if for any \( v \neq r \in V \) there exists a path from \( r \) to \( v \) in at most \( h \) hops (note that for the unbounded broadcast \( h = n - 1 \)).
- **Convergecast** — A graph \( G = (V, E) \) is convergecasting to a root node \( r \in V \), if for any \( v \neq r \in V \) there exists a path from \( v \) to \( r \).

If a graph holds some property \( P \), we say \( P \in P(G) \). A graph property \( P \) is said to be \( k \)-fault resistant (tolerant) in \( G \) if it still holds after an extraction of at most \( k \) – 1 nodes. That is given any subset of nodes \( X \), where \( |X| < k \), it holds \( P \in P(G \setminus X) \). We will refer to the \( k \) parameter as the fault resistance parameter. We use an abbreviate terminology of \(<property-name> \) problem, e.g. \( k \)-broadcast. We omit \( k \), if \( k = 1 \). This paper addresses the following topology problems:

- **Minimum Energy \( k \)-Fault Resistant Multicast** ([MEkM]) — Given a set of nodes \( T \) in \( \mathbb{R}^d \), a node \( r \in \mathbb{T} \) and a subset of nodes \( M \subseteq T \), find a power assignment \( A(T) \) so that \( H_A \) is \( k \)-multicasting from \( r \) to \( M \) and \( C_A \) is minimized.

- **Minimum Energy \( k \)-Fault Resistant Broadcast** ([MEkB]) — Given a set of nodes \( T \) in \( \mathbb{R}^2 \) and a node \( r \in T \), find a power assignment \( A(T) \) so that \( H_A \) is \( k \)-broadcasting from \( r \) and \( C_A \) is minimized.

- **Minimum Energy \( k \)-Fault Resistant h-Bounded-Hop Broadcast (MEkBH)** — Given a set of nodes \( T \) in \( \mathbb{R}^d \) \((d = 1, 2)\), a fault resistance parameter \( k \) and a maximal number of allowed hops \( h \), find a power assignment \( A(T) \) so that \( H_A \) is bounded-hop broadcasting from a root node \( r \in V \) in at most \( h \) hops and \( C_A \) is minimized.

Some properties are stronger than others, that is satisfying some topological property is sufficient for other properties to hold. For example strong connectivity is stronger than broadcast because any graph that holds the strong connectivity is also broadcasting from every node chosen as root. Formally, property \( P_1 \) is stronger than \( P_2 \) if for any graph \( G \) it holds \( P_1 \subseteq P(G) \Rightarrow P_2 \subseteq P(G) \). We denote \( P_1 > P_2 \) if \( P_1 \) is stronger than \( P_2 \). Note that if \( P_1 > P_2 \) and \( P_1 \) is \( k \)-fault resistant in \( G \), then \( P_2 \) is also \( k \)-fault resistant in \( G \).

Given a topology property \( P \) we wish to find a power assignment \( A_k^* \) so that \( P \) is \( k \)-fault resistant in \( H_{A_k^*} \) and \( C_{A_k^*} \) is minimized. Let \( A^* \) be the optimal power assignment so that the communication graph \( H_{A^*} \) holds the \( P \) property (the fault-resistance parameter is 1). Note that providing high order fault resistance costs more than providing lower order fault resistance. That is \( C_{A_{k_1}^*} \leq C_{A_{k_2}^*} \) iff \( k_1 < k_2 \).

In this paper we use the notations \( A_k^* \) and \( A^* \) for numerous topology properties. The appropriate property will be easily understood from the context.

Due to the distance-power gradient \( \alpha \) the edge costs do not hold the triangle inequality. However the relaxed triangle inequality is satisfied. In our cost model, for any \( u, v, w \in T \) there exists a parameter \( r \) so that:

\[
c(u, w) < r(c(u, v) + c(v, w)),
\]

where \( r = 2^{\alpha - 1} \) for every \( \alpha \geq 2 \).

### 1.2 Previous Work

Topology control in wireless networks is a relatively new field of interest. Nevertheless a wide area of problems has already been studied. Most of the problems are aimed at computing a low energy power assignment that meets global topological constraints. Two versions of the problem arise: symmetric and asymmetric. In the symmetric version for any two nodes \( t, s \in T \), \( p(t) \geq d(t, s)^\alpha \Leftrightarrow p(s) \geq d(s, t)^\alpha \), that is a node \( t \) can reach node \( s \) if and only if \( s \) can reach node \( t \), we can also refer to it as an undirected model. The asymmetric version allows directed links between two nodes. Kirousis et al. [29] introduced the MinRange(3C) problem, which is the \( k \)-strong connectivity problem for \( k = 1 \). They proved it to be NP-Hard for the three dimensional Euclidean space for any value of \( \alpha \). The same paper provided a 2-approximation algorithm for the planar case and an exact \( O(n^4) \) time algorithm for the one dimensional case. In the planar case, the NP-Hardness of the problem for every \( \alpha \) has been proved in [20] and a simple 1.5-approximation algorithm for the case of \( \alpha = 1 \) has been provided in [5]. See also [4, 30, 18, 29].
Wieselthier et al. in [38, 39] were the first to study the broadcast problem in wireless ad-hoc networks for the 2-dimensional case and when \( \alpha = 2 \). In this work, the performances of three heuristics, namely the minimum spanning tree (MST), the shortest path tree (SPT) and the broadcasting incremental power (BIP) have been experimentally compared (one to each other) on the random uniform model without providing theoretical results. The approach taken in [38, 39] is to build a source rooted spanning tree by adjusting transmit powers of nodes, followed by a sweep operation to remove redundant transmissions. Wan et al. in [37] present the first analytical results for this problem by exploring geometric structures of Euclidean MSTs. In particular, they prove that the approximation ratio of MST is between 6 and 12. Wan et al. [37] show a constant factor approximation for the minimum power asymmetric multicast problem. Cagalj et al. [11] give a proof of NP-Hardness of the minimum-energy broadcast problem in a Euclidean space. Many researchers provided analytic results of the minimum-energy broadcast algorithm based on computing an MST, see also [2, 6, 9, 16, 24]

We can also add an additional constraint parameter to the problem, the bounded diameter \( h \) of the induced communication graph. For the linear case node disposition, Kirousis et al. [29] develop an optimal power assignment algorithm in \( O(n^{1.5}) \) time. In the Euclidean case, [21] obtains constant ratio algorithms for the bounded-hop strong connectivity for well spread instances. Beier et al. [7] discuss the problem of finding a bounded-hop path between pairs of nodes with minimized power consumption. They find an optimal path in \( O(hn \log n) \) time. In [12] the authors obtain \( (O(\log n), O(\log n)) \) bicriteria approximation algorithms for the bounded-hop broadcast, bounded-hop connectivity and bounded-hop symmetric connectivity problems. In their output there are at most \( h \log n \) hops with \( \log n \) times the optimal cost for \( h \) hops. In [3] the authors present an exact algorithm for solving the 2-hop broadcast problem with a running time of \( O(n^2) \) as well as a PTAS with a running time of \( O(n^\mu) \) where \( \mu = O((h^2/\epsilon)^{1/2}) \). Funke and Laue [25] provide a PTAS for the k-hop broadcast algorithm in time linear in \( n \). Additional results for bounded range assignments can be found in [19, 23].

A natural generalization of any topology property requirement is \( k \)-fault resilience, which increases the network’s fault tolerance and provides multi-path redundancy for load balancing. The hardness of topology problems for the case of \( k = 1 \) implies that topology problems with a higher value of \( k \) are also NP-Hard. A first non trivial result for planar asymmetric \( k \)-strong connectivity was presented by Shpun- gin and Segal in [34]. They derived an approximation factor of \( O(k^2) \) for the planar case and some results for the linear case. Carmi et al. [15] improved the approximation ratio to \( O(k) \). Another possible connectivity property is \( k \)-edge connectivity, which implies that the removal of less than \( k \) edges leaves the graph strongly connected. In [14], Calinescu and Wan proved NP-Hardness of the symmetric two-edge and two-node strong connectivity and then provided a 4-approximation algorithm for both symmetric and asymmetric strong biconnectivity (\( k = 2 \)) and a 2k-approximation for both symmetric and asymmetric \( k \)-edge strong connectivity. Hajigraphi et al. [26] give two algorithms for symmetric \( k \)-strong connectivity, with \( O(k \log k) \) and \( O(k) \)-approximation factors and also some distributed approximation algorithms for \( k = 2 \) and \( k = 3 \) in geometric graphs. Jia et al. in [28] present various approximation factors (depending on \( k \)) for the symmetric \( k \)-strong connectivity, such as 3k-approximation algorithm for any \( k \geq 3 \) and 6-approximation for \( k = 3 \). Segal and Shpungin [33] extend static algorithms for \( k \)-connectivity to support dynamic node insert/delete operations. Additional results can be found in [1, 10, 13, 17, 22, 27, 31, 35].

1.3 Our Contribution

We develop a general approximation framework for various topology problems and obtain an \( O(k^2) \) approximation ratio for three \( k \)-fault tolerant topology problems: multicast, broadcast and convergecast. In addition, we consider the problem of \( k \)-fault tolerant bounded-hop broadcast. We derive an \( O(k^3) \) approximation factor for this problem. All our algorithms run in polynomial time.

2. PLANAR \( K \)-FAULT RESISTANT TOPOLOGY FRAMEWORK

We have developed a framework which provides \( k \)-fault resistance to a family of topology problems. In particular we show approximation algorithms for the MEkm, MEkB and MECk problems. For simplicity we assume \( \alpha = 2 \), but our results can be easily extended for any fixed \( \alpha \). The framework consists of two stages. Let \( MST \) be a minimum spanning tree of \( T \). First we construct a Hamiltonian cycle with a cost \( O(C_{MST}) \). Then each node is assigned a transmission range along the cycle with respect to the topology property \( \mathcal{P} \) and the fault resistance parameter \( k \). The cost of the resulting power assignment \( \Lambda_k \) is in \( O(k^2C_{MST}) \). We explain these stages in detail.

2.1 The Framework

The Hamiltonian cycle construction is based on the TSP-APPROX algorithm in [8]. The authors use the fact that the square of every 2-strongly connected graph is Hamiltonian to find a low cost Hamiltonian cycle. We use the MST-Augmentation algorithm presented in [14] for constructing an undirected 2-strongly connected graph. For every non-leaf node \( v \in T \), the algorithm constructs a local Euclidean minimum spanning tree \( MST_v \) over all its neighbors. As a result we obtain an undirected graph \( H \) with edges of both the initial \( MST \) and all the local minimum spanning trees \( MST_v \).

Hamiltonian Cycle Stage (I)

1. Find \( MST \).
2. Apply the MST-Augmentation algorithm to obtain a 2-strongly connected graph \( H \).
3. Apply the TSP-APPROX algorithm over \( H \) to obtain a Hamiltonian cycle \( h \) of the square of \( H \).

Figure 1 demonstrates these steps. In Fig. 1.a we see the initial MST. Then in Fig. 1.b the edges in bold are added as a result of the MST-Augmentation algorithm. Finally in Fig. 1.c a Hamiltonian cycle is created according to the TSP-APPROX algorithm.
Denote the resulting power assignment by $h$. The first two are defined in [8] and [14] respectively.

In our analysis we make use of several lemmas and theorems.

**Power Assignment Stage (II)**

For simplicity of annotation, let us denote clockwise and counterclockwise traverses in the Hamiltonian cycle originating at node $t$ by $(t_{+i})_{i=0}^{n} = (t = t_{+0}, t_{+1}, \ldots, t_{+n} = t)$ and $(t_{-i})_{i=0}^{n} = (t = t_{-0}, t_{-1}, \ldots, t_{-n} = t)$, respectively. Note that $t_{j} = t_{n-j}$ for $0 \leq j \leq n$.

The same construction technique is used for every topological property. In order to obtain $k$-node-disjoint paths from node $s$ to node $t$ we will assign transmission powers so that there are $k/2$ node-disjoint paths in the clockwise and counterclockwise directions of the Hamiltonian cycle. This is achieved by assigning each node with sufficient power to reach $k/2$ neighboring nodes in both directions of the cycle. Denote the resulting power assignment by $A_{h}^k$. In [34] we use a similar construction for providing $k$-strong connectivity.

### 2.2 Analysis

In our analysis we make use of several lemmas and theorems. The first two are defined in [8] and [14] respectively.  

**Theorem 2.1 (Theorem 1 [8]).** For the relaxed triangle inequality parameter $r$, the algorithm TSP-APPROX finds a Hamiltonian cycle $h$ of cost $C_{h} \leq 2rC_{H}$.

**Lemma 2.2 (Lemma 11 [14]).** Let $E_1$ be the set of all edges of $MST$ incident to leaves. Let $E_2$ be the set of all edges of the trees $MST$ for all non-leaf nodes $v \in T$. For $H' = (T, E_1 \cup E_2)$ and $\alpha = 2$, it holds $C_{H'} \leq 4C_{MST}$.

**Lemma 2.3.** For $\alpha = 2$, $C_{h} \in O(C_{MST})$.

Proof. From the construction of $H'$ we can see that $C_{H} \leq C_{H'} + C_{MST}$. According to Lemma 2.2 $C_{H'} \leq 4C_{MST}$ and therefore $C_{H} \leq 5C_{MST}$. For $\alpha = 2$, the relaxed triangle inequality parameter is $r = 2$. According to Theorem 2.1 $C_{h} \leq 4C_{H} \in O(C_{MST})$.

Denote by $P_{SC}$ the strong connectivity topology property. In [34] we prove that $P_{SC}$ is $k$-fault resistant in $H_{A_{h}^k}$ and derive the following cost bound.

1. Node $t$ is assigned a range, which is the sum of distances along the Hamiltonian cycle.
2. We altered the description of every external lemma or theorem to match the paper model and definitions.

**Lemma 2.4 ([34]).** $C_{A_{h}^k} \in O(k^2C_{h})$.

Note that a topology property $P$ is $k$-fault resistant in $H_{A_{h}^k}$ if $P < P_{SC}$. Recall that for some topology property $P$, we denote by $A_{h}^k$ the optimal power assignment so that $P$ is $k$-fault resistant in the induced communication graph $H_{A_{h}^k}$.

We are ready to present our main theorem.

**Theorem 2.5.** Given a set $T$ of $n$ nodes in the plane and a topology property $P$, if $C_{A_{h}^k} \geq C_{MST}$ and $P < P_{SC}$ then $C_{A_{h}^k} \in O(k^2C_{A_{h}^k})$ and $P$ is $k$-fault resistant in $H_{A_{h}^k}$ for any fault resistance parameter $k$.

Proof. Easy to see that $P$ is $k$-fault resistant in $H_{A_{h}^k}$.

From Lemmas 2.3 and 2.4 we have $C_{h} \in O(C_{MST})$ and $C_{A_{h}^k} \in O(k^2C_{h})$, respectively. Given $C_{A_{h}^k} \geq C_{MST}$ we can conclude $C_{A_{h}^k} \in O(k^2C_{A_{h}^k})$.

### 2.3 Approximating MEkM, MEkB and MEkC

Next we provide approximation algorithms for the problems MEkM, MEkB and MEkC, namely $k$-multicast, $k$-broadcast and $k$-convergecast. For each problem we first describe the power assignment and then prove the approximation ratio. Since all three problems have a root node $r \in T$, we will use a Hamiltonian cycle traverse originating in $r$, $(r_{+i})_{i=0}^{n}$ and $(r_{-i})_{i=0}^{n}$. Figure 2 demonstrates the power assignment stage (II) for the three topologies with $k = 2$. The arrows in the graph denote the transmission traffic we wish to obtain.

#### 2.3.1 $k$-Broadcast

First we obtain the power assignment $A_{h}^k$ which induces a broadcast tree rooted at node $r \in T$ and there are $k$ node-disjoint paths from $r$ to any other node.

1. $k$-Broadcast($T, r$)

Take the Hamiltonian cycle $h$ according to stage I of the framework. We follow the construction in stage II of the framework and form $k/2$ disjoint paths from $r$ to $r_{+1}$ in the clockwise direction of the cycle and $k/2$ disjoint paths from $r$ to $r_{-n-1}$ in the counterclockwise direction of the cycle.

Next we analyze the approximation ratio of the algorithm. We make use of the following Lemma from [36] to obtain a lower bound for the optimal solution $A^r$ for the MEkM problem.

**Lemma 2.6 (Lemma 4 [36]).** For any node set $T$ in the plane, the total energy required by broadcasting from any node in $T$ is at least $\frac{1}{6}C_{MST}$, where $6 \leq e \leq 12$.

We immediately conclude $C_{A^r} \in O(C_{MST})$. Note that our construction is cheaper than $A_{h}^k$. The next theorem is now easily derived.
Theorem 2.7. Given a set of nodes $T$ in the plane and a root node $r \in T$, the power assignment $A^{M}_{h}$ induces a $k$-broadcasting communication graph from $r$ and it holds
\[ C_{A^{M}_{h}} \in O(k^{2}C^{*}_{A}). \]

Proof. Since there are $k/2$ node-disjoint paths from $r$ to any other node in the clockwise direction of the cycle and $k/2$ node-disjoint paths in the clockwise direction we can conclude that the induced communication graph is $k$-broadcasting. For example, in Fig. 2.a, there is one path from $r$ to any other node in clockwise and counterclockwise direction. The approximation ratio follows immediately from Lemma 2.6 and Theorem 2.5.

2.3.2 $k$-Multicast

Let us describe the power assignment $A^{M}_{k}$ which induces a multicast tree rooted at node $r \in T$ and there are $k$ node-disjoint paths from $r$ to any node in $M$. We will use an approximation algorithm presented in [37]. The authors developed a heuristic called SPF (shortest-path-first), which finds a multicast tree in an asymmetric network model such that the total transmission power is no more than 24 times the optimum.

2 $k$-Multicast($T,r,M$)

Let $S \subseteq T$ be a set of nodes reachable from $r$ as a result of using SPF. We construct a Hamiltonian cycle $h_S$ traversing all nodes in $S$ according to stage I of the framework. Let $1 \leq f \leq l \leq |S| + 1$ be the indexes of the first and last nodes in $M$ as they appear in the hamiltonian cycle traverse $(r = r_0, r_1, \ldots, r_{n-1}, r_n = r)$. That is $r_{l+i}, r_{l+i} \in M$ and $\forall i: 0 < i < f$ or $i < n, \ r_{l+i} \notin M$. We now follow the construction in stage II of the framework and form $k/2$ disjoint paths from $r$ to $r_{i},$ in the clockwise direction of the cycle and $k/2$ disjoint paths from $r$ to $r_{(n-i)}$ in the opposite direction.

The next theorem proves the correctness of the assignment $A^{M}_{k}$ and the approximation ratio of $O(k^2)$ times the optimum for the MEkM problem. Again our construction is cheaper than $A^{h}_{k}$.

Theorem 2.8. Given a set of nodes $T$ in the plane, root node $r \in T$ and a multicast destination set $M$, the power assignment $A^{M}_{k}$ induces a $k$-multicasting communication graph from $r$ to $M$ and it holds
\[ C_{A^{M}_{k}} \in O(k^{2}C^{*}_{A}). \]

Proof. In a similar way, like the broadcast case, there are $k/2$ node-disjoint paths in both the clockwise and counterclockwise direction from $r$ to every node in $M$. For example, in Fig. 2.b there is one path in both directions from $r$ to all nodes in $M$. Let $A_{1}$ be the power assignment produced by SPF. Let $MST_{B}$ be the minimum spanning tree of $S$. Let $A^{*}_{k}$ be the optimal power assignment for $1$-broadcasting from $r$ to nodes in $S$. Obviously $C_{A_{1}} \geq C_{A^{*}_{k}}$. From Lemma 2.6 it is easy to see that $C_{A^{*}_{k}} \in \Omega(C_{MST_{B}})$. In [37] the authors proved $C_{A_{1}} \leq 24C^{*}_{A}$, where $A^{*}$ is the optimal power assignment for 1-multicasting from $r$. Since $A^{*}_{k}$ is cheaper than $A^{*}_{1}$ then $C_{A^{M}_{k}} \in O(k^2C_{MST_{B}})$. We conclude $C_{A^{M}_{k}} \in O(k^2C_{A})$, and finally obtain $C_{A^{M}_{k}} \in O(k^2C^{*}_{A})$.

2.3.3 $k$-Convergecast

A similar technique is proposed in case of $k$-convergecast. Power assignment $A^{C}_{k}$, which forms a $k$-convergecast tree to some root node $r$ will have $k$ node-disjoint paths from every node to $r$. We first describe the construction and then analyze the approximation ratio.

3 $k$-Convergecast($T,r$)

Take the Hamiltonian cycle $h$ according to stage I of the framework. We follow the construction in stage II of the framework and form $k/2$ disjoint paths from every $r_{i}$ to $r$ in the clockwise direction of the cycle and $k/2$ disjoint paths in the counterclockwise direction.

In order to obtain the approximation ratio of $O(k^2)$ we first prove the following bound.

Lemma 2.9. Given a set of nodes $T$ in the plane and a root node $r \in T$, the total energy required by convergecasting from any node to $r$ is at least $C_{MST}$.

Proof. Let $A^{*}$ be an optimal power assignment so that $H^{*}_{A}$ is $1$-convergecasting to $r$. Remove edges from $H^{*}_{A}$ leaving only paths towards the root, so there is a unique path from every node to $r$. Then replace all directed edges with undirected ones. Let the resulting tree be $ST$. There is only one path from any node to $r$, therefore every edge is created by assigning power to a different node. As a result $C_{MST} \leq C_{ST} \leq C^{*}_{A}$.

Since $C^{*}_{A}$ is cheaper than $A^{h}_{k}$, the following Theorem follows.

Theorem 2.10. Given a set of nodes $T$ in the plane and a root node $r \in T$, the power assignment $A^{C}_{k}$ induces a $k$-convergecasting communication graph to $r$ and it holds
\[ C_{A^{C}_{k}} \in O(k^{2}C^{*}_{A}). \]

Proof. There are $k/2$ node-disjoint paths in both directions of the cycle, from any node to $r$. For example, in Fig. 2.c there is one path in both directions from every node to $r$. The approximation ratio follows immediately from Lemma 2.9 and Theorem 2.5.

3. PLANAR BOUNDED-HOP BROADCAST

The general idea for a power assignment, which induces a $k$-fault resistant bounded-hop broadcast graph in the plane, is first to obtain a bounded-hop broadcast graph and then make it $k$-fault resistant. In [25] the authors present a PTAS algorithm for the MEKBB problem with fault resistance parameter $k = 1$. We use this construction as a basis for $k$-fault resistant bounded-hop broadcast. We first explain the technique used for $k$-fault resistant strong connectivity suggested in [15], which is then used to obtain a $k$-fault resistant broadcast tree.
For each edge \( e = (t,s) \) in the \( \text{mst} \) we denote \( r_e^* = r_e^{\ast \ast} \). Therefore each node \( t' \) is assigned a transmission range which is the maximum between \( r_e^* \), and its obligation to some node \( t \), where \( t' \in N_t \) and there is an edge \( e \in \text{mst} \) so that \( r_e^* > r_e^* \) (see Figure 3). Therefore,

\[
C_{A_k} = \sum_{e \in \text{mst}} p(t') \leq \sum_{e \in \text{mst}} \max\{r_e^*, \max_{e \in \text{mst}} r_e^*\}.
\]

From Lemma 3.1 and the fact that geometrical \( \text{mst} \) has a bounded degree of \( 6 \),

\[
C_{A_k} = O(k) \left( \sum_{e \in \text{mst}} (r_e^*)^2 + \sum_{e \in \text{mst}} |e|^2 \right).
\]

According to \([29]\) \( C_{\text{mst}} \leq C_{A_k} \leq C_{A_k}^* \). We can conclude, \( C_{A_k} = O(k) \left( C_{A_k}^* + C_{\text{mst}} \right) = O(k)C_{A_k}^* \).

### 3.2 The algorithm

Given a set of nodes \( T \) and a root node \( r \), we wish to construct a power assignment \( A_k^r \), so that the induced communication graph \( H_{A_k} \) is \( k \)-h-broadcast rooted at \( r \). As before, for each node \( t \in T \), let \( N_t \subseteq T \) be a set of \( k \)-closest nodes to \( t \), and put \( r_t^* = \max_{e \in N_t}(d(t,t')) \). Let \( A_k \) be a power assignment constructed in \([25]\) for some constant \( h \), so that \( H_{A_k} \) is a 1-\text{h-broadcast} graph. We are ready to describe the power assignment algorithm.

We start by constructing a directed spanning tree of \( H_{A_k} \) by running a BFS from the root node \( r \). Denote the resulting tree by \( \text{bht} \) and by level-1 nodes to be the nodes at distance \( i \) from the root. Clearly for each node \( t \in T \) there is a unique directed path of at most \( h \) hops from \( r \) to \( t \) in the \( \text{bht} \). Note that the power assignment \( A_{\text{bht}} \) required to induce this tree has a cost \( C_{A_{\text{bht}}} \leq C_{A_k} \). Next we decrease the depth of \( \text{bht} \) to \( h-1 \) by adding a directed edge from \( r \) to every level-2 node and remove all edges between level-1 nodes and level-2 nodes (see Figure 4). Call this tree \( \text{bht} \) and by \( A_{\text{bht}} \) the power assignment required to induce this tree. Easy to see that \( C_{A_{\text{bht}}} \leq 2C_{A_{\text{bht}}} \leq 2C_{A_k} \) by using the following observation.

**Observation 3.3.** For any \( x_1, x_2, \ldots, x_m \in \mathbb{N} \) it holds \((\sum_{i=1}^m x_i)^2 \leq m \sum_{i=1}^m x_i^2\).

Similar to the case of strong connectivity, we would like to create \( k \)-node disjoint paths along the edges of \( \text{bht} \), from \( r \) to any node other \( t \in T \). We start by assigning the root node \( r \) with a power \( p(r) = (r_e^*)^2 \), so that it can reach its \( k \)-closest neighbors. Next, for each directed edge \( e = (t,s) \) (from \( t \) to \( s \)) in \( \text{bht} \) we increase the power (if required) of each node \( t' \in N_t \) so it could reach all nodes in \( N_t \cup \{s\} \). Let \( A_k^r \) denote the resulting power assignment. It is easy to see that the resulting (directed) communication graph \( H_{A_k} \) is \( k \)-fault resistant \( h \)-bounded-hop broadcast rooted at \( r \). That is, for any node \( t \in T \) there are \( k \)-node disjoint paths from \( r \) to \( t \), that "follow" the path from \( r \) to \( t \) in \( \text{bht} \). And each of these paths has at most \( h \) hops.
3.3 Analysis

In order to analyze the cost of the power assignment $A^h_k$ we need to take a closer look at the power increase stage of each node. All nodes (except for the root) start with no power and it is increased if required. The power of $t' \in N_t$ is increased only to satisfy the demand of some outgoing edge $e = (t, s)$ from $t$ in the BHT, that is to reach any node in $N_s \cup \{s\}$. Since node $t'$ can be a member in many sets of k-closest neighbors, it might be required to increase its power many times, but eventually its power will be dominated by some outgoing edge $e' = (t, s)$, where $t' \in N_t$. Recall that for an edge $e = (t, s)$ we denote by $r^*_t = r^*_t(i)$ the range node $t'$ has to be assigned to reach $s$ and all the $k$-closest neighbors of $s$.

To simplify the notation, for any node $t'$, let $e_t = (t(i), s(i))$ be the edge which dominates the power assignment of $t'$, where $i(t) \neq s(i)$ are some nodes in $T$. Note that it is possible that $t(i) = t(j)$ for $i \neq j$. The root node might not have a dominating edge since its initial power is greater than 0. However, we will assume it has one and later show that it does not influence our analysis at all.

**Lemma 3.4.** $C_{A^h_k} \in O(k) \left( \sum_{t \in T} (r^*_t)^2 + C_{A_h} \right)$.

**Proof.** From Lemma 3.1 we have the following inequality: $r^*(t(i), s(i)) \leq r^*(t(i)) + d(t(i), s(i)) + r^*(i)$. From Observation 3.3, $p(t(i)) \leq 3 \left( (r^*(t(i))^2 + d(t(i), s(i))^2 + (r^*(i))^2) \right)$.

Let $p'(t)$ be the power node $t$ is assigned in $A_{BHT}$. Then $p'(t(i)) \leq d(t(i), s(i))^2$. We can write,

$$C_{A^h_k} = \sum_{t \in T} p(t) \leq 3 \sum_{t \in T} \left( (r^*(t))^2 + p'(t(i)) + (r^*(i))^2 \right).$$

For any node $t \in T$, only for $t' \in N_t$ we have $i(t) = t$. For any node $s \in T$, let $c_t = (t(s), s)$ be an incoming edge of $s$ in BHT. Then only for $t' \in N_t$, we have $s(i) = s$. As a result, $\sum_{t \in T} (r^*(t))^2 \leq k \sum_{t \in T} (r^*_t)^2$, $\sum_{t \in T} (r^*(t))^2 \leq k \sum_{t \in T} (r^*_t)^2$, and $\sum_{t \in T} p'(t(i)) \leq k \sum_{t \in T} p'(t) = k C_{A_{BHT}} \leq 2k C_{A_h}$. Therefore, $C_{A^h_k} \in O(k) \left( \sum_{t \in T} (r^*_t)^2 + C_{A_h} \right)$. Note that if the root has a dominating edge it does not affect the analysis.

In [25] the authors show a PTAS for a power assignment construction algorithm to obtain $A_h$. Therefore it holds $C_{A_h} \leq (1 + \epsilon) C_{A^*_1} \in O(C_{A^*_1})$, where $A^*_1$ and $A^*_h$ are the optimal power assignments that induce a $1-h$-broadcast and $k-h$-broadcast communication graphs respectively. We are ready to prove the main Theorem.

**Theorem 3.5.** $C_{A^h_k} \in O(k^3) C_{A^*_h}$.

**Proof.** Let every node $t \in T$ be assigned a transmission range $r^*_t$. Call this power assignment $A_1$. The induced communication graphs holds the following property, every node has at least $k$ neighbors. We also claim that $A_1$ is an optimal power assignment that induces a communication graph with such a property.

In section 2 we showed an approximation algorithm for a power assignment $A^*_h$, which induces a $k-(n-1)$-broadcast communication graph (there is no constant bound on the path length), with a cost $C_{A^*_h-1} \in O(k^2) C_{A^*_h-1}$, where $A^*_h-1$ is the optimal power assignment for the $k$-fault resistant unbounded broadcast.

The induced communication graph $H_{A^*_h-1}$ maintains the property that every node has at least $k$ neighbors (we use the power assignment for strong-connectivity, which has the same cost as the power assignment for broadcast). Therefore we can conclude that $C_{A_1} \leq C_{A^*_h-1}$. Easy to see that $C_{A^*_h-1} \leq C_{A^*_h}$, since forcing a constant maximal number of hops increases the power assignment. As a result, $\sum_{t \in T} (r^*_t)^2 = C_{A_1} \in O(k^2) C_{A^*_h}$. In conjunction with Lemma 3.4,

$$C_{A^h_k} \in O(k) \left( \sum_{t \in T} (r^*_t)^2 + C_{A_h} \right) = O(k^3) C_{A^*_h}.$$  

Which ends our proof.

A simple $O(h)$ approximation for very high fault resistance - Instead of forming $k$-node disjoint paths from $r$ to any other node, we could simply assign the root with enough power to reach all nodes in a single hop. Clearly such a power assignment is very fault resistant since the transmission between $r$ and any other node does not rely on relay nodes.

**Lemma 3.6.** Let $t$ be the most distant node from $r$. Let $A_r$ be a power assignment where the root is assigned $p(r) = d(r, t)^2$. Then $C_{A_r} \in O(h) C_{A^*_h}$.

**Proof.** There is a path $y = (r = u_1, u_2, \ldots, u_{i+1} = t)$ of at most $l \leq h$ hops from $r$ to $t$ in $H_{A_h}$. Let $r_u$ be the range assigned to node $u$ in $A_h$. Clearly $p(r) \leq \left( \sum_{i=1}^l r_u^2 \right)^2$ and $\sum_{i=1}^l (r_u)^2 \leq C_{A_h}$. According to Observation 3.3 we have $p(r) \leq \left( \sum_{i=1}^l r_u \right)^2 \leq l \sum_{i=1}^l (r_u)^2 \leq h C_{A_h}$. We conclude $C_{A_r} \in O(h) C_{A_h}$, and as a result $C_{A_r} \in O(h) C_{A^*_h}$.

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