A sublinear bound on the chromatic index of multigraphs

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Abstract

The integer round-up $\phi(G)$ of the fractional chromatic index yields the standard lower bound for the chromatic index of a multigraph $G$. We show that if $G$ has even order $n$, then the chromatic index exceeds $\phi(G)$ by at most $\max\{\log_{1/2} n, 1 + n/30\}$. More generally, we show that for any real $b, 2/3 < b < 1$, the chromatic index of $G$ exceeds $\phi(G)$ by at most $\max\{\log_{1/b} n, 1 + n(1 - b)/10\}$. This is used to show that for $n$ sufficiently large, $\chi'(G) < \phi(G) + 1 + \sqrt{n}$ in $n/10$. Thus the difference between the chromatic index and its lower bound $\phi(G)$ is eventually sublinear; that is, for any real $c > 0$, there exists a positive integer $N$ such that $\chi(G) - \phi(G) < cn$ for any multigraph $G$ with order $n > N$. © 1999 Elsevier Science B.V. All rights reserved

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1. Introduction

The order and size of a multigraph $G$ with vertex set $V(G)$ and edge set $E(G)$ are the cardinalities of $V(G)$ and $E(G)$, respectively. Otherwise, we follow the terminology and notation of [4].

Recall that the chromatic index $\chi'(G)$ of the multigraph $G$ is the minimum number of matchings that are required to cover the edges of $G$. Clearly $\chi'(G)$ is at least as great as the maximum degree $\Delta(G)$. Moreover, note that if $H$ is any nontrivial multigraph with odd order, then $\chi'(H) \leq 2|E(H)|/(|V(H)| - 1)$, because any matching in $H$ can contain at most $(|V(H)| - 1)/2$ edges; we denote this lower bound on $\chi'(H)$ by $t(H)$. Now for any subset $S$ of the vertices of $G$ we let $\langle S \rangle$ denote the subgraph of $G$ induced by the vertices in $S$. Then because $\chi'(G) \leq \chi'(H)$ for any subgraph $H$ of $G$, we have a new lower bound for $\chi'(G)$ given by $\Gamma(G) = \max\{t(\langle S \rangle)\}$, where the maximum is taken over all subsets $S$ of $V(G)$ for which $|S|$ is odd and at least 3.

Combining the lower bounds from the previous paragraph, we get an improved lower bound $\phi(G) = \max\{\Delta(G), \Gamma(G)\}$ for the chromatic index of $G$, where $[\Gamma(G)]$

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denotes the integer round-up of \( \Gamma(G) \). Goldberg [2,3] and Scymour [10] independently conjectured that this lower bound is quite tight, in the following sense (Goldberg’s conjecture was a bit stronger than the one stated here).

**Conjecture A.** For any multigraph \( G \), \( \chi'(G) \leq \max\{\Delta(G) + 1, \lceil \Gamma(G) \rceil\} \).

We often find it convenient to work with the following slightly weaker form of Conjecture A; this form of the conjecture also appeared in [10].

**Conjecture B.** For any multigraph \( G \), \( \chi'(G) \leq 1 + \max\{\Delta(G), \lceil \Gamma(G) \rceil\} \).

Although Conjecture B is somewhat weaker than Conjecture A, it still reduces the possibilities for \( \chi'(G) \) to two: either \( \chi'(G) = \phi(G) \) or \( \chi'(G) = \phi(G) + 1 \). Conjecture B would thus be an important result that achieves to some extent what Vizing’s theorem (see below) achieves for simple graphs.

The following three theorems give upper bounds for the chromatic index in terms of different invariants. As \( \Delta(G) \leq \phi(G) \) by definition, we restate the theorems in terms of the lower bound \( \phi(G) \) in corollaries to Theorems A and B.

**Theorem A** (Vizing [12]). If \( G \) is a simple graph, \( \chi'(G) \leq \Delta(G) + 1 \). More generally, for any multigraph \( G \) with maximum edge multiplicity \( m \), \( \chi'(G) \leq \Delta(G) + m \).

**Corollary.** For any multigraph \( G \) with maximum edge multiplicity \( m \), \( \chi'(G) \leq \phi(G) + m \).

The next result is the most recent in a string of similar earlier results by Shannon [11], Andersen [1] and Goldberg [2,3]

**Theorem B** (Nishizeki and Kashiwagi [7]). Let \( G \) be a multigraph. If \( \chi'(G) > (11\Delta(G) + 8)/10 \), then \( \chi'(G) = \phi(G) = \lceil \Gamma(G) \rceil \).

**Corollary.** For any multigraph \( G \), \( \chi'(G) \leq \phi(G) + (\Delta(G) + 8)/10 \).

**Theorem C** (Plantholt [9]). For any multigraph \( G \) of order \( n \), \( \chi'(G) \leq \phi(G) + |n/8| - 1 \).

Note that in the two corollaries and Theorem C above, we have a bound for the amount by which the chromatic index can surpass the lower bound \( \phi(G) \) in terms of three different invariants: the maximum edge multiplicity, the maximum degree, and the order of the multigraph. However, the amount by which the upper bound exceeds \( \phi(G) \) is in each case linear in the chosen invariant, even though Conjecture B states that this difference should be bounded by a constant (indeed, the constant 1). We obtain a new order-based upper bound on the chromatic index of a multigraph \( G \) (Theorem 1 below) and extend that result to a more general one (Theorem 2 below).
Theorem 1. For any multigraph $G$ with even order $n$, $\chi'(G) \leq \phi(G) + \max\{\log_{3/2} n, 1 + n/30\}$.

Theorem 2. For any multigraph $G$ with even order $n$ and any real number $b, 2/3 \leq b < 1$, $\chi'(G) \leq \phi(G) + \max\{\log_{1/b} n, 1 + n(1 - b)/10\}$.

Note that in Theorem 2, by choosing values of $b$ close to 1, we can make the coefficient of $n$ in the linear term in the bound arbitrarily small. By appropriate choice of $b$ for $n$ sufficiently large, we are then able to achieve Theorem 3 below, from which Theorem 4, which states that the difference between $\chi'(G)$ and $\phi(G)$ can be guaranteed to be eventually sublinear, follows immediately.

Theorem 3. For any multigraph $G$ with even order $n \geq 572$, $\chi'(G) \leq \phi(G) + 1 + \sqrt{n \ln n/10}$.

Theorem 4. For any real $c > 0$, there exists a positive integer $N$ such that $\chi'(G) < \phi(G) + cn$ for any multigraph $G$ with order $n > N$.

We note that Theorem 4 corresponds closely to the following recent result of Jeff Kahn [5].

Theorem (Kahn [5]). For any real $c > 0$ there exists a positive integer $D$ so that $\chi'(G) < \max\{\Delta(G), \Gamma(G)\} + c \max\{\Delta(G), \Gamma(G)\}$ for any multigraph $G$ with $\max\{\Delta(G), \Gamma(G)\} > D$.

There are some key differences between our results and those of Kahn. Our result describes the spread between the lower and upper bounds in terms of the order $n$ rather than in terms of the fractional chromatic index $\max\{\Delta(G), \Gamma(G)\}$, and in this sense Kahn's result seems to be the more natural one. However, our Theorems 1–3 give actual upper bounds on the chromatic index that can be applied to specific multigraphs. Finally, the methods of proof are quite different; Kahn uses probabilistic arguments, while the proofs in this paper are purely combinatorial.

2. Definitions and background

Let $G$ be a multigraph and let $S$ be a subset of $V(G)$. We let $\overline{S}$ denote $V(G) - S$, and let $\langle S \rangle$ denote the subgraph of $G$ that is induced by the vertices in $S$. We let $\delta(S)$ denote the coboundary of $S$, that is, the set of all edges that are incident with exactly one vertex of $S$. Recall that $\Delta(\langle S \rangle)$ denotes the maximum degree of $\langle S \rangle$. If we wish to stress that $\langle S \rangle$ is being considered as an induced subgraph of $G$, we use the notation $\Delta(\langle S \rangle; G)$ in order to clarify the host multigraph. We use the same convention for other invariants. For example, if $F$ is a 1-factor of $G$, then $\delta(S; G - F)$ gives the coboundary of the set of vertices $S$ within $G - F$. 

If $H$ is any multigraph with odd order at least three, we let $t(H) = 2|E(H)|/(|V(H)| - 1)$. An induced subgraph $\langle S \rangle$ of $G$ is said to be overfull if $t(\langle S \rangle) > \Delta(G)$, and full if $t(H) = \Delta(G)$. Recall that $\Gamma(G) = \max\{t(\langle S \rangle)\}$, where the maximum is taken over all nontrivial odd order subsets $S$ of $V(G)$. The fractional chromatic index of $G$ is given by $f(G) = \max\{\Delta(G), \Gamma(G)\}$, and recall that the lower bound $\phi(G)$ for the chromatic index is given by $\phi(G) = \max\{\Delta(G), \lceil \Gamma(G) \rceil\}$. Finally, following terminology introduced in [10], we define an r-graph to be an r-regular multigraph $G$ for which $\Gamma(G) \leq r$ (in fact, if $G$ is an r-graph, we must have $\Gamma(G) = r$ because for any vertex $v$, $t(G - v) = r$).

We will require the following substantial number of background lemmas. Most appear in [8,9], with many appearing first in [10] or [6]. By a 'nontrivial' multigraph we mean one with more than one vertex.

**Lemma A.** Let $S$ be a nontrivial odd order subgraph of $G$. If $\langle S \rangle$ is overfull then $|\delta(S)| < \Delta(G)$; if $\langle S \rangle$ is full then $|\delta(S)| = \Delta(G)$.

**Lemma B.** Let $G$ be r-regular. Then $G$ is an r-graph if and only if $|\delta(S)| \geq r$ for each odd order subset $S$ of $V(G)$.

**Lemma C** (Seymour [10]). Let $G$ be any multigraph, and let $\phi(G) = r$. There exists a multigraph $G^+$ containing $G$ such that
(i) $G^+$ is an r-graph, and
(ii) $|V(G^+)| = |V(G)|$ if $|V(G)|$ is even, $|V(G^+)| = |V(G)| + 1$ if $|V(G)|$ is odd.

**Lemma D** (Seymour [10]). Any r-graph has a 1-factor.

**Theorem D** (Tutte’s 1-factor Theorem). Let $G$ be a multigraph. There is a 1-factor of $G$ if and only if for each set $K$ of vertices of $G$, the number of odd-order components of $G - K$ does not exceed $|K|$.

Let $G$ be a multigraph, and let $S$ be a non-empty proper subset of $V(G)$. The multigraph $G_S$ with vertex and edge sets described below is called the multigraph obtained from $G$ by shrinking $S$: 1. The vertex set of $G_S$ is given by $V(G_S) = V(G) - S \cup \{s\}$, where $s$ is the vertex replacing $S$. 2. Each vertex $u \neq s$ in $V(G_S)$ has exactly $|\delta(u, S)|$ edges in $G_S$ joining it to $s$, where $\delta(u, S)$ denotes the set of edges in $G$ that join $u$ with a vertex of $S$. These edges are the only edges of $G_S$ that are incident with $s$, and so these comprise $\delta(\{s\})$ completely in $G_S$. 3. The edge set of $G_S$ is given by $E(G_S) = E(G - S) \cup \delta(\{s\})$.

**Lemma E.** Let $G$ be a multigraph, and let $S$ be a non-empty proper subset of $V(G)$. Then $\chi'(G) \leq \max\{\chi'(G_S), \chi'(G_S)\}$.
We now introduce some additional terminology. If $H$ is a nontrivial odd-order subgraph of a multigraph $G$, the excess of $H$ (denoted $\text{ex}(H)$) is given by $|E(H)| - \Delta(G)(|V(H)| - 1)/2$; thus the excess of $H$ gives the number of edges by which $H$ is overfull. Note that $\text{ex}(H)$ may be negative, and in fact $\text{ex}(H)$ is 0 (positive) if and only if $H$ is full (overfull). The slack of $H$ (denoted $\text{sl}(H)$) is given by $[(\Delta(G) + 1)(|V(H)| - 1))/2 - |E(H)|$. Note that $\text{sl}(H)$ gives the number of additional edges that $H$ would need in order to achieve $t(H) = \Delta(G) + 1$.

The proof of the main result will follow from a sequence of 1-factor removals and shrinkings of odd-order subgraphs, combined with a careful count of the effect that these operations have on the excess of odd-order subgraphs. The following two simple observations will be of use throughout the proof.

**Observation 1.** Let $G$ be a multigraph of even order $n$, and let $S$ be a nontrivial odd-cardinality subset of vertices of $G$. If $F$ is any 1-factor of $G$, then $\text{ex}((S); G - F) \leq \text{ex}((S); G) + \min\{(n - |S| - 1)/2, (|S| - 1)/2\}.$

**Proof.** Since $\Delta(G) = \Delta(G - F) + 1$, it follows that

$$\text{ex}((S); G - F) = \text{ex}((S); G) + (|S| - 1)/2 - k,$$

where $k$ gives the number of edges of $F$ which are incident with two vertices of $S$. But $F$ has $n/2$ edges, and at most $n - |S|$ of them do not have both incident vertices in $S$. Thus $k \geq |S| - n/2$, and the result follows. $\square$

**Observation 2.** Let $G$ be a multigraph, and suppose that $S$ is a nontrivial full or overfull odd-cardinality subset of $V(G)$ such that $\Delta(G_S) = \Delta(G) = \Delta$. Let $R^*$ be a nontrivial odd-cardinality set of vertices in $G_S$ and let $s$ be the vertex of $G_S$ that replaces $S$.

If $s \not\in R^*$, then $\text{ex}((R^*); G_S) = \text{ex}((R^*); G)$, and

if $s \in R^*$, then $\text{ex}((R^*); G_S) = \text{ex}((R); G) - \text{ex}((S); G)$, where $R = R^* - \{s\} \cup S$.

**Proof.** Because $\Delta(G_S) = \Delta(G)$, to find $\text{ex}((R^*); G_S)$ we need only check to see how the numbers of edges and vertices in $\langle R^* \rangle$ are changed by the shrinking procedure. If $s \not\in R^*$, the result is more than clear, so assume $s \in R^*$. Let $R$ be the set of vertices $R^* - \{s\} \cup S$ in $G$. Then $|R| = |R^*| + |S| - 1$, while $|E((R^*); G_S)| = |E((R); G)| - |E((S); G)|$.

Thus

$$\text{ex}((R^*); G_S) = |E((R^*); G_S)| - \Delta(|R^*| - 1)/2 = |E((R); G)| - |E((S); G)| - \Delta(|R^*| - |S| + |S| - 1)/2 = [|E((R); G)| - \Delta(|R^*| + |S| - 2)/2] - |E((S); G)| + \Delta(|S| - 1)/2$$

$$= \text{ex}((R); G) - \text{ex}((S); G),$$

as desired. $\square$
In the following lemma, note that because $G$ is $r$-regular, $\Gamma(G) \geq r$. It is possible, though, that $|S| = |V(G)| - 1$, in which case we would get $G_S$ to be isomorphic to $G$.

**Lemma F** (Plantholt [8]). Let $G$ be regular and $\langle S \rangle$ be an induced full or overfull subgraph which has maximum excess in $G$. Then $\Delta(G_S) \leq \Delta(G)$ and $\Gamma(G_S) \leq \Gamma(G)$, and similarly $\Delta(G_S) \leq \Delta(G)$ and $\Gamma(G_S) \leq \Gamma(G)$.

**Lemma 1.** Let $G$ be a multigraph of even order, with $\Delta(G) \leq \Gamma(G) \leq \Delta(G) + 1$. Let $\langle S \rangle$ be full or overfull in $G$, and among all such induced subgraphs assume that $sl(S)$ is a minimum. Then

$$\Delta(G_S) \leq \Delta(G), \quad \Gamma(G_S) \leq \Delta(G) + 1,$$

$$\Delta(G_S^+) \leq \Delta(G), \quad \Gamma(G_S^+) \leq \Delta(G) + 1.$$

**Proof.** Let $M$ be a matching, its vertices in $S$, such that $|E(M)| = sl(S)$ (so $t(\langle S \rangle \cup M) = \Delta(G) + 1$). Since adding a matching can increase $t(H)$ for any subset $H$ by at most 1, and since $S$ has minimum slack among all full or overfull subgraphs of $G$, $\Gamma(G \cup M) = \Delta(G) + 1$. By Lemma C, we can add edges to expand $G \cup M$ to a multigraph $G^+$ which is a $(\Delta(G) + 1)$-graph. Then $\langle S \rangle$ has maximum excess 0 in $G^+$, as does $\langle S \rangle$. Thus by Lemma F, $\Gamma(G_S^+) \leq \Delta(G) + 1$ and $\Gamma(G_S^+) \leq \Delta(G) + 1$, so $\Gamma(G_S) \leq \Delta(G) + 1$ and $\Gamma(G_S^+) \leq \Delta(G) + 1$. Finally, as $\langle S \rangle$ is full or overfull in $G$, by Lemma A we have $|\delta(S; G)| = |\delta(S; G)| \leq \Delta(G)$, so that $\Delta(G_S) \leq \Delta(G)$ and $\Delta(G_S^+) \leq \Delta(G)$, as desired.  

3. Main Theorem

Our main theorem is stated only for multigraphs of even order. Of course, we can always add an isolated vertex to a multigraph that has odd order, and then apply the theorem to achieve a similar result.

**Theorem 1.** For any multigraph $G$ with even order $n$,

$$\chi'(G) \leq \phi(G) + \max\{\log_{3/2} n, \ 1 + n/30\}.$$

**Proof.** We proceed by induction on $n$ and $\phi(G)$. In [8] it was shown that $\chi'(G) = \phi(G)$ whenever $G$ has order $n \leq 8$, so we assume that $n \geq 10$. Assume the result is true for all multigraphs $H$ with even order less than $n$, and all multigraphs $H$ which have order $n$ but for which $\phi(H) < \phi(G)$.

By Lemma C, we may assume that $G$ is an $r$-graph, with $\Delta(G) = \phi(G) = r$. By Lemma D, $G$ has a 1-factor $F$. Clearly $G - F$ is $(r - 1)$-regular; if $\phi(G - F) \leq r - 1$, we are done by the induction hypothesis. Thus, we assume that $\Gamma(G - F) > r - 1$, that is, $G - F$ contains an overfull subgraph.
In the argument to follow, we construct a set $Z$ of $k$ or $k+1$ smaller multigraphs $G_1, \ldots, G_k$ and possibly $G_{k+1}$ such that

$$\chi'(G) = \max\{r_i + \chi'(G_i), \ldots, r_k + \chi'(G_k), \text{ if } G_{k+1} \in Z\},$$

where $r_i = \Delta(G) - \Delta(G_i)$ for each $i$, $1 \leq i \leq k+1$. We will have $r_i + \phi(G_i) \leq 1 + \phi(G)$ for each $i$, $1 \leq i \leq k+1$, so that we have ‘wasted’ only one color relative to the lower bound $\phi$ in going from $G$ to this set of smaller multigraphs. Moreover, for each $i \leq k$, $G_i$ has order at most $2n/3$, so that if $Z$ contains only $G_1$ through $G_k$, the result follows easily by induction. For, consider some $G_i$, $i \leq k$. By the induction hypothesis,

$$r_i + \chi'(G_i) \leq r_i + \phi(G_i) + \max\{\log_2 2n/3, 1 + 2/3(n/30)\}$$

(as is easily checked; in this line and the line above, the maximums of the two-element sets are attained by the log terms until $n$ is large (well over 400), so when either max value is attained by the linear term, clearly $1 + 2/3(n/30) < n/30$ for these large values of $n$.)

If $G_{k+1}$ does appear in set $Z$, it will have maximum degree at most $n/3$ and $\phi(G_{k+1}) \leq \Delta(G_{k+1}) + 1$. Thus applying the corollary of Theorem B (the Nishizeki–Kashiwagi Theorem), we get $\chi'(G_{k+1}) \leq \Delta(G_{k+1}) + 1 + n/30$, so that $r_{k+1} + \chi'(G_{k+1}) \leq \phi(G) + 1 + n/30$, and the result follows.

Let us return then to the multigraph $G - F$, and let $\langle S \rangle$ be an induced overfull odd order subgraph which has maximum excess among all such subgraphs; as $G - F$ is regular, $\text{ex}(\langle S \rangle, G - F) = \text{ex}(\langle S \rangle)$, so we may assume that $|S| \leq |\bar{S}|$. By Lemma F, $(G - F)_{\bar{S}}$ and $(G - F)_{S}$ both have maximum degree $r - 1$, and we have both $I((G - F)_{\bar{S}}) \leq r$ and $I((G - F)_{S}) \leq r$. As $|\bar{S}| \geq |S|$, we have $|V((G - F)_{\bar{S}})| \leq 1 + (n/2) < 2n/3$; we place $(G - F)_{\bar{S}}$ in $Z$. If by chance we have also $|V((G - F)_{S})| \leq 2n/3$, we place $(G - F)_{S}$ in $Z$ to complete the construction of $Z$, and the result follows by the argument at the start. Therefore, we assume now that $|V((G - F)_{S})| > 2n/3$, so that $|S| < n/3 + 1$.

Let $G^*$ denote $(G - F)_{\bar{S}}$ and let $p = |V(G^*)|$. We note that one vertex $s$ in $G^*$ (the vertex replacing the shrunken set $S$) has degree less than $r - 1$ (by Lemma A), and all other vertices of $G^*$ have degree $r - 1$.

Claim 1. Any overfull induced odd subgraph $\langle R \rangle$ of $G^*$ has excess $\text{ex}(\langle R \rangle) \leq (n - |R| - 1)/2$.

Proof of Claim 1. If $s \notin R$, then $\text{ex}(\langle R \rangle; G^*) = \text{ex}(\langle R \rangle; G - F) \leq (n - |R| - 1)/2$, by Observation 1. If $s \in R$, let $Q$ denote the subset of $V(G)$ given by $R - \{s\} \cup S$. The multigraph $\langle Q \rangle$ is not overfull in $G$, so by Observation 1, $\text{ex}(\langle Q \rangle; G - F) \leq (n - |Q| - 1)/2 = (n - |R| - |S|)/2$. Thus by Observation 2, $\text{ex}(\langle R \rangle; G^*) = \text{ex}(\langle Q \rangle; G - F) - \text{ex}(\langle S \rangle; G - F) \leq (n - |R| - |S|)/2 \leq (n - |R| - 1)/2$. □
Let $\Delta^*$ denote the maximum degree of $G^*$. At this point, we have a multigraph $G^*$ with the following properties:

(i) $G^*$ has $p > 2n/3$ vertices,

(ii) all vertices in $G^*$ have degree $\Delta^*$, except one vertex $s$ that may have degree less than $\Delta^*$ (indeed the degree of $s$ is less than $\Delta^*$ in the current $G^*$),

(iii) $\phi(G^*) \leq \Delta^* + 1$, and

(iv) any induced overfull subgraph $\langle R \rangle$ of $G^*$ has excess at most $(n - |R| - 1)/2$.

To complete the construction of the set $Z$, we will perform a sequence of 1-factor removals and pairs of shrinkings of an induced subgraph and its induced complement. Each pair of shrinkings will result in two smaller multigraphs. One of those smaller multigraphs will have order less than $2n/3$, and be added to $Z$. The other shrunken multigraph will then become our focus; we will try to show that it has the four properties listed above, and continue this process until the set $Z$ is fully constructed.

**Claim 2.** The excess of $G^* - s$ is $(r - 1 - \deg(s))/2$.

**Proof of Claim 2.** Clearly $G^* - s$ has $[(p - 1)(r - 1) - \deg(s)]/2$ edges. Thus the excess in $G^* - s$ is given by $[(p - 1)(r - 1) - \deg(s)]/2 - (p - 2)(r - 1)/2 = [r - 1 - \deg(s)]/2$, as desired. □

Clearly if $\deg(s) = 0$, we cannot remove a 1-factor from $G^*$. We therefore consider two cases.

**Case 1.** $\deg(s) > 0$.

**Subcase A.** In $G^* - s$ there is no overfull odd order induced subgraph $\langle R \rangle$ for which $\sl(\langle R \rangle) < \sl(G^* - s)$.

**Claim 3.** In Subcase A, if $\langle R \rangle$ is overfull in $G^*$, then $|R| \geq p/2$.

**Proof of Claim 3.** We have $\sl(\langle R \rangle) \geq \sl(G^* - s)$, so $(|R| - 1)/2 - \ex(\langle R \rangle) \geq (p - 2)/2 - \ex(G^* - s)$, and so

$$\ex(G^* - s) - \ex(\langle R \rangle) \geq (p - |R| - 1)/2.$$ (**)  

Now by Claim 1, $\ex(G^* - s) < n/6$ because $p > 2n/3$. Now suppose that $|R| < p/2$. Then $(p - |R| - 1)/2 \geq (p/2)/2 > n/6$, so that from (**) above we get $n/6 - \ex(\langle R \rangle) \geq n/6$, a contradiction since $\ex(\langle R \rangle) > 0$. □

**Claim 4.** In Subcase A, any nontrivial odd subset of $V(G^*)$ containing $s$ is not overfull.

**Proof of Claim 4.** Suppose $R$ is an odd order subset of $V(G^*)$ containing $s$, and that $\langle R; G^* \rangle$ is overfull. Then

$$|\delta(R)| < \Delta^* - (\Delta^* - \deg(s)) = \deg(s),$$
so $|\delta(\overline{R})| < \deg(s)$, so

$$\text{ex}(\overline{R}; G^*) \geq (\Delta^* - \deg(s))/2 = \text{ex}(G^* - s).$$

But $\overline{R}$ is a proper subset of $G^* - s$, so then $\text{sl}(\overline{R}) < \text{sl}(G^* - s)$, a contradiction because we are in Subcase A. \(\Box\)

**Claim 5.** In Subcase A there exists a 1-factor $F^*$ of $G^*$.

**Proof of Claim 5.** Let $K$ be any subset of $V(G^*)$, and suppose the number of odd components of $G^* - K$ is $t > |K|$. All odd components of $G^* - K$, except possibly the one containing $s$ and at most one which is overfull (and therefore by Claim 3 contains at least half the vertices of $G^*$) have at least $\Delta^*$ edges in their coboundary in $G^*$. Since $\deg(s) > 0$, by Claim 4 the odd component containing $s$ would have at least 1 coboundary edge. Thus, the number of edges in the coboundary of $K$ in $G^*$ is at least $(t - 2)\Delta^* + 1 > |K|\Delta^* + 1$, since $t$ and $K$ have the same parity (because $p$ is even). This yields a contradiction, as each vertex in $K$ has degree at most $\Delta^*$. Thus, by Tutte's theorem, $G^*$ contains a 1-factor $F^*$. \(\Box\)

**Claim 6.** Let $R$ be an odd order subset of $V(G^*)$ such that $\langle R \rangle$ is overfull in $G^*$, and let $F^*$ be a 1-factor of $G^*$. If we are in Subcase A, then $\text{ex}(\langle R \rangle; G^* - F^*) \leq (n - p)/2 \leq (n - |R| - 1)/2$.

**Proof of Claim 6.** For any nontrivial odd order subset $R$ of $V(G^*)$, $\text{ex}(\langle R \rangle; G^*) \leq \text{ex}(\langle G^* - s \rangle; G^*) - (p - |R| - 1)/2$, because we are in Subcase A. But by Claim 1, $\text{ex}(\langle G^* - s \rangle; G^*) \leq (n - p)/2$, so by Observation 1, $\text{ex}(\langle R \rangle; G^* - F^*) \leq ((n - p)/2 - (p - |R| - 1)/2) + (p - |R| - 1)/2$. Thus $\text{ex}(\langle R \rangle; G^* - F^*) \leq (n - p)/2 \leq (n - |R| - 1)/2$, as desired. \(\Box\)

**Claim 7.** For any 1-factor $F^*$ of $G^*$, $\phi(G^* - F^*) \leq \Delta^*$.

**Proof of Claim 7.** Clearly $\Delta(G^* - F^*) = \Delta^* - 1$. We need to show that if $R$ is any nontrivial odd order subset of $V(G^*)$, then $t(\langle R \rangle; G^* - F^*) \leq \Delta^*$. If $t(\langle R \rangle; G^*) \leq \Delta^*$, this is certainly true. So, assume that $\langle R \rangle$ is overfull in $G^*$. By Claim 6, $\text{ex}(\langle R \rangle; G^* - F^*) \leq (n - p)/2 \leq n/6$, because $p > 2n/3$. But because $\langle R \rangle$ is overfull in $G^*$, by Claim 3 we have $|R| \geq p/2 > n/3$. Thus $\text{ex}(\langle R \rangle; G^* - F^*) \leq (|R| - 1)/2$, so $t(\langle R \rangle; G^* - F^*) \leq (\Delta^* - 1) + 1 = \Delta^*$, as desired. \(\Box\)

Thus in Subcase A, there is a 1-factor $F^*$ such that $G^* - F^*$ satisfies the four properties (i)--(iv) that were satisfied by $G^*$. We therefore replace $G^*$ by $G^* - F^*$ and repeat the procedure. Eventually we get down to either a multigraph with no edges (that can be added to $Z$ to complete its construction), or a multigraph $G^*$ where we are in either Subcase B or Case 2.
Subcase B. There exists an overfull odd order subgraph \( \langle R \rangle \) of \( G^* - s \) such that 
\[ \text{sl}(\langle R \rangle) < \text{sl}(G^* - s). \]

Let \( R \) be a subset of \( V(G^*) \) such that \( \langle R \rangle \) has minimum slack among all such subsets. By Lemma 1, \( G^*_R \) and \( G^*_s \) each have maximum degree at most \( A^* \), and \( \phi \)-values which are at most \( A^* + 1 \). At least one of these multigraphs has order at most \( 2n/3 \), and we place that one in set \( Z \). If by chance the other also has order at most \( 2n/3 \), we would place it in \( Z \) also, and the construction of \( Z \) would be complete. Therefore, let us assume that either \( |V(G^*_R)| > 2n/3 \) or \( |V(G^*_s)| > 2n/3 \). We wish to show that that multigraph has property (iv).

**Claim 8.** In both \( G^*_R \) and \( G^*_s \), any nontrivial induced odd order subgraph \( \langle Q \rangle \) has excess at most \( (n - |Q| - 1)/2 \).

**Proof of Claim 8.** First consider \( G^*_R \) and let \( s_R \) be the vertex of \( G^*_R \) that was formed by shrinking \( R \). Let \( \langle Q \rangle \) be any nontrivial odd order induced subgraph of \( G^*_R \). If \( s_R \notin Q \), then 
\[ \text{ex}(\langle Q \rangle; G^*_R) = \text{ex}(\langle Q \rangle; G^*) \leq (n - |Q| - 1)/2, \]
because condition (iv) is satisfied within \( G^* \).

If \( s_R \in Q \), then 
\[ \text{ex}(\langle Q \rangle; G^*_R) < \text{ex}(\langle Q - s_R \cup R \rangle; G^*) \]
(by Observation 2, since \( \langle R \rangle \) is overfull)
\[ \leq [n - ((|Q| - 1 + |R|) - 1)]/2 \]  
(by (iv) in \( G^* \))
\[ < (n - |Q| - 1)/2, \]
as desired.

Now consider \( G^*_s \), and let \( s_R \) denote the vertex that replaces \( \langle R \rangle \) upon shrinking. Let \( \langle Q \rangle \) be any nontrivial odd order induced subgraph of \( G^*_R \). If \( s_R \notin Q \), then as before 
\[ \text{ex}(\langle Q \rangle; G^*_R) = \text{ex}(\langle Q \rangle; G^*) \leq (n - |Q| - 1)/2. \]

So, assume that \( s_R \in Q \).

If \( s \notin \langle R \rangle \), then \( \langle R \rangle \) must be overfull in \( G^* \), because \( \text{ex}(\langle R \rangle); G^*) \geq \text{ex}(\langle Q \rangle; G^*) \) and \( \langle R \rangle \) is overfull in \( G^* \). Thus 
\[ \text{ex}(\langle Q \rangle; G^*_R) < \text{ex}(\langle Q - s_R \cup \langle R \rangle \rangle; G^*) \]
\[ \leq (n - (|Q| - 1 + |\langle R \rangle|) - 1)/2 \]
\[ < (n - |Q| - 1)/2, \]
as desired.

Finally, assume that \( s_R \in Q \) and that \( s \in \langle R \rangle \). Then \( s_R \) is the only vertex with degree less than \( A^* \) in \( G^*_R \). Suppose that \( \text{ex}(\langle Q \rangle; G^*_R) > (n - |Q| - 1)/2 \). Then 
\[ \text{ex}(\langle Q \rangle; G^*_R) \geq \text{ex}(\langle Q \rangle; G^*_R) > (n - |Q| - 1)/2 \]
(\( G^*_R \)) \( \geq (\langle Q \rangle - 1)/2 \) (the first inequality because \( s_R \in Q \)), the third because \( |Q| + |\langle R \rangle| \leq n \). Thus \( r(\langle Q \rangle; G^*_R) > A^* + 1 \), yielding a contradiction. Claim 8 now follows. \( \square \)
Let us temporarily denote by $G^{**}$ the multigraph from \{G*, G**\} that has order greater than $2n/3$. We have verified that $G^{**}$ satisfies each of conditions (i)–(iv), except that $G^{**}$ could have two vertices with degree less than $A*$ if $s$ is not a vertex in the subgraph that was shrunk to obtain $G^{**}$ from $G^*$. If so, let these two vertices be $s$ and $w$.

Start adding edges of the form $sw$ to $G^{**}$ until either $s$ or $w$ has degree $A*$, or until there is an odd order subgraph $(W)$ containing both $s$ and $w$ for which $t(\langle W \rangle) = A^* + 1$. In the latter case, since $\text{ex}(\langle W \rangle) > \text{ex}(\langle W^* \rangle)$ but $t(\langle W \rangle) \leq A^* + 1$, we must have $|W| > |V(G^{**})|/2$. Letting $G^{**}$ now include the added edges, we then use the shrinking operation to form multigraphs $G^{**}_W$ and $G^{**}_W$; of these two multigraphs, only $G^{**}_W$ could have order greater than $2n/3$. We therefore place $G^{**}_W$ in set $Z$. If by chance $|V(G^{**})| \leq 2n/3$, we place that multigraph in $Z$ and the construction of $Z$ is complete.

Therefore, assume that $|V(G^{**})| > 2n/3$. Note that $G^{**}_W$ has each of the key properties (i)–(iii); we wish to show it also has property (iv). To do so, we show that as we were adding edges, we kept the property that any induced overfull odd subgraph $(Q)$ has $\text{ex}(\langle Q \rangle) < (n - |Q| - 1)/2$. If not, then since $G^{**}$ originally satisfied property (iv), both $s$ and $w$ must be in $Q$, and so $\text{ex}(\langle Q \rangle) \geq \text{ex}(\langle Q \rangle)$. But if $\text{ex}(\langle Q \rangle) > (n - |Q| - 1)/2$, we must have $\text{ex}(\langle Q \rangle) > (n - |Q| - 1)/2 > (|Q| - 1)/2$, so that $t(\langle Q \rangle) > A^* + 1$, a contradiction. Thus, property (iv) is retained.

We conclude that $G^{**}_W$ has each of the key properties (i)–(iv). We rename $G^{**}_W$ as $G^*$, and iterate the argument on this updated $G^*$.

**Case 2:** $\deg(s) = 0$.

Let $A^*$ denote the maximum degree of $G^*$. In this case, our multigraph $G^*$ has the following properties:

(i) $G^*$ has $p > 2n/3$ vertices,

(ii) all vertices in $G^*$ have degree $A^*$, except one vertex $s$ has degree 0.

(iii) $\phi(G^*) = A^* + 1$, and

(iv) Any induced overfull subgraph $(R)$ of $G^*$ has excess at most $(n - |R| - 1)/2$.

Now because $G^* - s$ is $A^*$-regular, by simple counting we see it has excess $A^*/2$. But by property (iv), $\text{ex}(G^* - s) \leq (n - (p - 1) - 1)/2$. Combining these two pieces of information gives $A^* < n - p$; but as $p > 2n/3$, we thus have $A^* < n/3$.

By the Nishizeki–Kashiwagi theorem, we have $\chi'(G^*) \leq \max\{\phi(G^*), (11A^* + 8)/10\}$. But $\phi(G^*) \leq A^* + 1$ by property (iii). Also, because $A^* < n/3$, we have that $(11A^* + 8)/10$ exceeds $A^*$ by at most $n/30 + 1$. Thus, $\chi'(G^*) \leq A^* + n/30 + 1$. Thus, applying the argument on the multigraphs in $Z$ that was given at the very beginning of the proof, we get the desired result, and the proof of Theorem 1 is complete.

**4. Generalization, variations and possible improvements**

An examination of the proof of Theorem 1 shows that the proof can be easily modified to yield a more general result. The requirement that in set $Z$ each of the
multigraphs \( G_1, \ldots, G_k \) has order at most \( 2n/3 \) can be replaced by the requirement that each of \( G_1, \ldots, G_k \) in \( Z \) has order at most \( bn \), where \( b \geq 2/3 \), without changing any other arguments. In this change, the multigraph \( G_{k+1} \), if it appears in the construction of \( Z \), will have maximum degree at most \((1 - b)n\). Thus we obtain the following more general result.

**Theorem 2.** For any multigraph \( G \) with even order \( n \), and any real \( b \), where \( 2/3 \leq b < 1 \), \( \chi'(G) \leq \phi(G) + \max\{\log_{1/b} n, 1 + n(1 - b)/10\} \).

Naturally when using the bound given in Theorem 2, we would seek a choice of \( b \) that would minimize the maximum of the two terms in the bound. For \( n \) sufficiently large, such a choice yields the following explicit upper bound.

**Theorem 3.** For any multigraph \( G \) with even order \( n \geq 572 \),

\[
\chi'(G) \leq \phi(G) + 1 + \sqrt{n \ln n/10}.
\]

**Proof.** We choose \( b = 1 - \sqrt{10 \ln n/n} \); it is easy to check that \( b \geq 2/3 \) when \( n \geq 572 \). With this choice of \( b \), we have for the first term in the upper bound \( \log_{1/b} n = \ln n/(-\ln b) = \ln n/(-\ln(1 - \sqrt{10 \ln n/n})) \leq \ln n/\sqrt{10 \ln n/n} = \sqrt{n \ln n/10} \). On the other hand, the second term in the upper bound yields \( 1 + n(1 - b)/10 = 1 + n(\sqrt{10 \ln n/n})/10 = 1 + \sqrt{n \ln n/10} \). The result now follows from Theorem 2. \( \square \)

Because the term in the upper bound in Theorem 3 is sublinear, we can immediately conclude the following.

**Theorem 4.** For any real \( c > 0 \), there exists a positive integer \( N \) such that \( \chi'(G) - \phi(G) < cn \) for any multigraph \( G \) with order \( n > N \).

It is natural to ask if the proof of Theorem 1 yields an algorithm that can be implemented in polynomial time. Each time we shrink a multigraph \( G \) to get the pair of lesser order multigraphs \( G_S, G_{\overline{S}} \), the sum of the orders of \( G_S, G_{\overline{S}} \) exceeds that of \( G \) by exactly \( 2n \). It follows that the sum of the orders of the multigraphs in \( Z \) is at most \( 2n \). However, it is unclear whether the process of finding the induced subgraph of a multigraph which has minimum slack can be carried out in polynomial time. In order to overcome this problem, it is natural to shrink instead the induced multigraph \( \langle S \rangle \) for which the odd set quotient \( t(\langle S \rangle) \) is a maximum. This can be implemented in polynomial time, and the conditions of min slack versus max \( t(\langle S \rangle) \) value have similar properties (for a fixed order subgraph, the lower its slack, the higher its odd set quotient, and so in many cases the two rules would select the same set of vertices to shrink). Indeed, in Marcotte [6], Theorem 1.2 would provide a result corresponding to our Claim 5, and her Lemma 1.7 would provide a result corresponding to our Lemma F and Lemma 1. However, there would be added complications in the induction
process, because a parallel to our Claim 7 would not hold; deleting a 1-factor could fail to yield a corresponding decrease in $\phi(G^* - F)$.

Finally, it is clear that there is room for improvement of our Theorem 1. In particular, improving the treatment of $G_{k+1}$ holds the promise of removing the linear term from the upper bound in the theorem statement, so that the chromatic index cannot exceed $\phi(G)$ by more than $\log n$, for some appropriate base.

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References