An Isomorphic Fourier Transform Approach to Light Propagation in AWGs, FBGs, and Photonic Crystals

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Abstract
We have identified a comprehensive isomorphism between arrayed-waveguide grating (AWG), fibre Bragg grating (FBG) and photonic crystal (PC) devices, which allows the design characteristics of the former transverse geometry to be mapped on to the latter longitudinal structures. In this paper, we present a unified Fourier transform (FT) approach to the study of these important passive optical grating-based devices.

1. Introduction
Devices such as the arrayed-waveguide grating (AWG) [1] or the fibre Bragg grating (FBG) [2] are assuming increasing importance in the areas of fibre point-to-point communication and networking [3]. In the particular context of dense wavelength-division multiplexing (DWDM), these devices play a well-established role as wavelength-selective elements [4]. Recently, photonic bandgap effects and their embodiment in a new class of optical device, photonic crystals (PCs), have also received considerable attention [5], because they also feature wavelength selectivity, but may be realized in a more compact form on photonic integrated circuits (PICs) [6]. It is also well known that the AWG succumbs to a Fourier transform (FT) treatment, in the limit of the paraxial approximation combined with the far-field Fraunhofer diffraction [7]. It is also well known, in the case of a weakly-coupled FBG, that the first Born approximation offers a direct FT relationship between the reflection spectrum and the spatial distribution of scatterers for a weakly scattering medium [8]. However, this is in contrast to the strong scattering effects (i.e. high reflectivity) present in photonic crystals, and also the equivalent strongly-coupled FBG case. However, we have identified a comprehensive isomorphism between AWG and FBG/PC devices, which allows the design characteristics of the former transverse geometry to be mapped on to the latter longitudinal structure [9]. We should not be surprised that light propagation through a grating is ultimately independent of the orientation of the grating with respect to the light, since one of the paradoxes of the quantum mechanical approach is the assumption that a photon can be considered to be both localised (i.e. particle-like) and also distributed in space (i.e. spherical-wave-like) and passing through all the ‘slits’ associated with a grating, without any particular directional preference. From filter theory it is also well known that the AWG is a FIR filter, in contrast to the FBG/PC, which is an IIR device [10]. Our analysis appears to indicate that a passive IIR device can be related to a passive FIR device by making the FIR transfer function the argument of an appropriate hyperbolic function.

2. Coupled Mode Analysis
Out of the many methods for the mathematical one-dimensional (1D) modeling of these passive optical structures coupled-mode theory (CMT) offers a physically intuitive approach. The following analysis of the transfer function for light propagation is conducted for the longitudinal grating FBG case; but is also appropriate for the transverse grating AWG, in the limiting case where the coupling strength tends towards zero, and there is no feedback. We use the grating parameters κ and φ as defined by Kogelnik [8], but the coupled-mode formalism of Yariv [11]:

$$\frac{\partial A_f}{\partial z} = -j\kappa(z)A_f e^{j\phi} \quad (1a)$$
$$\frac{\partial A_b}{\partial z} = j\kappa^*(z)A_f e^{j\phi} \quad (1b)$$

where $A_f$ and $A_b$ are respectively the forward and backward propagating electric-field amplitudes of the light. The spatially-varying coupling coefficient is given by $\kappa(z)$, where $z$ is the longitudinal space variable, as shown in figure 1. This spatial variation in coupling coefficient can be expressed by the function $\kappa(z) = \kappa_0 g(z) e^{j\phi(z)}$, where $\kappa_0$ is the maximum coupling strength along the length of the grating, $g(z)$ is the purely real, normalised spatial...
Optical Axis,

The equation thus indicates that the transfer function consists of a FT relationship within a hyperbolic array (Fourier plane) of the AWG been defined with respect to a FBG. Shown in figure 2, the spatial amplitude distribution of light in the waveguide to make the appropriate substitutions (i.e. to transform between the two isomorphs) for the quantities, which have particular reflectivity: a solution to the inverse problem. To apply equation (5) to an AWG, we need to use Table 1 take the inverse FT of equation (5) to yield an expression for the required distributed coupling coefficient with a

Hence, we can consider the Fourier plane of a FBG to lie along its length. Since equation (5) is a FT, we can also

longitudinal and transverse grating structures:

Differentiating the reflection coefficient with respect to , and appropriately substituting the coupled-mode equations (1), allows the following Riccati equation to be derived:

\[ \frac{\partial \rho}{\partial z} = -\kappa v g(z) (1 - \rho^2) + j \left( \Delta \beta + \frac{\partial \phi}{\partial z} \right) \rho. \]  

(3)

For a FBG of length centered at \( z = 0 \), Jaggard et al. [13] have shown that the Riccati equation (3) can be analytically solved to yield the reflectivity at the front face of the grating:

\[ \rho(\Delta \beta) = j \tanh \left( -\frac{1}{2} \kappa v g(z) e^{i \phi(z)} \right) \int_{0}^{\infty} e^{i \Delta \beta z} dz = -j \tanh \left( \int_{0}^{\infty} \kappa(z) e^{i \Delta \beta z} dz \right). \]  

(4)

In contrast to [13] we have also introduced another simple \( \pi/2 \) phase shift to the overall reflectivity, i.e. multiplied the RHS of (4) by \( j \), to maintain consistency with [8]. In the case of strong coupling, it has been noted that equation (4), whilst being exact at the band centre, becomes inaccurate due to a phase error accumulation [14,15], so that an overly-narrow bandgap is predicted. We discuss the appropriate modifications to CMT for the strong-coupling case (e.g. as applied to photonic crystals) in section 3. For a grating of overall length \( L \), we can assume that the function \( g(z) = 0 \) for \( |z| > L/2 \), such that we can rewrite (4) as a generalised transfer function \( H(\Delta \beta) \), applicable to both longitudinal and transverse grating structures:

\[ H(\Delta \beta) = -j \tanh \left( \int_{0}^{\infty} g(z) e^{i \phi(z)} \right) e^{i \Delta \beta z} dz \].  

(5)

Equation (5) thus indicates that the transfer function consists of a FT relationship within a hyperbolic \( \tanh \) function. Hence, we can consider the Fourier plane of a FBG to lie along its length. Since equation (5) is a FT, we can also take the inverse FT of equation (5) to yield an expression for the required distributed coupling coefficient with a particular reflectivity: a solution to the inverse problem. To apply equation (5) to an AWG, we need to use Table 1 to make the appropriate substitutions (i.e. to transform between the two isomorphs) for the quantities, which have been defined with respect to a FBG. Shown in figure 2, the spatial amplitude distribution of light in the waveguide array (Fourier plane) of the AWG \( a(\epsilon) \) is equivalent to the spatial variation in coupling strength \( g(z) \) in the FBG.

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Figure 1: Schematic of a regular FBG, with an apodised (windowed) coupling strength \( g(z) \).

Figure 2: Schematic of a regular AWG, with an apodised (windowed) electric-field amplitude \( a(\epsilon) \).
Constrained by power conservation (i.e. Parseval’s Theorem) the amplitude function \( a(\epsilon) \) may take on any shape, provided \( \int |a| \, d\epsilon = \lambda R M / W \), where \( W \) is the centre-to-centre separation of the arrayed waveguides at the free-propagation region (FPR) entrance, \( R \) is the length of the FPR, and \( M \) is the number of waveguides in the array, and \( \epsilon \) (equivalent to \( 2z \)) is the transverse spatial co-ordinate of the arrayed-waveguide section. The phase function \( \Phi(\epsilon) \) across the arrayed-waveguide section of a regular AWG, equivalent to the phase-shift \( \phi(z) \), is given by:

\[
\Phi(\epsilon) = \frac{2\pi n \Delta \epsilon}{\lambda_0 W} - \beta_0 \epsilon \cos \psi,
\]

where the first component on the RHS is the (stepped) linear phase inherent to the echelon grating structure. The propagation constant \( \beta_0 \) corresponds with the wavelength \( \lambda_0 \) at the centre of the AWG filter response. For an AWG, we also define the direction cosine \( \cos \psi = x / R \), where \( x \) is the distance of the output waveguide from the optical axis of the second FPR, as shown in figure 2. The angle \( \psi \) is equivalent to the angle \( \theta \) in a FBG. \( \Delta l \) is the incremental pathlength difference in the arrayed-waveguide section of the AWG, and is equivalent to twice the geometric period \( \Lambda \) of a FBG. We note that the length \( L \) of a FBG is isomorphic with both \( M \Delta l \) (i.e. half the ‘length’ of an AWG) and \( MW \) (the AWG ‘width’). This arises due to the 2D planar geometry of an AWG, in contrast to the essentially 1D linear structure of a FBG. However, there is also a factor 2 multiplying the \( \Lambda \)-coordinate, due to the reflective nature of the FBG, which relates only to the ‘length’ of the grating, but not to its ‘width’. We can assume that the AWG ‘coupling strength’ (equivalent to \( \kappa_0 \)) is given by \( \kappa_0 = \sqrt{\pi r / \lambda R} \), which is small, since \( R \gg r, \lambda \) where \( r \) is the spotsizes of the waveguide mode. Thus, somewhat paradoxically, the AWG, whilst being a FIR filter, can also be regarded as an asymptotically-uncoupled IIR device. Hence, equation (5) derived for the IIR case, is also applicable to the transmissivity, \( t(\Delta \beta) \), of AWG devices. \( \Omega \) is the grating order, while \( FSR \) is the grating free-spectral range (FSR). The table below shows the correspondence between the equivalent physical and geometric parameters of these transverse and longitudinal devices:

<table>
<thead>
<tr>
<th>AWG</th>
<th>( n(\Delta \beta) )</th>
<th>( a(\epsilon) )</th>
<th>( C_0 )</th>
<th>( \Phi(\epsilon) )</th>
<th>( \cos \psi \times R )</th>
<th>( \Delta l )</th>
<th>( M \Delta l )</th>
<th>( MW )</th>
<th>( \Omega = n \Delta l / \lambda_0 )</th>
<th>( FSR = \lambda_0 / 2n \Delta l )</th>
</tr>
</thead>
<tbody>
<tr>
<td>DBR</td>
<td>( \tanh^{-1}(\lambda / \Delta \beta) )</td>
<td>( 2z )</td>
<td>( \kappa_0 )</td>
<td>( \phi(z) )</td>
<td>( \cos \theta )</td>
<td>( 2\Lambda )</td>
<td>( 2L )</td>
<td>( L )</td>
<td>( \Omega = 2n \Lambda / \lambda_0 )</td>
<td>( FSR = \lambda_0 / 2n \Lambda )</td>
</tr>
</tbody>
</table>

Table 1: Isomorphic physical and geometric parameters for AWGs and FBGs.

It is straightforward to show by substituting in the AWG parameters from Table 1, and assuming that \( C_0 \) is a small quantity, such that \( \tanh C_0 \to C_0 \), the transfer function (5) can be shown to be equivalent to the appropriate application of the Fresnel-Kirchhoff diffraction integral [16]:

\[
E_{\text{far-end}}(x) = \frac{-j}{\sqrt{\lambda R}} \int_{-\infty}^{\infty} E_{\text{entrance}}(\epsilon) e^{-j(2\pi \lambda / \Delta \beta)x} \, d\epsilon
\]

which is the more conventional treatment for light transmission through an AWG [3]. Of interest is the implication that the diffraction of light in free-space can therefore also be considered to be a weak coupled-mode problem.

### 3. Photonic Crystals and Strong Coupled Devices

The CMT can also be employed to describe light propagation in a photonic crystal, or strong-coupled longitudinal grating. However, detuning from the Bragg resonance is reduced in the case of strong coupling, leading directly to the observed broader bandwidth of the main lobe. Thus the detuning parameter \( \Delta \beta \) needs to be appropriately modified, in order to take strong coupling into account, and avoid the phase accumulation error [14,15]. For a weak to moderately strong-coupled device, the geometric distance \( z \), and optical pathlength \( l \) have an almost linear relationship, so that we can assume \( l = \pi z \); but for a grating with a high refractive index contrast (RIC), \( z \) and \( l \) are strongly divergent. The divergence means that the relationship between geometric (or optical) space and reciprocal space, which is conventionally related via \( \pi \), also requires modification. Instead, we need to define vectors in optical space and optical reciprocal space as the appropriate Fourier conjugate variables [17]. This means that the optical spatial frequency \( k_\phi \) at the Bragg condition is given by \( k_\phi = 2\pi / \pi \Lambda \), (as opposed to the conventional \( 2\pi / \Lambda \)) where \( \Lambda \) remains the geometric period of the grating structure. Thus we define \( k = k_\phi + \Delta k \) to be a vector...
in optical reciprocal-space, in contrast to the conventional geometric reciprocal-space. The phase $\phi$ of the propagating wave with respect to the grating phase (i.e. the detuning from the Bragg condition) can be understood to be the ratio $\phi = \Delta\beta / 2\kappa$. For a high RIC (i.e. strongly-coupled) grating, this phase needs to be modified thus:

$$\langle \phi \rangle = \frac{\langle \Delta\beta \rangle}{2\langle \kappa \rangle} \quad \text{(8)}$$

so that the triangle brackets mean the spatial average over the length of the grating, rather than a simple geometric average denoted by a bar across the top of the variable. In a somewhat recursive manner, spatially-averaged and geometrically-averaged quantities are related via the modified phase parameter $\langle \phi \rangle$ in the following equation, where as an example quantity, we have used the detuning parameter $\beta$:

$$2\beta = \Delta = \frac{2}{\kappa} \int \frac{dz}{L} L dz - \frac{dz}{L} \cdot = \frac{dz}{L} \cdot \int (9)$$

A modified Debye-Waller factor [18] can be recognized within the curly brackets. The refractive index quantities $n$ and $\bar{n}$ are similarly related, as well are $\langle \beta \rangle$ and $\bar{\beta}$ etc. Conventionally, the coupling coefficient is also defined in terms of the Fourier series coefficient associated with the permittivity of an infinitely-periodic grating [11]. However, as evident below in equation (11b), the spatially averaged coupling coefficient $\langle \kappa(z) \rangle$ is equivalent to the FT of the grating refractive index, and thus also has a spectral dependency, so that it can be considered as $\hat{\kappa}(\Delta k)$:

$$\kappa(z) = -\frac{j}{2n(z)} \frac{\partial n(z)}{\partial z} e^{ik_0 z} \quad \text{(11a)}, \quad \hat{\kappa}(\Delta k) = \langle \kappa(z) \rangle = -\frac{j}{2L} \int \frac{dz}{n(z)} \frac{\partial n(z)}{\partial z} e^{ik_0 z} dz \quad \text{(11b)}$$

Local coupling, as defined in (11a), is simply due to changes in refractive index with an appropriate phase change, i.e. equivalent to a Fresnel reflection. We can decompose the modified detuning parameter $\langle \Delta\beta \rangle$ into its quadrature components, $\langle \Delta\beta \rangle = \langle \Delta\beta_\parallel \rangle + j \langle \Delta\beta_\perp \rangle$ , such that by appropriate substitution of (8) into (10) we find:

$$\langle \Delta\beta_\parallel \rangle = \Delta\beta \pm \bar{\kappa} \left( \langle \Delta\beta \rangle \right)^{\frac{1}{2}} \left( 1 - \exp \left[ -\left( \frac{\Delta\beta}{\bar{\kappa}} \right)^2 \right] \right) \quad \text{(12a)}$$

and

$$\langle \Delta\beta_\perp \rangle = \mp 2\bar{\kappa} \left( \langle \Delta\beta \rangle \right)^{\frac{1}{2}} \left( 1 - \exp \left[ -\left( \frac{\Delta\beta}{\bar{\kappa}} \right)^2 \right] \right) \quad \text{(12b)}$$

where we assume $\Delta\beta = \bar{n}\Delta k$. For the coupled mode equations, we are only interested in the real part of the modified detuning parameter $\langle \Delta\beta \rangle$, which corresponds to a real phase-detuning from the resonant Bragg phase-matching condition. Multiplying the two conjugate solutions for $\langle \Delta\beta_\parallel \rangle$ together, we find that:

$$\langle \Delta\beta_\parallel \rangle^2 = \Delta\beta^2 - 2\bar{\kappa} \left( \langle \Delta\beta \rangle \right)^2 \left( 1 - \exp \left[ -\left( \frac{\Delta\beta}{\bar{\kappa}} \right)^2 \right] \right) \quad \text{(13)}$$

The substitution of $\langle \Delta\beta_\parallel \rangle$ for $\Delta\beta$ in section 2 is all that is required to correct for strong-coupled devices, with equation (13) used to relate $\langle \Delta\beta_\parallel \rangle$ to the experimentally observed detuning parameter $\Delta\beta$.

4. Application of Theory to a 1D Photonic Crystal

For a photonic crystal as shown in figure 3, with Bragg wavelength $\lambda_b$, constant coupling strength $\kappa_b$ along the length $L$ of the grating, the conventional CMT equation describing the reflection spectrum for the structure is [11]:

$$H_0 \left( \Delta\beta \right) = \frac{-jk_s \sinh sL}{s \cosh sL + j \frac{\Delta\varepsilon}{\varepsilon_0} \sinh sL} \quad \text{(15a)}$$

where $s^2 = \kappa_b^2 - \left( \frac{\Delta\beta}{\bar{\kappa}} \right)^2$ \quad \text{(15b)}, and $\kappa_b = \frac{\varepsilon_1 - \varepsilon_2}{\Delta\varepsilon \bar{n} \lambda_b}$. \quad \text{(15c)}

From the definition given in equation (11b), the same regular square grating of finite length $L$ consisting of $M$ periods of period $\Lambda$, with dielectric contrast $\Delta\varepsilon = \varepsilon_1 - \varepsilon_2$, has a spectral coupling coefficient closely given by:

$$\hat{\kappa}(\Delta k) = \frac{\varepsilon_1 - \varepsilon_2}{\bar{n} \lambda_b} \sin \left( \frac{\Delta\varepsilon L}{\bar{n} \lambda_b} \right) e^{-\Delta\varepsilon L / \bar{n} \lambda_b}, \quad \text{(16a)}$$

with $\hat{\kappa}(\Delta k) \rightarrow \kappa_b \delta(k - k_b)$ as $M \rightarrow \infty$. \quad \text{(16b)}

At the Bragg resonant frequency (i.e. $\Delta k = 0$), the spectral coupling coefficient has the same magnitude as given in conventional CMT. As the number of periods tends to infinity (i.e. the implicit assumption of an infinitely repeating
grating in conventional CMT) the limiting function for the coupling coefficient \( \kappa(\Delta k) \) becomes a delta function in the spectral domain, as indicated by (16b), so that as \( M \to \infty \), equation (13) tends towards:

\[
\left( \langle \Delta \beta \rangle / 2 \right)^2 = \left( \Delta \beta / 2 \right)^2 - \kappa_n^2.
\]  

(17)

Assuming the parameter \( s \) can be defined as \( s \equiv j \langle \Delta \beta \rangle / 2 \), we find that (17) is equivalent to (15b). From equation (5) the reflection spectrum for a photonic crystal with constant coupling strength \( \kappa_n \) along its length \( L \) is:

\[
H_p(\Delta \beta) = -j \tanh \left[ \kappa_n L \text{sinc} \left( \frac{\langle \Delta \beta \rangle L}{2} \right) \right].
\]  

(18)

Both equations (15a) and (18) yield the same reflectivity at the bandgap centre, \( H_p(0) = -j \tanh \kappa_n L \). Figure (4) plots them against each other, for a strong coupling example [20], where \( n_1 = 1.5 \), \( n_2 = 2.5 \), and the grating has \( M=5 \) periods. The figure shows excellent agreement between the two equations, with similar bandgap widths.

**5. Conclusions**

Optical devices such as the AWG and FBG discussed in this paper play an important role in current DWDM photonic networking. More recently, PC research has shown that these devices may play a pivotal role in future device integration. In this paper, we have shown that all three classes of device may be studied using a unified FT analysis. This not only aids in understanding the characterisation of currently available devices, but offers a powerful tool for the prediction of new modes of operation and the design of novel optical components.

**6. References:**