Motivated by ideas from the study of abstract data types, we show how to interpret non-well-founded sets as fixed points of continuous transformations of an initial continuous algebra. We consider a preordered structure closely related to the set HF of well-founded, hereditarily finite sets. By taking its ideal completion, we obtain an initial continuous algebra in which we are able to solve all of the usual systems of equations that characterize hereditarily finite, non-well-founded sets. In this way, we are able to obtain a structure which is isomorphic to HF̂, the non-well-founded analog of HF.

1. INTRODUCTION

The prototypical non-well-founded set is Ω = {Ω}; its existence is prohibited by the Foundation Axiom of Zermelo–Fraenkel set theory (ZFC).

Sixty years after being banished from set theory, non-well-founded sets have returned. This is mostly due to the work of Peter Aczel (1988), who not only developed an elegant and unified theory, but also showed how

non-well-founded sets can be applied to the study of communication and computation. He investigated an Anti-Foundation Axiom (AFA) that contradicts the usual Foundation Axiom (FA). (FA is also known as the Axiom of Regularity.) Aczel then used AFA to build a model of Milner's Synchronous Calculus of Communicating Systems (SCCS) (Milner, 1983). Shortly thereafter, Jon Barwise and John Etchemendy (1988) used AFA in conjunction with the ideas of situation theory to give a new diagnosis and treatment of the Liar Paradox ("This sentence is false"). A very different application is found in Barwise (1988), where non-well-founded sets are used to study common knowledge (better called "mutual information"), which is an important and problematic notion in distributed computation. The common thread in all these applications is that, by using a set theory countenancing self-membership, it is possible to develop natural solutions to problems involving self-reference and circularity.

Many problems involving self-reference, such as the problems of giving the semantics of the untyped lambda calculus and of programming languages supporting concurrency, have been tackled with a large measure of success within ZFC by using domain theory. In an appendix to his book, Aczel states that

A natural way to try to understand non-well-founded sets is to view them as limits, in some sense, of their well-founded approximations. This approach is inspired by Scott's theory of domains, but it cannot be done in any simple minded way, as I found out.

Indeed, none of the works described above explores the connection between Aczel's conception of a set and domain theory. The purpose of this paper is to provide just such a connection.

In this paper our focus is on hereditarily finite sets, i.e., sets that are finite and all of whose elements are hereditarily finite sets. Of course, this describes "hereditarily finite" in an intuitive way. Later, we will give two different, precise definitions of hereditarily finite set. In ZFC, these definitions coincide and define the collection HF of well-founded, hereditarily finite sets. In Aczel's theory ZFA = ZFC - FA + AFA, the definitions are inequivalent; HF₁, the collection of non-well-founded, hereditarily finite sets, is strictly larger than HF. For instance, Ω ∈ HF₁ - HF.

In proving the consistency of ZFA, Aczel worked in ZFC⁻, i.e., in ZFC without either FA or AFA. His proof that ZFA has a class model in ZFC⁻ readily gives a set in ZFC⁻ that is isomorphic to HF₁ in ZFA. But this structure does not contribute to an understanding of non-well-founded sets as limits of well-founded entities. We also work in ZFC⁻, and our goal is to give an alternative to Aczel's construction that does provide such an understanding. We use ideas from both domain theory and from the study of abstract data types to build a new structure 𝒯 in ZFC⁻ that is
isomorphic to $HF_1$ in ZFA. Our use of order-theoretic and algebraic methods leads us to introduce the preordered algebra $\mathcal{F}$ of protosets and its ideal completion, the continuous algebra $\mathcal{C}$ of partial sets.

Of course, when one works in ZFC, every object is just a set, and, from an entirely formal point of view, that is that. But we mean the terms protoset and partial set to convey intuitions about how these objects relate to elements of $\mathcal{D}$. Partial sets can be thought of as possibly incompletely realized sets. The partial order on $\mathcal{C}$ has an information-theoretic character in that going up in the partial order corresponds to defining a partial set in more and more detail. We introduce a new membership relation $\in_\mathcal{C}$ on $\mathcal{C}$. Protosets are descriptions of partial sets that are hereditarily finite and well-founded with respect to $\in_\mathcal{C}$. The elements of $\mathcal{D}$ are those partial sets that are components of fixed points of continuous functions that correspond to certain directed graphs. Elements of $\mathcal{D}$ are clearly approximated by partial sets that are well-founded with respect to $\in_\mathcal{C}$. We also introduce an entirely new structure $\mathcal{M}$, the set of all maximal—hence, totally defined—partial sets. Each element of $\mathcal{D}$ is in $\mathcal{M}$.

Our starting point is the observation that the set $HF$ of well-founded, hereditarily finite sets can be characterized in algebraic terms. Consider $HF$ as an algebra whose operations are the constant $\emptyset$, the binary operation union, and the unary operation that forms a singleton set by taking $x$ to $\{x\}$. This structure satisfies familiar algebraic laws. In particular, it is a commutative monoid whose binary operation, union, is idempotent. In the appropriate category of algebras, $HF$ is an initial algebra. Our idea is to enlarge $HF$ to a structure satisfying the same laws and in which equations such as $x = \{x\}$ have unique solutions. The study of abstract data types provides a natural choice for such a structure, namely, an initial continuous algebra. In such an algebra, there are unique solutions to systems of equations of the kind that interest us. The initial continuous algebra we construct is our algebra $\mathcal{C}$ of partial sets.

There is a standard construction of initial continuous algebras in the literature (cf., ADJ (1976), Hennessey (1988), and Möller (1985)). This construction has been studied extensively, but not in set-theoretic contexts. However, the standard description is too abstract to be of much use, involving as it does congruence classes of trees. In part to avoid working with congruence classes, we introduce the preordered algebra $\mathcal{F}$ of protosets. As a set in ZFC, the collection $\mathcal{F}$ of protosets has a strikingly simple structure, being just a little more complicated than $HF$. As indicated earlier, the ideal completion of $\mathcal{F}$ is the initial continuous algebra $\mathcal{C}$ of partial sets needed here.
A MODEL OF NON-WELL-FOUNDED SETS

2. BACKGROUND ON SYSTEMS AND THE ANTI-FOUNDATION AXIOM

We review in this section the basic definitions concerning non-well-founded sets as developed in the first two chapters of Aczel (1988). Our notation differs slightly from Aczel's. Although we have included all of the definitions and statements of results that we use, we have not included much motivation. For this, cf. (Aczel, 1988; Barwise and Etchemendy, 1988).

A system is a pair $S = \langle |S|, \xrightarrow{S} \rangle$, where $|S|$ is a set or class whose elements are called the nodes of $S$, and $\xrightarrow{S}$ is a binary edge relation on $|S|$. We frequently omit subscripts on arrows. If $x \to y$, then we call $y$ a child of $x$. The family of children of each node of a system is required to form a set. A system $S$ is small if $|S|$ is a set. An accessible pointed graph, or apg, is a triple $G = \langle |G|, \xrightarrow{G}, \text{top}_G \rangle$ such that $\langle |G|, \xrightarrow{G} \rangle$ is a small system, $\text{top}_G$ is a distinguished node of $G$, and every node of $G$ is accessible from $\text{top}_G$ in the sense that, for each $n \in |G|$, there is a finite path $\text{top}_G \to m_1 \to m_2 \to \cdots \to n$. If $G$ is an apg, denote the small system obtained by forgetting the distinguished node of $G$ by $G^* = \langle |G|, \xrightarrow{G} \rangle$. If $S$ is a system, and $x \in |S|$, then the apg $S^x = \langle |S^x|, \xrightarrow{S^x}, x \rangle$ is obtained by letting $|S^x|$ be the set of nodes of $S$ accessible from $x$ via $\xrightarrow{S}$, and by letting $\xrightarrow{S^x}$ be the restriction of $\xrightarrow{S}$ to $|S^x|$.

There are several important examples of systems:

1. The class $V$ of all sets can be turned into a system $V$ by taking as nodes the class $V$ itself, and as edges the arrows $x \to y$ when $y \in x$. Note that, for each set $x$, the apg $\forall x$ has as its set of nodes $|\forall x|$, the smallest transitive set of which $x$ is a member.

2. A system $\forall$ is obtained by taking as nodes all accessible pointed graphs and by requiring $G \xrightarrow{\forall} H$ iff $H = G^*n$ for some child $n$ of $\text{top}_G$. Observe that, for each apg $G$, the system $\forall G$ whose nodes are the accessible pointed graphs accessible from $G$ is isomorphic to, but not necessarily equal to, $G$.

3. If $S$ is any system, $x$ is a node of $S$, then we can form first the apg $Sx$ and then forget the top to get the subsystem of $S$ defined by $x$: $(Sx)^*$.

A bisimulation is a relation $\equiv$ on the nodes of a system such that $x \equiv y$ implies that for each child $a$ of $x$ there is some child $b$ of $y$ such that $a \equiv b$, and for each child $b$ of $y$ there is some child $a$ of $x$ such that $a \equiv b$. A bisimulation is not necessarily an equivalence relation. For every system $S$, there is a maximal bisimulation $\equiv_S$ on $S$ which includes all bisimulation relations. (This is essentially because the family of bisimulations on $S$ is closed under unions.) The maximal bisimulation on a system is an
equivalence relation. Two nodes $x$ and $y$ of a system $S$ are bisimilar if $x \equiv_S y$.

Suppose $S$ and $T$ are systems. A map $f: |S| \to |T|$ is a system map from $S$ to $T$ provided that $f$ preserves the sets of children. That is, for all nodes $x$ of $S$,

$$\{f(y) : x \xrightarrow{S} y\} = \{z : f(x) \xrightarrow{T} z\}.$$  

Systems together with system maps as homomorphisms constitute a category.

Every function $f: |S| \to |T|$ induces an equivalence relation $\equiv_f$ on its domain. Explicitly, $x \equiv_f y$ iff $f(x) = f(y)$. If $f: S \to T$ is a system map, then $\equiv_f$ is a bisimulation. A system map $f: S \to T$ is a strongly extensional quotient of $S$ if $f$ is surjective on nodes and $\equiv_f$ is exactly $\equiv_S$. Every system has a strongly extensional quotient. If $f: S \to T$ and $f': S \to T'$ are two strongly extensional quotients of $S$, then $T$ and $T'$ are isomorphic systems.

If $G$ is an apg, then a decoration of $G$ is a system map $d: G^* \to \mathcal{V}$, and we say $G$ is a picture of the set $d(\text{top}_G)$. If $x$ is a set, then the apg $\mathcal{V}^x$ is called the canonical picture of $x$ because the inclusion function $d: |\mathcal{V}^x| \to |\mathcal{V}|$ is a system map from $(\mathcal{V}^x)^*$ to $\mathcal{V}$ such that $d(x) = x$.

The Anti-Foundation Axiom (AFA) states that for every apg $G$ there is a unique system map $d_G: G^* \to \mathcal{V}$. The AFA implies that there is a strongly extensional quotient $e: \mathcal{G} \to \mathcal{V}$ such that for all $G$, $e(G) = d_G(\text{top}_G)$, where $d_G$ is the unique decoration of $G$.

For each function $f: X \to Y$ and each subclass $Z$ of $X$, let $f[Z]$ denote the image of $Z$ under $f$. The following definitions are new, as are the results of Theorem 2.1.

**Definition.** If $S = (|S|, \rightarrow_S)$ is a system, then a subsystem of $S$ is a system $T = (|T|, \rightarrow_T)$ where $|T| \subseteq |S|$ and $\rightarrow_T = \twoheadrightarrow_S \cap (|T| \times |T|)$. A transitive subsystem of $S$ is a subsystem $T = (|T|, \rightarrow_T)$ also satisfying the property that, if $x \in |T|$, $y \in |S|$ and $x \twoheadrightarrow_S y$, then $y \in |T|$.  

**Theorem 2.1.** If $T$ is a transitive subsystem of $S$ and $f: S \to S'$ is a system map, then the restriction $f|_T: T \to f[T]$ is a system map. If $f: S \to S'$ is a strongly extensional quotient, then so is $f|_T: T \to f[T]$.

**Proof.** Using the fact that $T$ is a transitive subsystem, it is routine to check that $f[T]$ is a system and that $f|_T: T \to f[T]$ is a system map. It then follows that $\equiv_{f|_T}$ is a bisimulation on $T$, and so this is contained in the maximum bisimulation $\equiv_T$ on $T$. On the other hand, the maximum bisimulation $\equiv_T$ can be extended to all of $S$ as the relation $\mathcal{R}$ defined by $x \mathcal{R} y$ if and only if $x = y$ or $x, y \in |T|$ and $x \equiv_T y$. 


It is easy to check that $R$ is a bisimulation on $S$, and so it is contained in the maximum bisimulation $\equiv_S$. If $f$ is strongly extensional, then $\equiv_S = \equiv_f$, and so $\equiv_T \subseteq \equiv_{f|_T}$. Combining this with the first containment shows that $\equiv_T = \equiv_{f|_T}$ if $f$ is strongly extensional.

3. HEREDITARILY FINITE SETS

Many of the non-well-founded sets that arise in applications are hereditarily finite. That is, they are finite, all of their elements are finite, and so on. Of course, saying this does not precisely define what it means for a set to be hereditarily finite. In fact there are two natural, precise definitions of the family of hereditarily finite sets:

- $HF = \text{def} \, \{x \in V | \forall \alpha \in x. \alpha \in V\}$
- $HF_1 = \text{def} \, \{x \in V | \exists \alpha \in x. \alpha \in V\}$

Here $\mathcal{P}_{<\omega}(x)$ is the set of finite subsets of $x$.

Now assuming the axioms of set theory (notably Pairing, Union, Infinity, and some of the Replacement Axioms, but neither Foundation nor Choice), HF exists. It can be constructed in the following way: Define $V_0 = \emptyset$, and given $V_i$, let $V_{i+1} = \mathcal{P}(V_i)$. Then $HF = \bigcup_i V_i$. We use the standard notation of rank($x$) to denote the least $i$ such that $x \in V_{i+1}$.

In contrast to this, $HF_1$ cannot be shown to exist without some Axiom of Foundation or Anti-Foundation. Under the usual Axiom of Foundation, we show $HF_1 = HF$ by $\varepsilon$-induction. (If a set $x$ is finite and if every element of $x$ belongs to some $V_i$, then $x \in V_N$, where $N = 1 + \max\{\text{rank}(y) : y \in x\}$.) But without this axiom we have no principle of proof by $\varepsilon$-induction. In fact, under the AFA, HF is a proper subset of HF_1.

**Definition.** An apg $G$ is finitely branching if each node of $G$ has finitely many children. $\mathcal{G}$ is the subsystem of $\mathcal{S}$ whose class of nodes consists of all finitely branching accessible pointed graphs.

**Proposition 3.1.** Assume AFA. Then $HF_1$ is the set of all sets whose pictures are finitely branching accessible pointed graphs; i.e., $HF_1 = e[|\mathcal{G}|]$.

**Proof.** This result is implicit in (Aczel, 1988); we provide a direct argument for completeness. To save on notation, we denote $e[|\mathcal{G}|]$ by $W$ in this proof.

First we show that $W$ is a set and satisfies $x = \mathcal{P}_{<\omega}(x)$. The strongly extensional quotient $e: \mathcal{S} \to \mathcal{Y}$ has the property that if $G$ is bisimilar to $H$, then $e(G) = e(H)$. Now it follows that $W$ is a set (as opposed to a proper class), since every finitely branching apg is isomorphic to—an apg whose node set is a set of natural numbers.
To show $W \subseteq \mathcal{P}_{<\omega}(W)$, let $G \in |\mathcal{G}|$. By AFA, $G$ has a unique decoration $d_G$, which is a system map, so

$$e(G) = d_G(\text{top}_G) = \{z : z \in d_G(\text{top}_G)\} = \{z : d_G(\text{top}_G) \rightarrow z\} = \{d_G(n) : \text{top}_G \rightarrow n\}.$$ 

But, for each $n$, the restriction of $d_G$ to $G^*n$ is a decoration. So, by the uniqueness of decorations, each $d_G(n)$ in fact equals $e(G^*n)$. Hence $e(G) = \{e(G^*n) : \text{top}_G \rightarrow n\}$, proving the desired inclusion. For the reverse inclusion, let $e[X]$ be an arbitrary finite subset of $W$, where $X$ is a finite subset of $|\mathcal{G}|$. Form a new apg $H$ as follows: Let the node set $|H|$ be the disjoint union of $\{G : G \in X\}$ together with a new vertex $m$. Let $\text{top}_H = m$, and let the edge relation $\rightarrow_H$ be the union of the edge relations $\rightarrow_G$ together with the pairs $m \rightarrow \text{top}_G$ for all $G \in X$. Then $H \in |\mathcal{G}|$, and $e(H) = \{e(G) : G \in X\}.$

What remains is to show that if $x$ is any set satisfying $x = \mathcal{P}_{<\omega}(x)$, then $x \subseteq W$. If $x = \mathcal{P}_{<\omega}(x)$, then $x \in \mathcal{P}(x)$, and every element of $x$ is finite. The fact that $x \subseteq \mathcal{P}(x)$ tells us that $x$ is transitive. Therefore $|\forall^-x|$, the smallest transitive set including $x$, is just $x$ itself. Let $y \in x$, and let $\forall^-y$ be the canonical picture of $y$. The inclusion $|\forall^-y| \subseteq |\forall^-|$ is a system map. By AFA, decorations are unique, so $e(\forall^-y) = y$. The facts that $x$ is transitive, $y \in x$, and $|\forall^-y|$ is the smallest transitive set of which $y$ is a member then imply that the nodes of $\forall^-y$ are elements of $x$; thus they are finite sets. Hence $\forall^-y \in |\mathcal{G}|$, so $y = e(\forall^-y) \in W$. 

Note that an apg in $\mathcal{G}$ may have an infinite set of nodes. Moreover, it is easy to construct a family of $2^{\aleph_0}$ apg's from $\mathcal{G}$ which are pairwise non-bisimular. Proposition 3.1 then implies that the cardinality of $\mathcal{H}_1$ is at least $2^{\aleph_0}$. Since every finitely branching apg is isomorphic to an apg whose node set is a set of natural numbers, the cardinality of $\mathcal{H}_1$ is at most $2^{\aleph_0}$. Thus the cardinality of $\mathcal{H}_1$ is $2^{\aleph_0}$. (This was also proved by Fernando (1989)).

4. SET ALGEBRAS

Consider the signature $\Sigma$ containing a constant $e$, unary operation $s$, and binary operation $+$. Consider also the following set $E$ of $\Sigma$-equations:

$$x + e = x$$
$$x + x = x$$
$$x + y = y + x$$
$$x + (y + z) = (x + y) + z.$$
Intuitively, $e$ represents the empty set $\emptyset$, $s$ stands for the singleton operation $x \mapsto \{x\}$, and $+$ represents union.

**Definition.** A set algebra is a $\Sigma$-algebra which satisfies the equations $E$.

**Convention.** Throughout this paper, empty sums in set algebras equal $e$. This allows us to write $\sum S + \sum T = \sum (S \cup T)$, even if $S$ or $T$ is empty.

It should not come as a surprise that the initial set algebra is HF, the hereditarily finite sets with the operations interpreted as we have done above.

**Proposition 4.1.** HF, with the natural interpretations of the operators, is an initial set algebra.

**Proof.** Let $\mathcal{A}$ be a set algebra. We define a map $\varepsilon : HF \to \mathcal{A}$ by $\varepsilon$-recursion on HF. For all $y \in HF$, $y = \bigcup_{z \in y} \{z\}$. So we define $\varepsilon(y) = \sum_{z \in y} s(z)$. Here the sum refers to the operation in $\mathcal{A}$. Our convention on empty sums implies that $\varepsilon(\emptyset) = e$. This map $\varepsilon$ is a homomorphism of set algebras because both HF and $\mathcal{A}$ are commutative monoids, and $+$ is idempotent. An easy $\varepsilon$-induction shows that it is unique. 

We study set algebras which have some additional properties:

**Definition.** A preordered set algebra $\mathcal{A}$ together with a preorder $\leq$ on $\mathcal{A}$ satisfying the following properties:

- $\mathcal{A}$ has a unique minimum element, denoted by $\perp$.
- The operations $s$ and $+$ are monotone in all their arguments.

**Definition.** A continuous set algebra is a preordered algebra whose preorder is a directed-complete partial order, and whose operations are continuous in the sense that they preserve least upper bounds of directed subsets. Thus, the carrier of a continuous set algebra is a cpo.

A homomorphism of preordered set algebras is required to preserve the operations, the preorder, and the minimum element $\perp$. A homomorphism of continuous set algebras must additionally be continuous.

**Definition.** Let $\mathcal{A}$ be a set algebra. We write $x \in_{\mathcal{A}} y$ for the relation $s(x) + y = y$.

Obviously, this definition is motivated by the situation in HF. Every set algebra $\mathcal{A}$ now canonically gives rise to a system, also denoted by $\mathcal{A}$, by the members of $\mathcal{A}$ being taken as the set of nodes and by the converse of $\varepsilon_{\mathcal{A}}$ being taken as the edge relation. As we shall see, the systems derived...
from set algebras can be quite interesting. Whenever it causes no confusion, we shall simply write \(x \in y\) instead of \(x \in \mathcal{A} y\).

5. The Algebra \(\mathcal{F}\) of ProtoSets

We define a family of sets \(\mathcal{F}_i\) by recursion on \(i\):

\[
\begin{align*}
\mathcal{F}_0 &= \emptyset \\
\mathcal{F}_{i+1} &= 2 \times \mathcal{P}(\mathcal{F}_i).
\end{align*}
\]

Of course, \(2 = \{0, 1\}\). We then set \(\mathcal{F} = \bigcup_i \mathcal{F}_i\). Just as HF is the \(\subseteq\)-minimal set \(x\) satisfying \(x = \mathcal{P}_{<\omega}(x)\), \(\mathcal{F}\) is the \(\subseteq\)-minimal set \(x\) satisfying \(x = 2 \times \mathcal{P}_{<\omega}(x)\).

We write \(x^0\) for the first component of an element \(x\) of \(\mathcal{F}\) and \(x^1\) for the second component. Moreover, for \(x \in \mathcal{F}\), we let rank\((x)\) be the least \(i\) such that \(x \in \mathcal{F}_{i+1}\).

We make \(\mathcal{F}\) into a set algebra in the following way:

\[
x + \mathcal{F} y = (\min(x^1, y^1), x^1 \cup y^1).
\]

It is trivial to check that the commutative monoid laws are satisfied, and that + is idempotent. Hinting at the preorder on \(\mathcal{F}\) to be introduced in the next section, we define \(\bot = (\emptyset, \emptyset)\). When it is clear that we are working with \(\mathcal{F}\), we will omit the subscripts. So in this notation \(\mathcal{F}_1 = \{e, \bot\}\). We also have the following normal form for elements of \(\mathcal{F}\):

\[
x = \langle x^0, \emptyset \rangle + \sum_{y \in x^1} s(y).
\]

The large summation sign refers to the operation \(+\). When \(x^1 = \emptyset\), then our convention concerning sums in set algebras tells us that the big sum is \(e\). Notice that the term \(\langle x^0, \emptyset \rangle\) on the right is either \(e\) or \(\bot\). The representation of (1) is unique in the sense that \(x^0\) is the only element \(i \in 2\) and \(x^1\) is the only finite subset \(S \subset \mathcal{F}\) such that

\[
x = \langle i, \emptyset \rangle + \sum_{y \in S} s(y).
\]

By initiality, there is a unique morphism of set algebras \(\gamma : HF \to \mathcal{F}\). This map is given explicitly as follows:

\[
\tilde{a} = \langle 1, \{b : b \in a\} \rangle.
\]
We should remark that the way we calculate ranks in \( F \) implies that for all \( a \in HF \), \( \text{rank}_{HF}(a) = \text{rank}_{\mathcal{F}}(\tilde{a}) \); i.e., \( \tilde{\cdot} \) is rank-preserving. An easy induction on rank shows that \( \tilde{\cdot} \) is one-to-one.

**Proposition 5.1.** \( F \) is the free set algebra generated by \( \{1\} \).

*Proof.* For any set algebra \( \mathcal{A} \) and any \( a \in \mathcal{A} \), it is easy to show that there is a unique set algebra homomorphism \( \phi: \mathcal{F} \to \mathcal{A} \) satisfying \( \phi(\bot) = a \). 

Recall that we have a general notion of membership in set algebras. Specializing this to the case of protosets, this definition tells us that \( x \in_\mathcal{F} y \) (as protosets) if and only if \( x \in y^1 \) (as sets).

**Proposition 5.2.** \( F \) is well-founded with respect to \( \in_\mathcal{F} \).

*Proof.* If \( x \in_\mathcal{F} y \), then \( \text{rank}(x) < \text{rank}(y) \).

Proposition 5.2 implies the following Induction Principle for \( F \). Let \( \phi(x) \) be a property such that

\[
(\forall x \in \mathcal{F}) ([ (\forall y \in \mathcal{F} x) \phi(y) ] \Rightarrow \phi(x)).
\]

Then for all \( x \in \mathcal{F} \), \( \phi(x) \).

Following the notational convention of suppressing unnecessary subscripts, whenever we use the notation \( x \in_\mathcal{F} y \) for protosets \( x \) and \( y \), we mean that \( x \in_\mathcal{F} y \). Occasionally, we shall need to write \( x \in y^1 \), and here of course we mean the ordinary relation of set membership. Our definitions above give a model of the protosets as specific hereditarily finite sets, but we think of protosets as set-like objects in their own right, with a perfectly good relation of membership of their own. In fact, we believe a general theory of protosets can be developed; for a start in this direction, see (Mislove, Moss, and Oles, 1989).

6. **The Intuition Behind \( \mathcal{F} \)**

There is an intuition according to which \( \mathcal{F} \) is a natural extension of HF. In order to understand this, consider the following notions, whose names were suggested to us by Jon Barwise:

**Definition.** A protoset \( x \) is *clear* if \( x^0 = 1 \), and *murky* if \( x^0 = 0 \).

Think for a moment of a set in HF as a box which contains its elements, and these elements are other boxes. A protoset is a box which also contains
its elements, but in addition, it may contain a large amount of dense packaging material. In fact, the packaging can obscure the fact that there might be other boxes inside, and these boxes might themselves have a lot of packaging. On one hand, if \( x \) is clear, then there is no extra dense packaging; the only objects present are the elements of \( x \). In this way, we see that \( e \), the clear set with no elements, is the analog in \( \mathcal{F} \) of the empty set \( \emptyset \). We can similarly interpret the map \( \tilde{=} : \text{HF} \to \mathcal{F} \). It takes hereditarily finite sets to "hereditarily clear" protosets with the same structure. On the other hand, a murky protoset is like a box which is full of dense packaging. So \( \bot \) is a box with lots of packaging but no definite elements. The murky protoset \( \langle 0, \{3, 4\} \rangle \) containing 3 and 4 is a protoset which definitely contains 3, 4 and it possibly contains any other protoset. The sum of two protosets \( x \) and \( y \) is a protoset \( z \) with all the elements in either \( x \) or \( y \); and, if either \( x \) or \( y \) is murky, so is \( z \). Thus adding \( \bot \) to a clear protoset \( x \) to get \( \bot + x \) gives a murky version of \( x \). The singleton operation \( s \) on protosets takes on arbitrary protoset \( x \) and creates a clear protoset that definitely contains \( x \) as its only element. Note that writing \( x \in \mathcal{F} y \) corresponds to the intuition that \( x \) is definitely an element of \( y \).

Continuing to develop our box metaphor, we can say that one protoset \( x \) is clarified by a second protoset \( y \), and we correspondingly write \( x \sqsubseteq y \), if we can obtain \( y \) from \( x \) by taking some (or none) of the packaging inside \( x \) (or inside some box in \( x \)) and replacing this by other protosets. When we carry out this replacement, the packaging taken out need not be completely eliminated, and if any is left, it continues to hide other protosets. The murky version of \( x \) is always clarified by \( x \): \( \bot + x \sqsubseteq x \). For example,

\[
\bot = \langle 0, \emptyset \rangle \sqsubseteq \langle 0, \{\bot, 4\} \rangle \sqsubseteq \langle 0, \{3, 4\} \rangle
\sqsubseteq \langle 0, \{0, 1, 2, 3, 4\} \rangle \sqsubseteq \langle 1, \{0, 1, 2, 3, 4\} \rangle = 5.
\]

In the next section, we given a precise definition of the clarification order \( \sqsubseteq \).

### 7. The Minimum Monotone Preorder

In this section we equip \( \mathcal{F} \) with a preorder that is intimately related to its structure as a set algebra. A monotone preorder (mpo) is a reflexive and transitive relation \( \leq \) on \( \mathcal{F} \) such that for all \( x, y, \) and \( z \in \mathcal{F} \),

- \( \bot \leq x \).
- If \( x \leq y \), then \( s(x) \leq s(y) \).
- If \( x \leq y \), then \( x + z \leq y + z \).
Note that an arbitrary mpo does not make \( F \) a preordered set algebra because \( \bot \) may not be the unique minimum element.

When working with an mpo, it is tempting to assume that \( x \leq x + y \). However, this is usually not the case. It is true that \( \bot + x \leq \bot + x + y \), since

\[
\bot + x \leq (\bot + y) + x = \bot + (x + y).
\]

For a generalization of this observation, we have the following result.

**Lemma 7.1.** Let \( \leq \) be an mpo, and let \( S \) and \( T \) be any finite subsets of \( F \). Suppose that for all \( x \in S \) there is some \( y \in T \) such that \( x \leq y \), and for all \( y \in T \) there is some \( x \in S \) such that \( x \leq y \). Then \( \sum_{x \in S} x \leq \sum_{y \in T} y \).

**Proof.** Let \( g: T \to S \) be such that \( g(y) \leq y \) for all \( y \in T \). Let \( f: S \to T \) be such that \( x \leq f(x) \) for all \( x \in S \). Now we calculate, using the idempotence, commutativity, associativity, and monotonicity of + and the preorder properties of \( \leq \),

\[
\sum_{x \in S} x = \sum_{x \in S} x + \sum_{y \in T} g(y) \\
\leq \sum_{x \in S} f(x) + \sum_{y \in T} y = \sum_{y \in T} y. 
\]

It is trivial to check that the intersection of any family of mpo's is an mpo. The universal relation \( x \leq y \) for all \( x, y \in F \) is an mpo, so the family of mpo's is nonempty. Thus there is a minimum mpo. We use the symbol \( \leq \) to denote this preorder, and we read \( x \leq y \) as \( x \) is clarified by \( y \). We are interested in \( \leq \) because of its connection to the initial continuous set algebra.

Now the point of working with \( F \) is that we can get our hands on this preorder \( \leq \), but the definition makes it hard to see how one could ever tell, for example, whether \( \bot + s(s(\bot)) \leq s(s(e) + \bot) \) or not. We would like a characterization of \( \leq \) in terms of \( \in \). The key result in this direction is Theorem 7.3 below. It is based on a lemma which shows how to obtain one mpo from another. To state the lemma, we need a definition.

**Definition.** If \( \leq \) is any mpo, then we form a new relation \( \leq^+ \) on \( F \) by \( x \leq^+ y \) iff either (A) or (B) holds:

(A) \( x \) is murky, and

(A1) For all \( a \in x \) there is some \( b \in y \) such that \( a \leq b \).

(B) \( x \) and \( y \) are both clear, and both

(B1) For all \( a \in x \) there is some \( b \in y \) such that \( a \leq b \).

(B2) For all \( b \in y \) there is some \( a \in x \) such that \( a \leq b \).
Lemma 7.2. Let \( \leq \) be any mpo. Then \( \leq^+ \) is also an mpo, and \( \leq^+ \) is a suborder of \( \leq \).

Proof. That \( \leq^+ \) is a preorder follows from the assumption that \( \leq \) is a preorder. The minimality of \( \bot \) in \( \leq^+ \) is due to the fact that \( \bot \) is murky and has no elements. It is routine to check that \( s \) and \( + \) are monotone with respect to \( \leq^+ \).

Finally, we show that if \( x \leq^+ y \), then \( x \leq y \). Let \( S = \{ s(a) : a \in x \} \), and let \( T = \{ s(b) : b \in y \} \). On one hand, suppose that \( x \) is murky. Then \( \langle x^0, \emptyset \rangle = \bot \). So

\[
x = \bot + \sum S \leq \left( \langle y^0, \emptyset \rangle + \sum T \right) + \sum S = \left( \langle y^0, \emptyset \rangle + \sum (T \cup S) \right).
\]

By clause (A1), Lemma 7.1 applies to \( T \cup S \) and \( T \). So

\[
x \leq \left( \langle y^0, \emptyset \rangle + \sum (T \cup S) \right) \leq \langle y^0, \emptyset \rangle + \sum T = y.
\]

On the other hand, suppose that \( x \) and \( y \) are both clear. By clauses (B1) and (B2), Lemma 7.1 applies to \( S \) and \( T \). Therefore \( x = e + \sum S \leq e + \sum T = y \).

Theorem 7.3 (the Structure Theorem). For all \( x, y \in \mathcal{F} \), \( x \) is clarified by \( y \) (\( x \subseteq y \)) iff either (A) or (B) holds:

- (A) \( x \) is murky, and
  - (A1) For all \( a \in x \) there is some \( b \in y \) such that \( a \subseteq b \).
- (B) \( x \) and \( y \) are both clear, and both
  - (B1) For all \( a \in x \) there is some \( b \in y \) such that \( a \subseteq b \).
  - (B2) For all \( b \in y \) there is some \( a \in x \) such that \( a \subseteq b \).

Proof. We know from Lemma 7.2 that \( \subseteq^+ \) is a suborder of \( \subseteq \). However, since \( \subseteq^+ \) is an mpo and \( \subseteq \) is minimal, the two are equal.

Theorem 7.4. When preordered by \( \subseteq \), \( \mathcal{F} \) is an initial preordered set algebra.

Proof. The Structure Theorem implies that \( \bot \) is the minimum element of \( \mathcal{F} \) under \( \subseteq \). It also implies that \( s \) and \( + \) are monotone with respect to \( \subseteq \). Thus \( \mathcal{F} \) is a preordered set algebra. The verification of initality uses the Structure Theorem, and also the normal form (1).

Let us again consider the question raised earlier: is \( \bot + s(s(\bot)) \subseteq s(s(e) + \bot) \)? By (A1), we see that this is true iff \( s(\bot) \subseteq s(e) + \bot \). But a
clear protoset cannot be clarified by a murky protoset, so this last assertion is false. Therefore \( I + s(s(\bot)) \not\subseteq s(s(e) + \bot) \).

There are several other results which follow from the Structure Theorem.

**Corollary 7.5.** A protoset \( y \) is \( \preceq \) -maximal iff for some \( x \in HF \), \( y = x \).

**Proof.** We first argue by \( \varepsilon \) -induction on \( x \in HF \) that for all \( y \in \mathcal{F} \), if \( x \subseteq y \), then \( x = y \). Assume the result for all \( z \in x \), and let \( x \subseteq y \). Since \( x \) is clear, \( y \) is clear. For all \( w \in y \) there is some \( z \in x \) such that (by the induction hypothesis) \( w = z \). The converse also holds, by (B2) and the induction hypothesis. Therefore \( y = x \) by Eq. (2).

The converse can be proved similarly by applying the Induction Principle for \( \mathcal{F} \) to the protoset \( y \). \( \square \)

**Corollary 7.6.** If \( x \subseteq y \), then \( \text{rank}(x) \leq \text{rank}(y) \).

**Proof.** By induction on \( \text{rank}(x) \). If \( \text{rank}(x) = 0 \), then the statement is clearly true. Assume it for \( n \), and let \( x \) have \( \text{rank} \ n + 1 \). Suppose that \( x \subseteq y \). Then there is some \( z \in x \) such that \( \text{rank}(z) = n \). For this \( z \), there is some \( w \in y \) such that \( z \subseteq w \). It follows that \( \text{rank}(y) > \text{rank}(w) \geq n \).

**Corollary 7.7.** For all \( x \in \mathcal{F} \), \( \{ y : y \subseteq x \} \) is finite.

**Proof.** \( \{ y : y \subseteq x \} \subseteq \mathcal{F}_n \), where \( n = \text{rank}(x) \). \( \square \)

**Proposition 7.8.** Define a sequence \( \langle z_i : i \in \omega \rangle \) by the recursion \( z_0 = \bot \), and \( z_{i+1} = s(z_i) + \bot \). Then for all \( i \), \( \text{rank}(z_i) = i \), and if \( y \) is any protoset of rank at least \( i \), \( z_i \subseteq y \).

**Proof.** By induction on \( i \). \( \square \)

**Corollary 7.9.** Suppose that \( x \subseteq y \) and \( \text{rank}(x) < i < \text{rank}(y) \). Then there is a protoset \( z \) such that \( \text{rank}(z) = i \), and \( x \subseteq z \subseteq y \).

**Proof.** First we consider the case \( x = \bot \). If \( 0 < i < \text{rank}(y) \), let \( z = z_i \) in the notation of Proposition 7.8.

Now we use induction on the protoset \( x \). Assume the result for all \( x' \in x \), and suppose that \( x \subseteq y \) and \( \text{rank}(x) < i < \text{rank}(y) \). If \( x \) is clear, then \( y \) is also clear. Let \( k = \text{rank}(y) \). Since \( k > 0 \), there is some \( y' \in y \) of rank \( k - 1 \). Let \( x' \in x \) be such that \( x' \subseteq y' \). Note that

\[
\text{rank}(x') < i - 1 < k - 1 = \text{rank}(y').
\]

By induction hypothesis, there exists \( z' \) of rank \( i - 1 \) such that \( x' \subseteq z' \subseteq y' \). If \( z = x + s(z') \), then \( x \subseteq z \subseteq y \) and \( \text{rank}(z) = i \). On the other hand, if \( x \) is murky, let \( z = x + z_i \). Then \( \text{rank}(z) = i \), and \( x \subseteq z \subseteq y \).
DEFINITION. Protosets $x$ and $y$ are compatible if they have a common clarification, i.e., if there is a protoset $z$ such that $x \subseteq z$ and $y \subseteq z$.

Another consequence of the Structure Theorem is the following bisimulation-like criterion of compatibility. It is an important result for our work. Since the proof is straightforward, we omit it.

**Corollary 7.10.** Let $S$ and $T$ be finite subsets of $\mathcal{F}$. Then the following are equivalent:

1. $\sum_{x \in S} s(x)$ and $\sum_{y \in T} s(y)$ are compatible.
2. For all $x \in S$ there is some $y \in T$ such that $x$ and $y$ are compatible, and vice-versa.

**Corollary 7.11.** Let $x, y$ be elements of $\mathcal{F}$. If $z \in \mathcal{F}$ is such that $x \subseteq z$ and $y \subseteq z$, then there is such a $w \subseteq z$ such that $x \subseteq w, y \subseteq w$, and $\text{rank}(w) = \max(\text{rank}_{\mathcal{F}}(x), \text{rank}_{\mathcal{F}}(y))$.

**Proof.** By induction on the maximum of $\text{rank}_{\mathcal{F}}(x)$ and $\text{rank}_{\mathcal{F}}(y)$. 

This last result implies that two compatible protosets have a finite set of minimal upper bounds.

8. **The Continuous Algebra $\mathcal{C}$ of Partial Sets**

A finitely branching apg $G$ can be interpreted as a system of $\Sigma, E$-equations. For each node $n$ of $G$, one has the equation

$$n = \sum_{m \rightarrow n} s(m).$$

The assertion of AFA that each apg has a unique decoration implies that each such system of equations has a unique set-theoretic solution. To find a connection between domain theory and non-well-founded sets, it is natural to turn to an initial continuous set algebra, which is a cpo in which such systems of equations have distinguished solutions. The solutions can be viewed as least fixed points of continuous functions. Generally, in an initial continuous algebra, the non-maximal elements are usually viewed as partial or incompletely defined elements. Hence we call the elements of an initial continuous set algebra *partial sets*. The main problem is now to describe an initial continuous set algebra in a tractable way. For this we use the initial preordered algebra of protosets.

An ideal of $\mathcal{F}$ is a subset $I$ of $\mathcal{F}$ with the following properties:

- $I$ is downward closed: if $x \subseteq y \in I$, then $x \in I$.
- $I$ is directed: every finite subset of $I$ has an upper bound in $I$. 


In particular, the second condition implies that an ideal is necessarily non-empty. For example, given any \( x \in \mathcal{F} \), the set
\[
\downarrow x = \{ y \in \mathcal{F} : y \subseteq x \}
\]
is an ideal. An ideal of this form is called a principal ideal. It is easy to see that if a finite subset of \( \mathcal{F} \) is an ideal, then it is a principal ideal. Conversely, Corollary 7.7 says that every principal ideal of \( \mathcal{F} \) is a finite ideal. For every increasing sequence
\[
x_0 \subseteq x_1 \subseteq \cdots \subseteq x_i \subseteq \cdots
\]
we have an ideal \( X = \bigcup_i \downarrow x_i \). Since \( \mathcal{F} \) is countable, every ideal of \( \mathcal{F} \) can be represented by some (not necessarily unique) increasing sequence.

Let \( \mathcal{C} \) be the set of ideals of \( \mathcal{F} \). We make \( \mathcal{C} \) into a preordered set algebra in the following way:
\[
e_\mathcal{C} = \downarrow e = \{ \bot, e \}
\]
\[
\bot_\mathcal{C} = \downarrow \bot = \{ \bot \}
\]
\[
s_\mathcal{C}(I) = \bigcup \{ \downarrow s(x) : x \in I \}
\]
\[
I +_\mathcal{C} J = \bigcup \{ \downarrow (x + y) : x \in I, y \in J \}.
\]
(The constants and operations appearing on the right side are those of \( \mathcal{F} \).) It should be checked that \( \mathcal{C} \) is closed under these operations, that the commutative monoid laws are satisfied, and that + is idempotent. Moreover, the inclusion relation is a partial order on \( \mathcal{C} \). The unique \( \subseteq \)-minimum ideal is \( \bot_\mathcal{C} \), and the operations of \( s_\mathcal{C} \) and \( +_\mathcal{C} \) are monotone.

By initiality of \( \mathcal{F} \), there is a unique homomorphism of preordered set algebras \( i : \mathcal{F} \rightarrow \mathcal{C} \). Since the map \( x \mapsto \downarrow x \) is a homomorphism of preordered set algebras, \( i(x) = \downarrow x \) for all protosets \( x \).

Let \( X \) be a cpo. An element \( k \in X \) is compact if for every directed subset \( D \) of \( X \), \( k \leq \bigvee D \) implies that there is some \( x \in D \) such that \( k \leq x \). The set of compact elements of \( X \) is denoted \( K(X) \). A cpo \( X \) is algebraic if, for every \( x \in X \), \( D = \{ k \in K(X) : k \leq x \} \) is a directed subset of \( X \), and \( x = \bigvee D \).

**Proposition 8.1.** \( \mathcal{C} \) is an algebraic cpo. The compact elements of \( \mathcal{C} \) are the principal ideals of \( \mathcal{F} \).

**Proof.** The union of any directed set of ideals is an ideal, so \( \subseteq \) is a complete partial order. Since \( \mathcal{C} \) has a minimum element \( \bot_\mathcal{C} \), \( \mathcal{C} \) is a cpo.

For any ideal \( I \) of \( \mathcal{F} \), \( \{ \downarrow x : x \in I \} \) is a directed subset of \( \mathcal{C} \), and \( I = \bigcup \{ \downarrow x : x \in I \} \). Hence we will know \( \mathcal{C} \) is an algebraic cpo once we check that \( K(\mathcal{C}) \) equals the set of principal ideals. Each principal ideal is
obviously compact. Conversely, if \( I \) is compact, then from \( I = \bigcup \{ \downarrow x : x \in I \} \), we see that for some \( y \in I \), \( I \subseteq \downarrow y \) and therefore \( I = \downarrow y \).

**Theorem 8.2.** \( \mathcal{C} \) is an initial continuous set algebra.

*Proof.* The operations \( s_\mathcal{C} \) and \( +_\mathcal{C} \) are continuous because their definitions employ \( \bigcup \). So \( \mathcal{C} \) is a continuous set algebra.

Let \( \mathcal{A} \) be a continuous set algebra, and let \( \varepsilon : \mathcal{F} \to \mathcal{A} \) be the unique homomorphism of preorder set algebras whose existence is guaranteed by the initiality of \( \mathcal{F} \). For each ideal \( I \) of \( \mathcal{F} \), the monotonicity of \( \varepsilon \) insures that \( \{ \varepsilon(x) : x \in I \} \) is a directed subset of \( \mathcal{A} \). Define \( \phi : \mathcal{C} \to \mathcal{A} \) by

\[
\phi(I) = \bigcup \{ \varepsilon(x) : x \in I \}.
\]

Using Proposition 8.1, it is a routine exercise to check that \( \phi \) is the unique homomorphism of continuous set algebras from \( \mathcal{C} \) to \( \mathcal{A} \).

\( \mathcal{C} \) is a set algebra, so we automatically have a membership relation \( \varepsilon_\mathcal{C} \) given by

\[
I \varepsilon \mathcal{C} J \quad \text{iff} \quad s(I) + J = J. \tag{4}
\]

This membership relation makes \( \mathcal{C} \) a system. That is, we put an edge \( I \to J \) between ideals \( I \) and \( J \) whenever \( J \varepsilon I \). We are interested in the set-theoretic properties of \( \mathcal{C} \) and of a particular subsystem \( \mathcal{D} \) that we define in the next section.

9. Ideals Associated to Graphs

Let \( G \in |\mathcal{B}| \). That is, \( G \) is a finitely branching apg. We associate to each node \( n \) of \( G \) a sequence \( \langle n_i : i \in \omega \rangle \) of elements of \( \mathcal{F} \). The sequences are defined by recursion on \( i \), for all nodes of \( G \). For all \( n \), set \( n_0 = \bot \). Given \( n_i \) for all \( n \in G \), we set

\[
n_{i+1} = \sum_{n \prec m} s(m_i). \tag{5}
\]

This last sum refers to the finitely many children of \( n \) in \( G \). The fact that we are working with finitely branching apgs is important here, since \( \mathcal{F} \) does not in general have infinite sums. By our conventions concerning empty sums, if \( n \) has no children in \( G \), then the sum is \( e \).

Note that we are suppressing the graph \( G \) in this notation. Fortunately, the underlying graph is usually clear from the context. Recall from Section 2 that, for an apg \( G \) and a node \( n \in |G| \), \( G^*n \) is the apg whose nodes
are those nodes of $G$ accessible from $n$. One point worth mentioning is that the sequence $\langle n_i : i \in \omega \rangle$ defined using $G$ is the same as the sequence $\langle n_i : i \in \omega \rangle$ defined using $G^n$.

**Proposition 9.1.** For each node $n$ of a finitely branching apg $G$, and for each $i \in \omega$, $n_i \subseteq n_{i+1}$.

**Proof.** By induction on $i$. For all $n$, $n_0 = \bot \subseteq n_1$. Assume that $n_i \subseteq n_{i+1}$ for all $n$. Then by the Structure Theorem, for all $n$,

$$n_{i+1} = \sum_{n \rightarrow m} s(m_i) \subseteq \sum_{n \rightarrow m} s(m_{i+1}) = n_{i+2}.$$

For each $n$, let $I_n = \bigcup_{i \in \omega} \downarrow n_i$. Then Proposition 9.1 shows that each $I_n$ is an ideal. Hence $I_n \in \mathcal{G}$. We will use the notation $I_{G,n}$ when we need to refer to the underlying graph containing $n$.

**Example.** Consider the apg $G$ with one node $n$ and an edge $n \rightarrow n$. This apg is a picture of the set $\Omega = \{0\}$. For all $i \geq 1$, $n_i = s'(1)$. Consequently, $I_n = \bigcup_i s'(1)$.

Consider next the case where $G$ has two nodes, $n$ and $m$, and two edges $n \rightarrow n$ and $n \rightarrow m$. Then $m_0 = \bot$, $m_1 = e$, and for all $i \geq 2$, $m_i = e$. Turning to $n$, we see that $n_0 = \bot$, $n_1 = s'(\bot)$ and $n_2 = s(s'(\bot)) + s(e)$. More generally, for all $i \geq 1$, $n_{i+1} = s(n_i) + s(e)$.

**Proposition 9.2.** Let $x$ be a well-founded, hereditarily finite set. Consider $x$ as a node of its canonical picture, the finitely branching apg $V^x$. Then, for all $i > \text{rank}(x)$, $x_i = \check{x}$. Therefore $I_x = \downarrow \check{x}$.

**Proof.** By $\varepsilon$-induction on $x$.

**Definition.** $\mathcal{D}$ is the subset of $\mathcal{G}$ consisting of all ideals of the form $I_{G, \text{top}^G}$ for $G \in |\mathcal{G}|$.

**Proposition 9.3.** $\mathcal{D}$ is subalgebra of the set algebra $\mathcal{G}$. Consequently, $\mathcal{D}$ is a subsystem of $\mathcal{G}$.

**Proof.** Check that $\mathcal{D}$ is closed under the operations $s$ and $+$ of $\mathcal{G}$.

There is a natural map $d: |\mathcal{G}| \rightarrow \mathcal{D}$ given by

$$d(G) = I_{G, \text{top}^G}. \quad (6)$$

In some sense, $\mathcal{D}$ is the system we are after. It contains all of the ideals $\downarrow a$ corresponding to the sets in $\text{HF}$, and it also contains the solutions of systems of equations that give us the non-well-founded sets.
THEOREM 9.4. For all \( n \in G \), \( I_n = \sum_{n \rightarrow m} s(I_m) \). Moreover, if \( \{ J_n : n \in G \} \) is any collection of ideals such that \( J_n = \sum_{n \rightarrow m} s(J_m) \), then \( I_n \subseteq J_n \) for all \( n \).

Proof. Transferring Eq. (5) from \( \mathcal{F} \) to \( \mathcal{G} \), we see that for all \( n \)

\[
\downarrow n_{i+1} = \sum_{n \rightarrow m} s(\downarrow m_i).
\]

Using the continuity of the operations, we have

\[
I_n = \bigcup_i \downarrow n_{i+1} = \bigcup_i \sum_{n \rightarrow m} s(\downarrow m_i) = \sum_{n \rightarrow m} s\left(\bigcup_i \downarrow m_i\right) = \sum_{n \rightarrow m} s(I_m).
\]

For the second statement, fix the ideals \( \{ J_n : n \in G \} \). An easy induction on \( i \) shows that, \( n_i \in J_n \) for all \( n \). It follows that for all \( n \), \( I_n \subseteq J_n \).

COROLLARY 9.5. If \( G \xrightarrow{\gamma} H \), then \( d(H) \in \mathcal{C}_d(G) \).

PROPOSITION 9.6. If \( \equiv \) is any bisimulation on \( \mathcal{G} \) and \( G \equiv H \), then \( d(G) = d(H) \).

Proof. We use induction on \( i \) to show that, for all \( G \equiv H \), \( (\text{top}_G)_i = (\text{top}_H)_i \). For \( i = 0 \), this is trivial. Assume this proposition for \( i \), and suppose that \( G \equiv H \). Regarding \( G \) and \( H \) as nodes of \( \mathcal{G} \), for each child \( G' \) of \( G \), there is some child \( H' \) of \( H \) such that \( G' \equiv H' \). That is, for each child \( n \) to \( \text{top}_G \), there is some child \( m \) of \( \text{top}_H \) such that \( G^n = H^m \). By the inductive hypothesis, \( \{ n_i : \text{top}_G \rightarrow n \} \subseteq \{ m_i : \text{top}_H \rightarrow m \} \). The reverse inclusion is proved similarly, so \( \{ n_i : \text{top}_G \rightarrow n \} = \{ m_i : \text{top}_H \rightarrow m \} \). Thus,

\[
(\text{top}_G)_{i+1} = \sum_{\text{top}_G \rightarrow n} s(n_i) = \sum_{\text{top}_H \rightarrow m} s(m_i) = (\text{top}_H)_{i+1}.
\]

LEMMA 9.7. Suppose that \( G \) and \( H \) belong to \( |\mathcal{G}| \), and suppose that for some \( i \in \omega \), \( n \in G \), and \( m \in H \), \( n_i \) and \( m_i \) are compatible. Then \( n_i = m_i \).

Proof. By induction on \( i \), using Corollary 7.10.

THEOREM 9.8. The map \( d: \mathcal{G} \rightarrow \mathcal{D} \) is a system map. Moreover, it is a strongly extensional quotient of \( \mathcal{G} \).

Proof. Let \( G \in |\mathcal{G}| \). By Corollary 9.5,

\[
\{ d(G') : G \xrightarrow{\gamma} G' \} \subseteq \{ I \in \mathcal{D} : d(G) \xrightarrow{\gamma} I \}.
\]

Going the other way, suppose that \( I \in \mathcal{D} \) is such that \( I \in \mathcal{C}_d(G) \). Fix some \( H \in |\mathcal{G}| \) and some \( m \in |H| \) such that \( I = I_{H,m} \). Then \( I_{H,m} \in \mathcal{C}_d(G) \), so
AMODELOFNON-WELL-FOUNDEDSETS

s(I_{H,m}) + d(G) = d(G). Fix i for a moment, and note that s(m_i) + ⊥ belongs to s(I_{H,m}) + d(G). By Theorem 9.4, \( d(G) = \sum_{\text{top} \to n} s(I_{G,n}) \). Thus, there are some child \( n \) of \( \text{top}_G \) and some \( j \) such that \( m_i \subseteq n_j \). But then \( m_i \) and \( n_i \) are compatible. It now follows from Lemma 9.7 that \( m_i = n_i \). Now this holds for all \( i \), and since \( \text{top}_G \) has only finitely many children, there is a fixed child \( n \) such that for all \( i \), \( m_i = n_i \). Therefore, \( I_{H,m} = I_{G,n} \). Note that \( G \rightarrow (G^*n) \), so by Corollary 9.5, \( d(G^*n) = I_{G,n} \in \mathcal{G} d(G) \). We have shown that \( \{ I \in D : d(G') \models G \} \subseteq \{ d(G') : G \rightarrow G' \} \). This completes the verification that \( d \) is a system map.

To say that \( d \) is a strongly extensional quotient just means that \( d[\mathcal{G}] = \mathcal{D} \), and that the equivalence relation \( \equiv_d \) it induces on \( \mathcal{G} \) is exactly the relation of bisimilarity. The equivalence relation induced by a system map is always a bisimulation. We know by Proposition 9.6 that bisimilar graphs are decorated the same way by \( d \). In other words, \( \equiv_d \) is contained in \( \equiv_d \). But \( \equiv_d \) is the maximal bisimulation on \( \mathcal{D} \), so the two bisimulations are the same.

We now come to the central result of the paper.

**Theorem 9.9.** Assuming AFA, \( \mathcal{D} \) is isomorphic to \( HF_1 \).

**Proof.** By AFA, there is a strongly extensional quotient \( e : \mathcal{G} \rightarrow \mathcal{V} \), and Theorem 2.1 implies that the restriction of \( e \) to \( \mathcal{D} \) is also a strongly extensional quotient. But \( d : \mathcal{G} \rightarrow \mathcal{D} \) is a strongly extensional quotient by Theorem 9.8, and so the image \( HF_1 = e[\mathcal{G}] \) and the image \( D = d[\mathcal{G}] \) are isomorphic systems.

We also have the following strengthening of Theorem 9.8.

**Theorem 9.10.** \( \mathcal{D} \) is a transitive subsystem of \( \mathcal{C} \). That is, if \( I \) is an ideal, \( G \in [\mathcal{D}] \), and \( I \in \mathcal{D} d(G) \), then \( I \in \mathcal{D} \). Therefore the inclusion \( i : \mathcal{D} \rightarrow \mathcal{C} \) is a system map, as is the composition \( i \circ d : \mathcal{G} \rightarrow \mathcal{C} \) which extends the codomain of \( d \) from \( \mathcal{D} \) to \( \mathcal{C} \).

**Proof.** Suppose that \( I \) is any ideal such that \( I \in \mathcal{D} d(G) \). We can write \( I \) as \( \bigcup_{i \in I} x_i \) for some increasing sequence \( \langle x_i : i \in \omega \rangle \). For each \( i \), there is some \( j \) such that \( s(x_i) \subseteq \sum_{\text{top} \to n} s(n_j) \). So for each \( i \) there is some \( j \) such that \( x_i \subseteq n_j \). Since \( G \) is finitely branching, there is a fixed child \( n \) of \( \text{top}_G \) such that each \( x_i \) belongs to \( I_n \). Thus \( I \subseteq I_n \).

Let \( m \) and \( p \) be children of \( \text{top}_G \) such that \( I_m \neq I_p \). By Lemma 9.7, there is some \( i \) such that \( m_i \) and \( p_i \) are incompatible. Since \( G \) is finitely branching, there is some \( k \) such that for all \( i \geq k \), and all children \( m \) and \( p \) of \( \text{top}_G \), if \( I_m \neq I_p \), then \( m_i \) and \( p_i \) are incompatible. Fix some \( i \geq k \). Then \( \sum_{\text{top} \to m} s(m_i) \) belongs to \( s(I) + \sum_{\text{top} \to m} s(I_m) \). So there is some \( x \in I \) and
some child $m$ of $\text{top}_G$ such that $m_i \subseteq x$. But then $m_i \in I \subseteq I_n$. Since $n_i \in I_n$ as well, $m_i$ and $n_i$ are compatible and hence equal. As a result, $n_i \in I$.

This argument holds for all $i \geq k$, and therefore $I_n \subseteq I$. So $I_n = I$. \]

10. The System $\mathcal{M}$ of Maximal Ideals

Let $\mathcal{M}$ be the subset of $\mathcal{G}$ consisting of the maximal ideals of $\mathcal{G}$. Our main results are a proof that $\mathcal{D} \subseteq \mathcal{M}$ and a characterization of $\mathcal{M}$ along the lines of the definition of $\mathcal{D}$, as well as a proof that the well-founded part of $\mathcal{G}$ consists of principal ideals. The main idea is an association of maximal ideals with the nodes of arbitrary accessible pointed graphs.

**Definition.** A protoset $x$ is canonical of level $i$ if for some node $n$ of a finitely branching apg $G$ and some $i$, $x = n_i$.

Note that if $x$ is a canonical protoset of level $i$, then $x$ might well have rank less than $i$. For example, $e$ is a canonical protoset of level $i$ for all $i > 1$.

**Proposition 10.1.** If $x$ is a canonical protoset of level $i$, then $\text{rank}(x) \leq i$. Therefore, for each $i$, there are only finitely many canonical protosets of level $i$. Moreover, let $X$ be a set of canonical protosets of level $i$. Then there exists a finitely branching apg $G$ such that \((\text{top}_G)_{i+1} = \sum_{x \in X} s(x)\).

**Proof.** An easy induction on $i$ shows that for all canonical protosets $x$ of level $i$, $\text{rank}(x) \leq i$. It follows that for each $i$, there are only finitely many canonical protosets of level $i$. Let $X$ be any set of such protosets. If $X$ is empty, let $G$ be an apg with exactly one node $n$ and no edges. Then $n_{i+1} = e = \sum_{x \in X} s(x)$. Otherwise, let $X = \{x_1, ..., x_k\}$. For $1 \leq j \leq k$, let $G_j$ be a finitely branching apg, and let $n_j \in |G_j|$ be such that $(n_j)_i = x_j$. Let $H$ be the disjoint union of $G_1, ..., G_k$, with a new vertex $m$. Let the edge relation $\rightarrow_H$ be the union of the edge relations $\rightarrow_{G_j}$ together with the pairs $m \rightarrow_H n_j$ for $1 \leq j \leq k$. Then $H$ is finitely branching, and $m_{i+1} = \sum_{x \in X} s(x)$. \]

Let $S$ be any small system, finitely branching or not. We do not require $S$ to be an apg with a specified top. We can define protosets $n_i$ by recursion on $i$ just as we did earlier, by $n_{i+1} = \sum_{m \rightarrow n_i} s(m)$. The question is whether the $n_i$'s so defined are protosets; in fact, Proposition 10.1 can be used to give an inductive proof that each $n_i$ is a canonical protoset of level $i$. We shall also denote by $I_n$ the ideal obtained in this way. If $G$ is an apg, then $G$ comes with a top, and we write $d(G)$ for $I_{\text{top}}$. 


**Lemma 10.2.** Let $G$ be any apg, and let $n \in |G|$, $i \in \omega$, and $x \in \mathcal{F}$. If $\operatorname{rank}(x) \leq i$, and $x$ is compatible with $n_{i+1}$, then $x \subseteq n_{i+1}$.

**Proof.** By induction on the protoset $x$. Let $z$ clarify both $x$ and $n_{i+1}$. Since $i+1 \geq 1$, $n_{i+1}$ and $z$ are both clear. For every $y \in x$, there is some $w \in z$ such that $y \subseteq w$, and thus there is some child $m$ of $n$ such that $y$ and $m_i$ are compatible. The induction hypothesis applies to $y$. Since $\operatorname{rank}(y) \leq i - 1$, $y \subseteq m_i$. If $x$ is murky, then (A) of the Structure Theorem shows that $x \subseteq n_{i+1}$. If $x$ is clear, then the same argument as above shows that each $m_i$ clarifies some $y \in x$. So once again, $x \subseteq n_{i+1}$. !

For the next result, call a subset $S$ of $\mathcal{F}$ pairwise compatible if for every $x$ and $y$ from $S$ there is some $z \in \mathcal{F}$ which clarifies both $x$ and $y$. This common clarification $z$ need not belong to $S$.

**Theorem 10.3.** If $G$ is an apg and $n$ is a node of $G$, then $I_n$ is a maximal ideal. In fact, each $I_n$ is a maximal member of the family of subsets $S$ of $\mathcal{F}$ which are pairwise compatible.

**Proof.** If $x$ is compatible with each element of $I_n$, then in particular, it is compatible with $n_i$, where $i = \operatorname{rank}(x) + 1$. So $x \subseteq n_i$ by Lemma 10.2. !

**Corollary 10.4.** If $G$ is a finitely branching apg, then $d(G)$ belongs to $\mathcal{M}$. Therefore $\mathcal{D}$ is a subset of $\mathcal{M}$.

As a corollary to this last result, we have the following strengthening of Theorem 9.4.

**Corollary 10.5.** Let $G$ be finitely branching. The set of ideals $\{I_n : n \in G\}$ has the property that for all $n$, $I_n = \sum_{m \leq n} s(I_m)$. Moreover, if $\{J_n : n \in G\}$ is any collection of ideals such that $J_n = \sum_{m \leq n} s(J_m)$, then for all $n$, $I_n = J_n$.

We would like to prove a converse to Theorem 10.3, namely that each maximal ideal of $\mathcal{C}$ comes from some node of some apg. In order to do this, we need some facts about the maximal ideals. For example, Theorem 10.6 below implies that every maximal ideal contains a clear protoset. In addition, it gives a construction which may be of independent interest.

**Theorem 10.6.** There exists a unique $I \in \mathcal{M}$ such that every murky protoset $x$ belongs to $I$. $I$ contains some clear protoset. Moreover, $I \notin \mathcal{D}$.

**Proof.** Call a protoset $x$ congenial if $x$ is clear, and if for all $y \in \mathcal{F}$ there is some $z \in x$ such that $y$ and $z$ are compatible. Every protoset of the form
s(⊥) + s(z) is congenial, and s(e) + s(⊥ + s(⊥)) is also congenial. Let 
I = \bigcup \{ x: x \text{ is congenial} \}.

As the first step in showing that I is as desired, we check that I is an 
ideal. Let v and w be congenial. Then every v' \in v is compatible with some 
w' \in w, and vice versa. Let

A = \{ u: \text{for some } v' \in v \text{ and } w' \in w, u \text{ is a minimal upper bound of } v' \text{ and } w' \}.

Then A is finite by Corollary 7.11. So x = \sum_{u \in A} s(u) is a clear protoset, and 
x clarifies v and w. We claim that x is congenial and therefore belongs to 
I. Let y \in \mathbb{F}. There is some v' \in v and some a \in \mathbb{F} such that 
y \subseteq a and v' \subseteq a. There is then some w' \in w and some b such that 
a \subseteq b and w' \subseteq b. Thus b in an upper bound of v' and w'. Again by Corollary 7.11, there 
is some minimal upper bound z of v' and w' such that z \subseteq b. So y is 
compatible with z \in x, and therefore x is congenial.

This proves that I is an ideal. If x is a murky protoset, then 
x \subseteq s(⊥) + \sum_{y \in x} s(y). Since this last protoset is congenial, we conclude 
that every murky protoset belongs to I.

Next, we show that if J is any ideal which contains every murky 
protoset, then every clear element x \in J is congenial. From this we conclude 
that I is maximal, and that it is the only maximal ideal containing each 
murky protoset. Let y \in \mathbb{F}. Since s(y) + ⊥ \in J, x and s(y) + ⊥ are 
compatible. Therefore some z \in x is compatible with y. Thus x is congenial. 
This proves that J \subseteq I.

Finally, we show that I \notin \mathcal{D} by showing that if G is finitely branching, then 
d(G) does not contain all the protosets of the form ⊥ + s(\bar{a}). For if each 
\bot + s(\bar{a}) did belong to \bigcup_{\text{top } G} s(I_{G,n}), then for every a \in \text{HF} there would be 
some child n of top G such that \bar{a} \in I_{G,n}. But top G has only finitely many 
children, so some I_{G,n} contains infinitely many \bar{a}. But if a \neq b, then \bar{a} and 
b are incompatible. This is a contradiction.

\textbf{Corollary 10.7.} Let I be a maximal ideal, and let x \in I be murky. Then 
there exists some clear y \in I such that x \subseteq y.

\textbf{Proof.} By Theorem 10.6, the ideal of all murky protosets is not 
maximal. Thus I contains some clear protoset u. Let y \in I be a common 
clarification of x and u.

Next, we need a closure property of the set of maximal ideals.

\textbf{Lemma 10.8.} Let I be a maximal ideal. Then I is a maximal pairwise 
compatible subset of \mathbb{F}.
Proof. Write $I$ as a limit of an increasing sequence from $\mathcal{F}$, $I = \bigcup_i x_i$, as in (5). We may assume that every $x_i$ is clear, since Corollary 10.7 shows that every maximal ideal contains a clear protoset. Furthermore, the result holds for the maximal ideals of the form $\downarrow a$ for $a \in HF$, since each $a$ is maximal. So we may assume that $\text{rank}(x_i) \geq i$ for all $i$.

Suppose that $y$ is compatible with each $x_i$. We construct an ideal $J \supseteq I$ such that $y \in J$. We construct a sequence of protosets

$$y = y_0 \supseteq y_1 \supseteq \cdots \supseteq y_j \supseteq \cdots$$

such that, for all $j$, $x_j \subseteq y_{j+1}$, and $y_j$ is compatible with each $x_i$. Let $y_0 = y$. Suppose we have $y_j$ with this property. For each $i > j$, let $u_i$ be such that $y_j \subseteq u_i$ and $x_i \subseteq u_i$. By Corollary 7.11, there is a protoset $v_i$ whose rank is the maximum of the ranks of $x_j$ and $y_j$ such that $y_j \subseteq v_i \subseteq u_i$ and $x_j \subseteq v_i$. There are only finitely many possible $v_i$, so for some fixed $v$ there are infinitely many $i$ such that $y_j \subseteq v \subseteq u_i$ and $x_j \subseteq v$. It follows that $v$ is compatible with infinitely many $x_i$, and hence it is compatible with all of them. Take $y_{j+1} = v$.

Now that we have the sequence $\langle y_j : j \in \omega \rangle$, let

$$J = I \cup \bigcup_j \downarrow y_j.$$ 

Clearly, $J$ is an ideal. Since $I$ is maximal, $I = J$. But $y \in J$, so $y \in I$.

The well-founded part of $\mathcal{C}$ is the set of ideals $I$ of $\mathcal{C}$ such that there is no infinite descending chain

$$I = I_0 \supseteq I_1 \supseteq \cdots \supseteq I_n \supseteq \cdots.$$ 

(7)

We know that $\mathcal{C}$ contains $\mathcal{D} \cong HF_1$ as a subsystem, so it is not well-founded. We characterize the well-founded part of $\mathcal{C}$ in Theorem 10.10 as exactly the principal ideals. Not only is this an interesting result in itself, it is also a crucial step toward our promised converse to Theorem 10.3.

Lemma 10.9. Let $I$ be an infinite ideal, and suppose that $I$ contains some clear protoset. Let $z \in I$ and let $y \in \mathcal{F}$ $z$. Then there exists an ideal $J$ such that $J \in \mathcal{F}$ $I$ and $y \in J$. Moreover, if we further assume that $y = \bot$, then there exists an infinite ideal $J$ such that $J \in \mathcal{F}$ $I$.

Proof. Let $I$ be an infinite ideal, let $w \in I$ be clear, let $z \in I$, and let $y \in \mathcal{F}$. We first fix a certain clear protoset $r \in I$, as follows: if $y = \bot$, then let $r = s(\bot)$. Otherwise, let $v \in I$ be such that $z \subseteq v$ and $w \subseteq v$. Note that $v$ is clear, and consider the protoset $r = v + \sum_{t \in z} s(t)$. Then $r$ is clear, $r \subseteq v$, so $r \in I$. Also $v \in r$. This defines $r$. Theorem 10.3.
Write \( I \) as in (5), as \( \bigcup_{i} \downarrow x_i \) for some strictly increasing sequence \( \langle x_i : i \in \omega \rangle \). We may assume that \( x_0 = r \). So every \( x_i \) is clear. An easy induction shows that \( \text{rank}(x_i) \geq i \) for all \( i \). Finally, we may assume that if \( i \leq j \), \( a \in x_i \) and \( b \in x_j \), then \( a \subseteq b \), and \( b \subseteq a \), then \( a = b \).

Consider the small system \( S \) defined as follows: Let \( |S| = \{ a \in \mathcal{F} : (\exists i) a \in x_i \} \). Note that \( y \) belongs to \( |S| \), since \( y \in x_0 \). \( |S| \) is infinite since \( I \) is an infinite ideal. Let the relation \( \rightarrow \) be defined on \( |S| \) by

\[
a \rightarrow b \text{ iff } \text{ for some } i, a \in x_i, b \in x_{i+1}, \text{ and } a \subseteq b.
\]

This defines a system \( S = \langle |S|, \rightarrow \rangle \). Since the relation \( \subseteq \) is transitive but irreflexive, \( S \) is acyclic.

We next show by induction on \( i \) that if \( v \in \mathcal{F} \setminus x_i \), then for some \( w \in x_0 \), \( v \) is accessible in \( S \) from \( w \). This is obvious for \( x_0 \). Assume this for \( x_i \), and consider the clear protoset \( x_{i+1} \). Let \( w \in x_{i+1} \). There exists some \( u \in x_i \) such that \( u \subseteq v \). If \( u \subseteq v \), then \( u \rightarrow v \). By the induction hypothesis, \( u \) is accessible from some \( w \in x_0 \), and \( v \) is accessible from the same \( w \). If \( v \subseteq u \), then \( v = u \) is accessible by induction hypothesis.

Note that if \( y = \bot \), then \( x_0 = r = s(\bot) \). So under this assumption, \( |S_y| \) is infinite.

Consider the small system \( S_y \). There are two cases, depending on the cardinality of \( |S_y| \). Suppose first that \( |S_y| \) is finite, so \( S_y \) contains some cardliness node. Let \( w \) be such a node. Let \( k \) be such that \( w \in x_k \). We claim that, for all \( i \geq k \), \( w \in x_i \). For if not, then there would be some \( v \in \mathcal{F} \setminus x_i \) such that \( u \subseteq v \), and thus \( w \rightarrow v \). So for every \( i \geq k \), \( s(w) + x_i = x_i \). It follows that \( s(\downarrow w) \setminus I = I \). Since \( w \in |S_y| \), \( w \) is accessible from \( y \), so \( y \subseteq w \). It follows that the ideal \( \downarrow w \) is as desired.

Henceforth we assume that \( |S_y| \) is infinite. We will construct an infinite ideal \( J \) such that \( J \cap \downarrow I \) and \( y \in J \). This will complete the proof of the lemma. By König’s Infinity Lemma, we have two cases: Either there exists an infinite chain in \( S_y \),

\[
y = a_0 \rightarrow a_1 \rightarrow \cdots \rightarrow a_n \rightarrow \cdots,
\]

or else some node \( a \in |S_y| \) has infinitely many children in \( S \).

If there is an infinite chain as in (8), then let \( J = \bigcup_{n} \downarrow a_n \). Then \( J \) is an infinite ideal, since \( \text{rank}(a_n) \geq n \). It is easy to check that \( J \subseteq I \), and we omit this verification. Note that \( y \in J \) in this case.

In the other case, fix some \( a \in |S_y| \) such that \( a \) has infinitely many children. In particular, \( \uparrow a \cap |S_y| \) is infinite. Since each \( x_i \) is finite, there are infinitely many \( i \) such that for some \( b \in x_{i+1} \), \( a \rightarrow b \). Thus there are infinitely many \( i \) such that \( a \in x_i \).

Now we define protosets

\[
a = c_0 \subseteq c_1 \subseteq \cdots \subseteq c_n \subseteq \cdots
\]
such that rank(c_i) = rank(a) + i and \( \uparrow (c_i) \cap |Sy| \) is infinite. Let \( c_0 = a \). Given \( c_i \), let \( j = \text{rank}(a) + i + 1 \). Note that \( |Sy| \) has infinitely many elements of rank greater than \( j \). By Corollary 7.9, there exists some \( c_{i+1} \supseteq c_i \) of rank exactly \( j \) such that \( \uparrow (c_{i+1}) \cap |Sy| \) is infinite.

Let \( J = \bigcup_n c_n \). Note that \( J \) is an infinite ideal, since the \( c_i \) are of unbounded rank. We show that \( s(J) + I = I \).

Consider some \( x_i \). Since \( a \) belongs to infinitely many \( x_j \), there is some \( j \geq i \) such that \( a \in x_j \). Then \( x_i \subseteq x_j = s(a) + x_j \). This shows that each \( x_i \) belongs to \( s(J) + I \). Thus \( I \subseteq s(J) + I \).

Going the other way, consider some \( c_i \), and some \( x_m \). Since \( \uparrow (c_i) \cap |Sy| \) is infinite, there is some \( n \geq m \) and some \( b \in x_n \) such that \( c_i \subseteq b \). Then \( s(c_i) + x_n \subseteq s(b) + x_n = x_n \). This last protoset belongs to \( I \), so \( s(J) + I \subseteq I \). This shows that \( J \in G \). Finally, recall that \( a \in |Sy| \). Thus \( y \subseteq a \), and therefore \( y \in J \). \( \square \)

**Theorem 10.10.** The well-founded part of \( G \) is the set of principal ideals of \( \mathcal{F} \).

**Proof.** First, we observe that \( \mathcal{F} \) is well-founded below each principal ideal. To see this, suppose that \( I \in \mathcal{F} \downarrow x \). Then \( s(I) + \downarrow x = \downarrow x \). So for every \( y \in I \), there is some \( z \in \mathcal{F} x \) such that \( y \leq z \). In particular, \( \text{rank}(y) \leq \text{rank}(z) < \text{rank}(x) \). Thus, \( I \) is a finite ideal, and is therefore of the form \( \downarrow y \) for some \( y \) of smaller rank than \( x \). This implies that \( \mathcal{F} \) has no infinite descending chain below \( \downarrow x \).

For the other direction, let \( I \) be an infinite ideal. \( I \) is the union of the lower sets of a strictly increasing sequence

\[
x_0 \subseteq x_1 \subseteq \cdots \subseteq x_i \subseteq \cdots.
\]

Note that \( \text{rank}(x_i) \geq i \) for all \( i \). We only need to construct an infinite ideal \( J \in \mathcal{F} I \), as this implies that there is an infinite descending chain as in (7). If \( I \) contains a clear element, then we apply Lemma 10.9, where we take \( s(\perp) \) for \( x \) and \( \perp \) for \( y \). We henceforth assume that every element of \( I \) is murky.

We construct an infinite \( J \in \mathcal{F} I \). (Our proof shows that \( J \) belongs to every infinite \( I \) which contains only murky protosets.) Recall the sequence of protosets \( \{ z_i : i \in \omega \} \) from Proposition 7.8. Let \( J = \bigcup_i \downarrow z_i \). Note that \( J \) is an infinite ideal, since \( \text{rank}(z_i) = i \).

We show that \( s(J) + I = I \). First, let \( a \in s(J) + I \). Then for some \( i \) and \( j \),

\[
a \subseteq s(z_i) + x_j \subseteq s(z_k) + x_{k+1},
\]

where \( k = \max\{i, j\} \). But \( x_{k+1} \) has an element \( z \) of rank at least \( i \). Thus
\[ a \subseteq s(z_k) + x_{k+1} \subseteq x_{k+1}, \] so \( a \in I. \) We have shown that \( s(J) + I \subseteq I. \) The converse is obvious.

This result implies the following Induction Principle for \( \mathcal{C}: \) Suppose \( \phi(I) \) is a property of partial sets, and suppose that

1. If \( \phi(J) \) for all \( J \in \mathcal{C} I, \) then \( \phi(I). \)
2. If \( X \) is a directed subset of \( \mathcal{C} \) and \( \phi(J) \) for all \( J \in X, \) then \( \phi(\bigcup X). \)

Then for all \( I \in \mathcal{C}, \) \( \phi(I). \)

**Lemma 10.11.** Let \( J \) be a maximal ideal, \( y \in J, \) and \( z \in y. \) Then there is a maximal ideal \( K \) such that \( K \in \mathcal{C}_J \) and \( z \in K. \)

**Proof.** We first consider the case where \( J \) is finite. For some \( a \in H_{\mathcal{C}}, \)
\[ J = \downarrow a. \] Let \( z \in \mathcal{C}_J \subseteq \downarrow a. \) Recall that \( a = \sum_{b \in a} s(b). \) For some \( b \in a, \) \( z \subseteq b. \) Finally, \( \downarrow b \) is a maximal ideal, \( \downarrow b \subseteq \downarrow a, \) and \( z \subseteq \downarrow b. \)

Now we turn to the case where \( J \) is infinite. By Corollary 10.7, \( J \) contains a clear protoset. By Lemma 10.9, there is some ideal \( K \) such that \( K \in \mathcal{C}_J \) and \( z \in K. \) By Zorn's Lemma, there is a maximal ideal \( K' \supseteq K. \) Then \( J = s(K) + J \subseteq s(K') + J. \) By maximality, \( J = s(K') + J, \) so \( K' \in \mathcal{C}_J. \)

At long last, we have the converse of Theorem 10.3.

**Theorem 10.12.** Let \( J \in \mathcal{M}. \) Then there exists some apg \( G \) such that \( d(G) = J. \)

**Proof.** Consider \( \mathcal{M} \) as an induced subsystem of \( \mathcal{C}. \) That is, consider the system \( \mathcal{M} \) whose nodes are the maximal ideals, and where \( I \rightarrow \mathcal{M} J \) iff \( I \rightarrow \mathcal{C} J. \) For each \( J \in \mathcal{C}, \) consider the protosets \( J_i. \) We show

\[
\text{For all } J \in \mathcal{M} \text{ and all } i \in \omega, \quad J_i \in J. \quad (9)
\]

It follows from this that for all \( J \in \mathcal{M}, \)
\[
I_J = \bigcup_i \downarrow J_i \subseteq J.
\]

But \( I_J \) is maximal by Theorem 10.3, so we indeed see that \( I_J = J. \)

We prove (9) by induction on \( i. \) For \( i = 0, \) \( J_0 = \downarrow J \) for all \( J. \) Suppose (9) for some \( i, \) and let \( J \in \mathcal{M}. \) We claim that the protoset \( J_{i+1} = \sum \{ s(K_i): J \rightarrow \mathcal{C} K \} \) belongs to \( J. \) By Lemma 10.8, it is sufficient to show that \( J_{i+1} \) is compatible with each \( x \in J. \) By Corollary 10.7, we need only check that \( J_{i+1} \) is compatible with each clear \( y \in J. \)

Fix some clear \( y \in J. \) Let \( z \in y. \) Then by Lemma 10.9, there exists some maximal ideal \( K \) such that \( J \rightarrow \mathcal{C} K \) and \( z \in K. \) By induction hypothesis, \( K_i \in K. \) So \( z \) is compatible with some \( K_i. \) This holds for all \( z \in y. \)
Going the other way, fix some \( K \) such that \( J \rightarrow K \). Since \( y \in J = s(K) + J \), there is some \( u \in K \) and some \( z \in y \) such that \( z \subseteq u \). But \( K_i \subseteq K \) by induction hypothesis, so \( z \) and \( K_i \) are compatible. This holds for all children \( K \) of \( J \) in \( \mathcal{C} \).

The last two paragraphs, together with Corollary 7.10, show that \( y \) is compatible with \( J_{i+1} = \sum \{ s(K_i) : J \rightarrow K \} \).

**Corollary 10.13.** \( \mathcal{M} \) is subalgebra of the set algebra \( \mathcal{C} \). \( \mathcal{M} \) is a transitive subsystem of \( \mathcal{C} \). That is, if \( I \) is a maximal ideal and \( J \) is any ideal such that \( J \in \preceq I \), then \( J \in \mathcal{M} \).

**Proof.** Theorems 10.3 and 10.12 imply that that \( \mathcal{M} \) is closed under the operations \( s \) and \( + \) of \( \mathcal{C} \).

Suppose that \( J \in \preceq I \) and \( I \) is maximal. Let \( x \) be compatible with every element of \( J \); we show that \( x \in J \). Let \( i = \text{rank}(x) \). By Theorem 10.12, \( I \) contains a canonical protoset of level \( i + 2 \), say \( n_{i+2} \). Since \( s(J) + I = I \), there is some \( y \in J \) and some canonical \( m_{i+1} \in n_{i+2} \) such that \( m_{i+1} \subseteq y \). Thus \( m_{i+1} \in J \). But \( x \) and \( m_{i+1} \) are compatible, so \( x \subseteq m_{i+1} \) by Lemma 10.2. Thus \( x \in J \).

Since \( \mathcal{M} \) is a set algebra, we might wonder about the properties of the membership relation \( \in \mathcal{M} \).

**Proposition 10.14.** There exists an ideal \( I \in \mathcal{M} \) such that for all \( J \in \mathcal{M} \), \( J \in \preceq I \). So \( \mathcal{M} \) is a model of the "universal set" axiom: \( (\exists x)(\forall y) \ (y \in x) \).

**Proof.** Let \( I \) be the ideal from Theorem 10.6. Let \( J \) be an arbitrary maximal ideal, and consider the ideal \( s(J) + I \). Every murky protoset belongs to \( I \) and hence to \( s(J) + I \). By Corollary 10.13, \( s(J) + I \) is a maximal ideal. So by the uniqueness assertion of Theorem 10.6, \( s(J) + I = I \). Thus \( J \in \preceq I \).

11. **A Domain Equation for \( \mathcal{C} \)**

In this section, we present some material on the structure of \( \mathcal{C} \) as a domain. We first show that \( \mathcal{C} \) is an SFP-object (Corollary 11.7), which means that \( \mathcal{C} \) is an inverse limit of finite posets under embedding-projection pairs. Then we investigate the structure of \( \mathcal{C} \) as a domain, and we show that \( \mathcal{C} \) is a solution of the domain equation:

\[
D \simeq 1 + \mathcal{P}_\text{pl}(D),
\]

the separated sum of the one-point domain and the Plotkin powerdomain over \( D \).

We write \( \mathcal{HF} \) for the set \( \{ \bar{a} : a \in \mathcal{HF} \} \), and \( \mathcal{HF} \) for \( \{ \downarrow \bar{a} : a \in \mathcal{HF} \} \).
Throughout this section we will use the notation and terminology of (Gierz et al., 1980); the reader should consult, in particular, Chap. 0 of that work for those terms or notations which we do not explicitly define.

We already know that \( \mathcal{C} \) is an algebraic cpo, and the map \( x \mapsto \downarrow x: \mathcal{F} \to \mathcal{C} \) takes \( \mathcal{F} \) onto the lower set \( K(\mathcal{C}) \) of the compact elements of \( \mathcal{C} \). We exploit these facts in investigating the structure of \( \mathcal{C} \) and we begin with the following observation.

**Theorem 11.1.** \( \mathcal{C} \) is not a local lattice; i.e., there is an element \( \mathcal{Z} \in \mathcal{F} \) and there are pairwise inequivalent elements \( x, y, a, \) and \( b \) in the lower set of \( \mathcal{Z} \) such that \( x \) and \( y \) are minimal upper bounds in \( \mathcal{F} \) of \( a \) and \( b \), and \( a \) and \( b \) are maximal lower bounds in \( \mathcal{F} \) of \( x \) and \( y \).

**Proof.** We sketch the argument, omitting a great number of tedious verifications. First, we say that a protoset \( y \) covers \( x \) if \( x \subseteq y \), and whenever \( x \subseteq z \subseteq y \), either \( x = z \) or \( y = z \). Now it is an easy consequence of the Structure Theorem that if \( y \) covers \( x \), then \( s(y) \) covers \( s(x) \) in \( s(y) \), and \( s(x) + s(y) \) covers \( s(x) \).

Let \( y_0 = s(e) \) and \( x_0 = s(e) + s(\bot) \). Given \( x_n \) and \( y_n \), let \( y_{n+1} = s(y_n) \), and \( x_{n+1} = s(x_n) + s(y_n) \). An induction on \( n \) shows that \( y_n \) covers \( x_n \). Moreover, \( y_n \) and \( y_m \) are incompatible for \( n \neq m \), since they are distinct elements of \( \mathcal{H} \).

Let \( P \) be the set \( \mathcal{P}(\{0, 1, 2, 3\}) \) ordered by inclusion. Let \( \phi: P \to \mathcal{F} \) be given by

\[
\phi(S) = \sum_{i \in S} s(x_i) + \sum_{i \notin S} s(y_i).
\]

This map \( \phi \) has the property that \( S \subseteq T \) iff \( \phi(S) \subseteq \phi(T) \). Moreover, \( \phi(P) \) is a lattice in the sense that \( \phi(S \cup T) \) is the unique least upper bound in \( \mathcal{F} \) of \( \phi(S) \) and \( \phi(T) \). Similarly, \( \phi(S \cap T) \) is the greatest lower bound.

Note that there is nothing special about \( \phi \) here. We could just as well start with \( P \) as \( \mathcal{P}(\{0, 1, \ldots, n\}) \) for any \( n \). In this way, we get an isomorphic embedding of every finite boolean algebra into \( \mathcal{F} \).

Here are the five elements \( \mathcal{Z}, x, y, a, \) and \( b \) mentioned in the statement of the theorem:

\[
\begin{align*}
z &= s(\phi(\{0, 1, 2, 3\})) \\
x &= s(\phi(\{0, 3\})) + s(\phi(\{1, 2\})) \\
y &= s(\phi(\{0, 2\})) + s(\phi(\{1, 3\})) \\
a &= s(\phi(\{0\})) + s(\phi(\{1\})) \\
b &= s(\phi(\{2\})) + s(\phi(\{3\})).
\end{align*}
\]
Now we turn to a discussion of the Scott topology on $\mathcal{C}$. It has as a basis the sets
$$U_x = \{ I \in \mathcal{C} : \downarrow x \subseteq I \}$$
for $x \in \mathcal{F}$. Corollary 7.5 implies that the open sets of $\mathcal{C}$ of the form $U_a$ for $a \in HF$ are singletons. Theorem 11.5 below and the fact that $\gamma$ is one-to-one implies that these are the only open singletons.

**Proposition 11.2.** Let $\psi : \mathcal{F} \rightarrow \mathcal{F}$ be the unique set algebra homomorphism such that $\psi(\bot) = e$. For all $x \in \mathcal{F}$, $x \subseteq \psi(x)$, and $\psi(x) \in HF$. Hence $\mathcal{H}\mathcal{F}$ is a dense subset of $\mathcal{C}$.

**Proof.** The first result is proved by induction on rank($x$). Since Scott-open sets are upper sets and a basis for the Scott-open sets is $\{ U_x : x \in \mathcal{F} \}$, the second result follows.

Let $X_0 = \{ \bot \}$, and for each $n \geq 0$, let $X_{n+1} = \{ s(x) + \tilde{a} : x \in X_n \quad \& \quad a \in HF \}$. Then $X = \bigcup_n X_n$ is the smallest subset of $\mathcal{F}$ containing $\bot$ such that if $x \in X$ and $a \in HF$, then $s(x) + \tilde{a} \in X$.

**Proposition 11.3.** $\{ \downarrow x : x \in X \}$ is a dense subset of $\mathcal{C} - \mathcal{H}\mathcal{F}$.

**Proof.** We first show by induction on $y \in \mathcal{F} - \mathcal{HF}$ that there is some $x \in X$ such that $y \subseteq x$. On one hand, suppose $y$ is clear. Then for some $w \in y$, $w \notin HF$. By the induction hypothesis, there is some $z \in X$ such that $w \subseteq z$. But then
$$y = s(w) + \frac{\sum_{v \in y} s(v) \subseteq s(z) + \sum_{w \notin v \in y} s(\psi(v))},$$
where $\psi : \mathcal{F} \rightarrow \mathcal{F}$ is the map from Proposition 11.2. Since $HF$ is closed under $s$ and $+$, the last large summation is an element of $HF$. So, if $y$ is clear, then $y$ is clarified by an element of $X$. On the other hand, if $y$ is murky, then $y = \bot + y \subseteq s(\bot) + \sum_{v \in y} s(\psi(v))$, and this last element again belongs to $X$.

To complete the proof, let $I \in \mathcal{C} - \mathcal{H}\mathcal{F}$, and let $U$ be a Scott-open subset of $\mathcal{C}$ containing $I$. Then there is some $y \in \mathcal{F}$ with $I \subseteq U_y \subseteq U$. Since the elements of $HF$ are maximal in $\mathcal{F}$, the elements of $\mathcal{H}\mathcal{F}$ are maximal ideals of $\mathcal{F}$. Since $I \notin \mathcal{H}\mathcal{F}$, $y \notin HF$, for otherwise $I = \downarrow y \in \mathcal{H}\mathcal{F}$. So what we just showed above implies there is some element $x \in X$ with $y \subseteq x$. Then $\downarrow x \in U_y$, and so $\{ \downarrow x : x \in X \} \cap U \neq \emptyset$, which proves the claim.

**Proposition 11.4.** For every $x \in X$ there are incompatible protosets $y$ and $z$ in $\mathcal{F} - \mathcal{HF}$ such that $x \subseteq y$ and $x \subseteq z$. 
Proof. We have to prove something stronger in order to have a workable induction hypothesis. By induction on the least integer \( k \) such that \( x \in X_k \), we will show that every \( x \in X \) has incompatible clarifications \( y_{n,x} \) and \( z_{n,x} \) of rank \( > n \) for all \( n \geq \text{rank}(x) \).

We start with the case \( k = 0 \). Thus \( x = \bot \). Let \( y_{0,\bot} = s(s(\bot)) \), and let \( z_{0,\bot} = s(\bot) + s(e) \). To see that these are incompatible, we will use Corollary 7.10. First, \( s(\bot) \) and \( e \) are incompatible. It follows that \( s(s(\bot)) \) and \( s(\bot) + s(e) \) are incompatible. Now that we have \( y_{0,\bot} \) and \( z_{0,\bot} \), we define \( y_{n+1,\bot} = s(y_{n,\bot}) \) and \( z_{n+1,\bot} = s(z_{n,\bot}) \). These are also incompatible elements of \( \mathcal{F} - \text{HF} \), and their ranks are \( n + 1 \).

Now assume \( r \in X_{k+1} \), so \( r = s(x) + a \), where \( x \in X_k \) and \( a \in \text{HF} \). Let \( n \geq \text{rank}(x) \), so that also \( n \geq \text{rank}(x) \). By the induction hypothesis, there exist incomparable clarifications \( y_{n,r} \) and \( z_{n,r} \) of ranks greater than \( n \). Let \( y_{n,r} = s(y_{n-1,x}) + a \). Similarly, let \( z_{n,r} = s(z_{n-1,x}) + a \). The induction hypothesis tells us that \( r \subseteq y_{n,r} \) and similarly for \( z \). We check that \( y_{n,r} \) and \( z_{n,r} \) are incompatible. Suppose not. Then by Corollary 7.10, either \( y_{n-1,x} \) is compatible with \( z_{n-1,x} \), there is some \( b \in a \) such that \( y_{n-1,x} \) is compatible with \( b \). The first alternative is impossible, by the induction hypothesis. Each \( b \) is maximal in \( \mathcal{F} \) by Corollary 7.5. So, for some \( b \in a \), \( y_{n-1,x} \subseteq b \). By Corollary 7.6,

\[
n - 1 < \text{rank}(y_{n-1,x}) \leq \text{rank}(b) < \text{rank}(a) \leq n.
\]

This contradiction shows that \( y_{n,r} \) and \( z_{n,r} \) are incompatible.

Note that the set \( \mathcal{C} - \mathcal{HF} = \mathcal{C} - \bigcup \{ U_a : a \in \text{HF} \} \) is a closed subspace of \( \mathcal{C} \).

Theorem 11.5. \( \mathcal{C} - \mathcal{HF} \) has no isolated points. In fact, every neighborhood of every \( I \in \mathcal{C} - \mathcal{HF} \) contains a maximal ideal \( J \in \mathcal{C} - \mathcal{HF} \) such that \( J \neq I \).

Proof. Suppose that \( I \in \mathcal{C} - \mathcal{HF} \), and let \( U \) be an open set containing \( I \). Let \( x \in \mathcal{F} \) be such that \( I \in U_x \). Then \( x \notin \mathcal{HF} \). By Propositions 11.3 and 11.4, there exist incompatible clarifications \( y \) and \( z \) of \( x \) in \( \mathcal{F} - \mathcal{HF} \). Then the ideals \( \downarrow y \) and \( \downarrow z \) are incompatible in \( \mathcal{C} \), so either \( \downarrow y \not\subseteq I \) or \( \downarrow z \not\subseteq I \).

Suppose the first alternative holds. We only need to show how to extend \( \downarrow y \) to a maximal ideal \( J \in \mathcal{C} - \mathcal{HF} \); then automatically \( J \in U_x \). By repeated use of Proposition 11.4, there is a strictly increasing sequence

\[
y = y_0 \subset \cdots \subset y_n \subset \cdots
\]

above \( y \). Let \( J_0 = \bigcup_n \downarrow y_n \). Then \( J_0 \) is an ideal, so by Zorn's Lemma, it has
A MODEL OF NON-WELL-FOUNDED SETS

47

a maximal extension $J$. Since $J_0$ is infinite, so is $J$ and $J \in \mathcal{C} - \mathcal{H} \mathcal{F}$ because every ideal of the form $\downarrow \hat{a}$ is finite.

We now show that $\mathcal{C}$ is an SFP-object, and hence that the $\lambda$-topology on $\mathcal{C}$ is compact. The basic open sets in this topology are those of the form

$$U_x = \left( \bigcup_{y \in B} U_y \right),$$

where $x \in \mathcal{F}$ and $B$ is a finite subset of $\mathcal{F}$. Rather than confine our discussion to $\mathcal{C}$, we present some general results about SFP-objects which we have not been able to locate elsewhere in the literature, and we give the application of these results to $\mathcal{C}$ as corollaries of the general results.

If $D$ is an algebraic cpo, then the $\lambda$-topology on $D$ is defined as having a basis of sets

$$\uparrow k = \{ x \in D : k \leq x \text{ and } y \not\leq x \forall y \in F \},$$

where $k \in K(D)$ and $F \subseteq K(D)$ is a finite set. This topology is always Hausdorff: given $x \neq y \in D$, there is a compact element $k$ in $D$ below one of $x$ and $y$ but not the other. Then $\uparrow k$ and $D - \uparrow k$ are disjoint $\lambda$-open sets around $x$ and $y$. It is not always true that the $\lambda$-topology is compact. But, as we show below, it is true that the $\lambda$-topology on an SFP-object is compact.

First, given a pair of algebraic cpo’s $D$ and $E$, an embedding-projection pair $\langle e, p \rangle$ between $D$ and $E$ is a pair of Scott-continuous maps $e : D \to E$ and $p : E \to D$ satisfying $p \circ e = 1_D$ and $e \circ p \leq 1_E$. By a sequence of finite posets, we mean a sequence $D_n$ of finite posets and embedding-projection pairs $e_{mn} : D_m \to D_n$ and $p_{mn} : D_n \to D_m$, for each pair $m \leq n$, satisfying the property that $e_{nk} \circ e_{mn} = e_{mk}$ and $p_{mn} \circ p_{nk} = p_{mk}$ for $m \leq n \leq k$. An SFP-object is an algebraic cpo $D$ for which there is a sequence $D_n$ of finite posets and embedding-projection pairs $e_n : D_n \to D$ and $p_n : D \to D_n$ with $p_n \circ e_n = 1_D$ and $e_n \circ p_n \leq 1_D$ for each $n$, $1_D = \bigvee_n (e_n \circ p_n)$, and $p_m = p_{mn} \circ p_n$ and $e_m = e_n \circ e_{mn}$ for all $m \leq n$. If we forget the embeddings, the SFP-object $D$ is the inverse limit in the category of cpo’s of the inverse system $\langle D_n, p_{mn} : D_n \to D_m \rangle_{m \leq n}$.

**Theorem 11.6.** Let $D$ be an SFP-object, and let $D_n$ be the sequence of finite posets with $e_n : D_n \to D$ and $p_n : D \to D_n$ the embedding-projection pairs. Then the $\lambda$-topology on $D$ is the topology $D$ inherits from the natural inclusion map $e : D \to \prod_n D_n$, where we endow $\prod_n D_n$ with the Tychonoff topology. Consequently, the $\lambda$-topology on $D$ is compact and Hausdorff.

**Proof.** Since $D$ is the inverse limit of the $D_n$’s, the corestriction of the
map \( e: D \to \prod_n D_n \) given by \( e(d) = (p_n(d))_n \) is a bijection from \( D \) onto the set \( \{(x_n) \in \prod_n D_n : p_{mn}(x_n) = x_m \} \).

Fix \( n > 0 \). Since \( \langle e_n, p_n \rangle \) is an embedding-projection pair, it follows that \( e_n(x) = \min \{ y \in D : p_n(y) = x \} \) for each \( x \in D_n \). So, \( e_n(x) = p_n^{-1}(\uparrow x) \) for each \( x \in D_n \), and since \( p_n \) is Scott-continuous, we conclude that \( \uparrow e_n(x) \) is Scott-open in \( D \). That is, \( e_n(x) \in K(D) \) for each \( x \in D_n \) and each \( n \). Conversely, given a compact element \( k \in K(D) \), \( k = \bigvee_n e_n(p_n(k)) \), so there is some \( n \) with \( k = e_n(p_n(k)) \). Thus, \( K(D) = \bigcup_n e_n(D_n) \).

Now, again fixing \( n \), \( D_n \) is finite, so for a fixed element \( x \in D_n \), there are finitely many elements \( \{ y_1, \ldots, y_m \} \subseteq \uparrow x - \{ x \} \) such that \( \uparrow x = \{ x \} \cup \bigcup_{1 \leq i \leq m} \uparrow y_i \). Then, \( p_n^{-1}(x) - \uparrow e_n(x) - (\bigcup_{1 \leq i \leq m} \uparrow e_n(y_i)) \). Since \( e_n(D_n) \subseteq K(D) \), this last set is a basic open set in the \( \lambda \)-topology of \( D \). This shows that the map \( p_n: D \to D_n \) is \( \lambda \)-continuous, and so \( e \) is continuous from \( D \) with the \( \lambda \)-topology onto \( e(D) \) with the inherited Tychonoff topology from \( \prod_n D_n \).

On the other hand, each basic \( \lambda \)-open subset of \( D \) has the form \( \uparrow k - \uparrow F \), where \( k \in K(D) \) and \( F \subseteq K(D) \) is finite. Then, there is some \( n \) so that \( k \in e_n(K(D_n)) \) and \( F \subseteq e_n(K(D_n)) \). It is then easy to show that \( \uparrow k - \uparrow F = p_n^{-1}(k) \). Thus, the basic \( \lambda \)-open subsets of \( D \) all have the form \( p_n^{-1}(x) \), for some \( x \in D_n \) and some \( n \). Since the \( \lambda \)-topology on the finite poset \( D_n \) is the discrete topology, this implies that the image of each basic \( \lambda \)-open subset of \( D \) under the map \( e \) is open in the inherited Tychonoff topology on \( e(D) \), and so \( e \) is homeomorphism.

Finally, since \( e(D) \) is closed in the compact space \( \prod_n D_n \), this topology is compact and Hausdorff.

Of course, our interest is in the algebraic cpo \( \mathcal{C} \). For each \( n \), the poset \( \mathcal{C}_n = \{ \downarrow x : x \in \mathcal{F}_n \} \) is finite (since \( \mathcal{F}_n \) is finite). We now show that \( \mathcal{C} \) is an SFP-object using the sequence \( \langle \mathcal{C}_n \rangle_n \).

**Corollary 11.7.** \( \mathcal{C} \) is an SFP-object. In fact, \( \mathcal{C} \) is an inverse limit of the sequence of finite posets \( \langle \mathcal{C}_n \rangle_n \). Consequently the \( \lambda \)-topology on \( \mathcal{C} \) is compact.

**Proof.** For each \( n \), \( \mathcal{F}_n \) has only finitely many elements, and so the same is true of \( \mathcal{C}_n = \{ \downarrow x : x \in \mathcal{F}_n \} \). If \( n > 0 \) and \( I \in \mathcal{C} \), then the set \( \{ \downarrow x : x \in I \cap \mathcal{F}_n \} \) is a finite set of compatible elements (they are all in \( I! \)), so Corollary 7.11 implies this set has a largest element. This implies that the inclusion \( \iota_n: \mathcal{C}_n \to \mathcal{C} \) has an upper adjoint

\[
\pi_n(I) = \bigcup \{ \downarrow x : x \in I \cap \mathcal{F}_n \}
\]

which is a Scott-continuous projection. Thus \( \mathcal{C} \) is the inverse limit of the family \( \{ \mathcal{C}_n : n > 0 \} \) under embedding-projection pairs \( \langle \iota_n, \pi_n \rangle \). It
is clear that, for \( m < n \), \( \mathcal{C}_m \subseteq \mathcal{C}_n \), and that there is a natural projection map \( \pi_{mn}: \mathcal{C}_n \to \mathcal{C}_m \). Thus, \( \mathcal{C} \) is SFP. The last claim then follows from Theorem 11.6.

Our next goal is to show that the family \( \mathcal{M} \) of maximal ideals of \( \mathcal{C} \) is closed in the \( \lambda \)-topology. In fact, we will show more. We show that there is a canonical ultrametric on \( \mathcal{C} \), and that \( \mathcal{M} \) is the completion of \( \mathcal{K} \mathcal{F} \) in this metric. This rounds out our description of the relationship between \( \mathcal{K} \mathcal{F} \), \( \mathcal{D} \), \( \mathcal{M} \) and \( \mathcal{C} \), which was begun in Sections 8, 9, and 10. Once again, though, we present these results in the setting of SFP-objects in general, and then apply them to \( \mathcal{C} \).

If \( D \) is an SFP-object, then we showed in the proof of Corollary 11.7 that \( K(D) = \bigcup e_n(D_n) \). Now each \( D_n \) is finite, and so we can endow \( D_n \) with the discrete metric. Using this metric, we can then endow \( D \) with a metric in two ways. First, there is the inherited metric from the Frechet metric on \( \prod D_n \); this metric is defined on \( \prod D_n \) by \( d((x_n), (y_n)) = \sum d(x_n, y_n)/2^n \) (where \( d \) is the discrete metric on each \( D_n \)), and this metric then can be restricted to the set \( e(D) \). Another way to endow \( D \) with a metric is to use the projections \( p_n: D \to D_n \) directly to define the distance \( \delta \) on \( D \) by

\[
\delta(x, y) = \begin{cases} 0 & \text{if } x = y \\ 2^{-(n+1)} & \text{otherwise}, \end{cases}
\]

where \( n = \min\{m: p_m(x) \neq p_m(y)\} \). Now, it is a standard result that these metrics are equivalent on any countable product of finite spaces. Moreover, Corollary 11.7 implies these metrics both generate the \( \lambda \)-topology on \( D \). The advantage of the second metric is that it is an ultrametric; i.e., \( \delta(x, y) \leq \max(\delta(x, z), \delta(z, y)) \), \( \forall x, y, z \in D \).

Of course, all of this applies to \( \mathcal{C} \) by Corollary 11.7. In this setting, the ultrametric \( \delta \) is explicitly given by

\[
\delta(I, J) = \begin{cases} 0 & \text{if } I = J \\ 2^{-(n+1)} & \text{otherwise}, \end{cases}
\]

where \( n = \min\{m: \pi_m(I) \neq \pi_m(J)\} \). Moreover, \( \delta \) is a metric we can actually calculate, as we demonstrate below.

For the next result, recall that we associate sequences of protosets to the nodes of accessible pointed graphs using Eq. (5) from Section 9.

**Lemma 11.8.** Let \( a \in \text{HF} \) and consider \( a \) as a node of its canonical picture, the finitely branching apg \( \forall \alpha \). For each \( i \geq 0 \), \( a_{i-1} \subseteq \pi_i(\downarrow \hat{a}) \subseteq \downarrow a_i \).

**Proof.** For each \( i \in \omega \), the definition of \( \pi_i \) implies \( \pi_i(\downarrow \hat{a}) \) is the maximum element of \( \mathcal{C}_i \) below \( \downarrow \hat{a} \). Recall that by Proposition 10.1 \( \text{rank}(a_i) \leq i \).
Therefore \(\downarrow a_{i-1} \subseteq \pi_i(\downarrow \tilde{a})\). Let \(m_i\) be a \(\sqsubseteq\)-maximal element of \(\pi_i(\downarrow \tilde{a})\). Note that \(\text{rank}(m_i) \leq i - 1\). Also, \(m_i\) is compatible with \(a_i\), since they are both clarified by \(\tilde{a}\). Thus, Lemma 10.2 implies that \(m_i \subseteq a_i\), so that \(\pi_i(\downarrow \tilde{a}) \subseteq \downarrow a_i\).

**Lemma 11.9.** If \(I \in \mathcal{M}\) is a maximal ideal of \(\mathcal{C}\), then there is a Cauchy sequence \(\langle \downarrow \tilde{a}_i: i \in \omega \rangle\) which converges to \(I\).

**Proof.** Since \(I \in \mathcal{C}\), \(I = \bigcup_i \{\pi_i(I): i \in \omega\}\), and \(\pi_i(I) \in K(\mathcal{C})\) for each \(i\). If \(I\) is a finite ideal, then Proposition 11.2 implies that \(I = \downarrow \tilde{a}\) for some \(a \in \text{HF}\), and we can use the constant sequence \(\langle \tilde{a} \rangle\) in this case.

Suppose that \(I\) is infinite. For each \(i\), let \(\tilde{x}_i\) be a protoset satisfying \(\pi_i(I) = \downarrow x_i\). In the notation of Proposition 11.2, let \(a_i \in \text{HF}\) be such that \(x_i \subseteq \psi_\lambda(x_i) - \tilde{a}_i\). Since \(\mathcal{C}\) is compact in the \(\lambda\)-topology, the sequence \(\langle \downarrow \tilde{a}_i \rangle\) has some cluster point \(J \in \mathcal{C}\). Now, for each \(i \in \omega\), the set \(X\) of ideals which contain the ideal \(\pi_i(I)\) is a closed set in the \(\lambda\)-topology, and so \(J \in X\); i.e., \(\pi_i(I) \subseteq J\) for all \(i\). But \(I = \bigcup_i \{\pi_i(I): i \in \omega\}\), so \(I \subseteq J\) as well. Since \(I\) is maximal, it follows that \(I = J\). Thus, the only cluster point of \(\langle \downarrow \tilde{a}_i \rangle\) is the ideal \(I\), and so the sequence \(\langle \downarrow \tilde{a}_i \rangle\) is Cauchy and has \(I\) as its limit.

**Theorem 11.10.** \(\mathcal{M}\) is the completion of the space \(\mathcal{H} \mathcal{F}\) in the ultrametric \(\delta\).

**Proof.** We show that \(\mathcal{M}\) is the closure of \(\mathcal{H} \mathcal{F}\) in \(\mathcal{C}\); the result then follows since a closed subset of a complete space is complete. Also, Lemma 11.9 implies we only need to show that the limit of a Cauchy sequence from \(\mathcal{H} \mathcal{F}\) in \(\mathcal{C}\) is a maximal ideal.

So, let \(\langle \downarrow \tilde{a}_n \rangle\) be a Cauchy sequence from \(\mathcal{H} \mathcal{F}\). Since the \(\lambda\)-topology is compact, this sequence has a unique cluster point \(I \in \mathcal{C}\). To show that \(I\) is maximal, we show that, if \(x\) is compatible with each \(y \in I\), then \(x \in I\). Let \(i = \text{rank}(x)\). Note that, for each \(i\), the sequence \(\langle \pi_i(\downarrow \tilde{a}_n) \rangle\) converges, as \(n\) goes to infinity, to \(\pi_i(I)\). Since \(\mathcal{C}\) is finite, this sequence is in fact eventually equal to \(\pi_i(I)\). Fix \(n\) so that \(\pi_{i+1}(\downarrow \tilde{a}_n) = \pi_{i+1}(I)\). Lemma 11.8 implies that \(\downarrow (a_n) \subseteq \pi_{i+1}(\downarrow \tilde{a}_n)\). Since \(\pi_{i+1}(I) \subseteq I\), \((a_n) \in I\). In particular, \((a_n)\) and \(x\) are compatible. Now, Lemma 10.2 implies that \(x \subseteq (a_n)\), and so we conclude that \(x \in I\).

One method for finding a domain \(D\) which satisfies a given property is to derive a domain equation which the desired domain satisfies, and then to take for the domain in question an initial solution of the equation. As described in (Smyth and Plotkin, 1982), this approach works if the equation can be expressed in terms of a continuous endofunctor \(F\) on a suitable category of domains. The category of interest to us is \(\text{SFP}^k\), consisting of \(\text{SFP}\)-objects and embedding-projection pairs as morphisms. The domain
construction which led to the discovery of $SFPE$ is the Plotkin power-domain, which we now describe.

For a domain $D$ with compact elements $K(D)$, consider the family $\mathcal{P}_{<\omega}(K(D))$ of non-empty finite subsets of $K(D)$ in the Egli–Milner order: for finite sets $F, G \in \mathcal{P}_{<\omega}(K(D))$, we define

$$F \sqsubseteq_{EM} G \text{ if and only if } (\forall x \in F)(\exists y \in G) x \sqsubseteq y \& (\forall y \in G)(\exists x \in F) x \sqsubseteq y.$$ 

This is a preorder on $\mathcal{P}_{<\omega}(K(D))$, and the Plotkin powerdomain of $D$ is the ideal completion

$$\mathcal{P}_{pl}(D) = \{ I \subseteq \mathcal{P}_{<\omega}(K(D)) : I \text{ is a directed lower set} \}.$$ 

Since $\mathcal{P}_{pl}(D)$ is an ideal completion, the set of compact elements can be described as $K(\mathcal{P}_{pl}(D)) = \{ \downarrow F : F \in \mathcal{P}_{<\omega}(K(D)) \}$, the set of principal ideals of $(\mathcal{P}_{<\omega}(K(D)), \sqsubseteq_{EM})$.

Now, the Plotkin powerdomain functor $\mathcal{P}_{pl}$ is a continuous endofunctor on the category $SFPE$ (cf. Plotkin (1976)), and so there is an initial solution of the equation

$$D \simeq 1 + \mathcal{P}_{pl}(D),$$

where $1$ is the one-point domain, and $+$ represents the separated sum of domains. We now show that $\mathcal{C}$ satisfies this equation.

**Theorem 11.1**. The continuous set algebra $\mathcal{C}$ satisfies the domain equation

$$\mathcal{C} \simeq 1 + \mathcal{P}_{pl}(\mathcal{C}).$$

**Proof.** It is enough to show that the set of compact elements of the domain $\mathcal{C}$ is isomorphic to the set of compact elements of the domain $1 + \mathcal{P}_{pl}(\mathcal{C})$. Now, $K(\mathcal{C}) = \{ \downarrow x : x \in \mathcal{C} \}$ under inclusion. On the other hand,

$$K(1 + \mathcal{P}_{pl}(\mathcal{C})) = 1 + K(\mathcal{P}_{pl}(\mathcal{C}))$$

$$= 1 + \{ \downarrow F : F \in \mathcal{P}_{<\omega}(K(\mathcal{C})) \}$$

$$= 1 + \{ \downarrow F : F \in \mathcal{P}_{<\omega}(\{ \downarrow x : x \in \mathcal{C} \}) \},$$

where $+$ denotes the separated sum of posets. We now define mutually inverse isomorphisms

$$\phi : K(\mathcal{C}) \to K(1 + \mathcal{P}_{pl}(\mathcal{C}))$$

and

$$\psi : K(1 + \mathcal{P}_{pl}(\mathcal{C})) \to K(\mathcal{C}).$$
For $\phi$, we first define a map $\delta: K(\mathcal{C}) \to \{\emptyset, \{\bot, \bot\}\}$ by

$$
\delta(\bot x) = \begin{cases} 
\emptyset & \text{if } x \text{ is clear} \\
\{\bot, \bot\} & \text{if } x \text{ is murky}.
\end{cases}
$$

Then we define $\phi: K(\mathcal{C}) \to K(1 + \mathcal{P}_1(\mathcal{C}))$ by

$$
\phi(\bot x) = \begin{cases} 
1 & \text{if } x = e \\
\bot & \text{if } x = \bot \\
\{\bot x': x' \in \mathcal{C} \cup \delta(x)\} & \text{otherwise}.
\end{cases}
$$

It is routine to show that $\phi$ is an order preserving map.

Likewise, we define a map $\gamma: \{\bot F: F \in \mathcal{P}'_{<\omega}(\{\bot x: x \in \mathcal{C}\})\} \to \{e, \bot\}$ by

$$
\gamma(\bot F) = \begin{cases} 
e & \text{if } \bot F \\
\bot & \text{if } \bot F \in F.
\end{cases}
$$

We then define $\psi: K(1 + \mathcal{P}_1(\mathcal{C})) \to K(\mathcal{C})$ by

$$
\psi(y) = \begin{cases} 
e & \text{if } y = 1 \\
\bot & \text{if } y = \bot \\
\gamma(y) + \sum \{s(x): \bot x \in F\} & \text{if } y = \bot F.
\end{cases}
$$

Again, it is routine to show that $\psi$ is order-preserving, and that the composition of these maps in either order is the identity.

12. Comparison with Other Work

There have been several other constructions of the hereditarily finite non-well-founded sets as limits of well-founded sets. We would like to mention two efforts in this direction, those of Boffa (unpublished note) and Abramsky (unpublished note).

Boffa consider mappings $e_i: V_{i+1} \to V_i$ defined by recursion on $i \in \omega$: $e_0(x) = \emptyset$ for all $x$, and $e_{i+1}(x) = \{e_i(y): y \in x\}$. Consider the inverse limit

$$
M = \lim (V_{i+1}, e_{i+1}),
$$

and define a membership relation $\in_M$ on $M$ by $x \in_M y$ iff for all $i$, $x(i) \in y(i+1)$. There is a natural system map $f: \mathcal{G} \to M$. Moreover, there is a natural relation $\leq_M$ on $M$, and $(M, \leq_M)$ is a complete semilattice. But unlike $\mathcal{G}$, $M$ is not presented as the set of maximal elements of any domain.

Of course, it is clear that Boffa’s approach is different from ours. In particular, Boffa is not constructing a domain; rather he is constructing a completion of HF to construct a model of HF$_1$ directly. In addition, Boffa
shows that the well-founded part of $M$ is isomorphic to HF and $(M, \leq_M)$ is a local lattice.

The main reason why the two constructions differ is that they are based on different intuitions. Boffa's intuition is that sets have well-founded approximations, and the approximations of $x$ describe increasing parts of the exact membership structure of $x$. For us, the objects used to approximate sets are protosets, not sets. We feel that one of the important by-products of our work is the introduction of protosets, and the study of their approximation order $\subseteq$.

Abramsky (unpublished note) outlines a number of different ways in which a model of HF$_1$ can be constructed. Among the methods he employs is to solve a domain equation and then show that the solution contains a model for HF$_1$ within its maximal elements, much as we have done in Section 11. However, Abramsky's equation is different from ours; he solves

$$D \simeq 2 \oplus \mathcal{P}_l(D_\perp),$$

where 2 is the two-point domain and $\oplus$ denotes the coalesced sum of domains. He also shows that the set of maximal elements of $D$ is a complete ultrametric space, as we have shown in Theorem 11.10. Clearly the results which Abramsky has obtained are quite similar to ours. However, we have only seen an outline of those results, and, absent the details of the constructions, we are unable to draw precise conclusions about how his methods and ours differ.

There is a clear difference between our approach and Abramsky's. We have constructed the domain $C$ directly from HF and $\mathcal{F}$, so the internal structure of $C$ is clearly accessible, more so than it would have been if we had simply tried to solve a functorial isomorphism. Moreover, our approach has the advantage of being founded on the motivations provided by the structures HF and $\mathcal{F}$. The alternative of taking as a definition of $C$ a solution to a domain equation such as the one given above provides no such intuition. And there remains the question of which domain equation to use as the defining one for $C$. Since the equation our model satisfies differs from the one which Abramsky uses, there is the (open) question of whether the two models are isomorphic. In any case, $C$ is a continuous set algebra in which every element is the supremum of well-founded elements (with respect to this membership relation), and the well-founded elements of $C$ are completely characterized as the principal ideals of $\mathcal{F}$ (Theorem 10.10). Thus, $C$ is the best one could hope for in a domain relating well-founded objects and hereditarily finite, non-well-founded sets.

RECEIVED February 21, 1990; FINAL MANUSCRIPT RECEIVED November 14, 1990
REFERENCES

ABRAMSKY, S. (unpublished notes), A Cook's tour of the finitary non-well-founded sets.


BOFFA, M. (unpublished notes), Finite approximations of sets.


