Non-existence of monotone equilibria in games with correlated signals

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Abstract

Participation in economic games such as auctions is typically costly, which means that players (potential bidders) must consider whether to participate. Such decisions may be no less crucial than how to bid, and yet the literature has been mostly concerned with bidding, assuming an exogenously given number of bidders; see Krishna [Auction Theory, Academic Press, New York, 2002]. Landsberger and Tsirelson [Correlated signals against monotone equilibria, preprint SSRN 222308, Social Science Research Network Electronic Library, May 2000. Available online: http://papers.ssrn.com/abstract=222308] have shown that fundamental results established in symmetric auction theory with correlated signals, such as the existence of a monotone equilibrium, may not hold if participation decisions are part of equilibrium. The major goal of this paper is to illustrate and explain this result that may be considered counter intuitive given the emphasis placed on monotone equilibria in the auction literature.

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0. Introduction

A major goal of this paper is to illustrate and explain why a large class of economic games, such as auctions, with continuously distributed correlated signals may not have monotone
(symmetric) equilibria if participation is costly and is part of equilibrium; see, [7]. Since non-monotonicity emerges at the stage of participation it is constructive to consider first pure participation games. For a monotone equilibrium to take place it is necessary that there exists a threshold, $t$, such that players with signals $s < t$ do not participate and players with $s > t$ participate, or the converse. Understanding why threshold strategies do not support symmetric equilibria in pure participation games is a good start to understand why such strategies do not support symmetric equilibria in more general games, such as auctions. The latter introduce additional factors since participants may manipulate reported signals to affect their gains; therefore, although it is constructive to start with the analysis of pure participation games, one should not stop there.

Proceeding in this spirit, we consider in the first two sections pure participation games in which players privately observe their signals (types), and must decide whether they want to participate in the game. Participation is costly, signals are correlated and are truly disclosed to the principal who runs the game. Participants incur a cost, $c$, and the one with the highest signal gets a prize, $C > c$.  

- To illustrate the case, we consider in Section 1 a simple model (a toy model) with 2 players and a simple continuously distributed joint distribution of signals. It is shown that if player 1 adheres to a threshold strategy, $t$, then a best response to it cannot be any threshold strategy; $t$ or other. Hence, there cannot be a threshold equilibrium. It is also shown that higher signals can be bad news in the sense that (over some range) a higher signal corresponds to a lower probability of winning.

- We generalize the non-existence result in Section 2, where we consider a more general setting; no restrictions on the joint distribution of signals and allowing any number of players, $n$. Moreover, the dual role of signals (good and bad news) is identified independently of strategic considerations; the latter were invoked in the toy model.

- Extending the scope of our analysis, we consider in Sections 3 and 4 a second price auction. Restricting attention to a multi-normal distribution of signals, we show that monotone equilibria do not exist unless there are only a few players or when participation cost are small.  

- We show in Section 5 that threshold participation strategies may induce ‘wild’ distributions of participants, which is another reason why monotone equilibria may not exist.

Several authors established sufficient conditions under which monotone equilibria take place in a large class of games; see [1,10]. In spite of the generality of situations to which these papers refer, their sufficient conditions do not hold when participation is costly and is part of equilibrium. Participation games defy the single crossing conditions that are at the heart of the results derived in these, interesting, papers.

This paper is not the first to observe that costly actions coupled with correlated signals may be incompatible with monotone equilibria. By considering a very stylized example, Milgrom

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1 In pure participation games the winner gets a fixed prize, independent of his signal. When winners get a prize that depends on their signals we drop the term ‘pure’. In both cases it is assumed that signals are reported truthfully.

2 Similar results can be established for a first price auction.
and Weber [9] showed that this may be so. However, they downplayed the importance of their example, assuming the problem away by fiat. Krishna and Morgan [5] showed that an all-pay auction with correlated signals may not have a monotone equilibrium, and derived sufficient conditions for such an equilibrium to exist; they were fully aware of the fact that non-monotonicity is a serious problem and not just a fluke.

Finally, one may ask how should the results established in this paper affect future research that addresses games under incomplete information, such as auctions. Since we believe that participation decisions are of paramount importance in most, if not all, real life auctions (otherwise we would not bother ourselves and the readers with this topic), we contend that participation decisions should be addressed in future research. In particular, since once it is done, important results obtained for symmetric auctions with correlated signals may call for a revision. For example, single unit auctions may not be an efficient mechanism that ensures that the object is awarded to the player who values it most. The winner may be a player with a signal lower than another player who chose not to participate.

True, non-monotone equilibria are, as yet, an unexplored topic that requires a new approach which may be technically demanding, see [6]. However, there are many topics in economic theory that required significant technical sophistication and were gradually resolved. Hence, there are good reasons to believe that the difficulties of establishing non-monotone equilibria can be resolved too.

1. A toy model

To introduce the reader to the idea that costly actions and correlated signals are, in the large, at variance with monotone (threshold) equilibria, we chose in this section a setting that allowed to obtain an explicit, and still manageable, expression for the probability of winning function, $W(s)$, which is crucial in understanding the driving force of the results.

We assume 2 risk neutral and, initially, identical players who observe their private one-dimensional signals $S_1, S_2$. Each player chooses an action $A_k$, either 1 (participate) or 0 (quit). Both players choose their actions simultaneously and independently. Each participant incurs participation cost $c \in (0, 1)$, and the one with the highest signal $S_k = \max_i \{S_i | A_i = 1\}$ is the winner and gets the prize, $C = 1$. To focus attention on participation decisions we assume that signals are disclosed truthfully to whoever runs the game. Hence, the only decision to be undertaken by players is whether they want to participate in the game.

Let $S_1, S_2$ be distributed uniformly on the domain

$$
\begin{cases}
0 \leq S_1 \leq 1, & 0 \leq S_2 \leq 1, \\
S_2 > \rho S_1, & S_1 > \rho S_2,
\end{cases}
$$

(1.1)

where $\rho \in (0, 1)$; see Fig. 1. Note that $S_1, S_2$ are exchangeable and affiliated.

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3 Ties are excluded by assuming a non-atomic distribution of signals.
1.1. Winning probability

The information player 1 with signal $s$ has about the signal of player 2, $S_2$, is summarized by the conditional distribution of $S_2|S_1=s$; it is uniform on

$$\begin{cases} 
    (\rho s, 1) & \text{if } s \in (\rho, 1), \\
    (\rho s, s/\rho) & \text{if } s \in (0, \rho).
\end{cases} \quad (1.2)$$

Player 1 can win if player 2 does not participate, $A_2 = 0$, or if player 2 participates, $A_2 = 1$, but has a lower signal. Hence, the probability of winning, $W(s)$, is given by

$$W(s) = 1 - \mathbb{P}(A_2 S_2 < s) = 1 - \mathbb{P}(A_2 = 1, S_2 > s),$$

which can be written as

$$W(s) = 1 - \begin{cases} 
    \frac{1}{1 - \rho s} \int_s^1 p(x) \, dx & \text{if } s \in (\rho, 1), \\
    \frac{1}{1 - \rho^2} \int_s^1 p(x) \, dx & \text{if } s \in (0, \rho),
\end{cases} \quad (1.3)$$

where $0 \leq p(\cdot) \leq 1$ stands for the participation strategy.

Note that even in this simple model the probability of winning, $W(s)$, depends on an unknown strategy, $p(s)$, in a technically complex way. This interdependence, which is absent when participation decisions are not modeled, is the main source of the technical complexity involved in establishing non-monotone equilibria. Under monotone equilibria the probability of winning is given by a simple, exogenously given, expression which simplifies the problem immensely.
1.2. The failure of threshold strategies to support an equilibrium

1.4. Definition. A function $0 \leq p(s) \leq 1$ is a threshold (participation) strategy if there exists a threshold, $t$, such that a player participates if and only if $s \geq t$.

Under a threshold strategy $t$, (1.3) can be rewritten as

$$W(s) = 1 - \begin{cases} 
1 - \max(s, t) & \text{if } s \in (\rho, 1), \\
1 - \rho s & \text{if } s \in (t, \rho), \\
s/\rho - \max(s, t) & \text{if } s \in (0, t/\rho). 
\end{cases}$$ \hspace{1cm} (1.5)

1.6. Proposition. In our simple pure participation game if player 2 uses a threshold strategy, then there is no (non-trivial) best response to it in the class of threshold strategies.

Since the proof of the case where $t > \rho$ is a simple replica of the case where $t < \rho$ and, structurally, so are the results, we consider only the latter case; see Fig. 2a–c.

Proof of Proposition 1.6. It can be easily shown that if player 2 uses a threshold strategy $t \in (0, 1)$, then the probability of winning function of player 1, $W(s)$, as given in (1.5), has an ‘almost’ U-shape, see Fig. 2c and d, which proves the proposition since player 1 should participate if, and only if, $W(s) > c/C$. The $W(s)$ function dictates either a trivial equilibrium when $c$ is small and player 1 always participates, or player 1 participates when his signals are small or large. Intermediate signals dictate non-participation. The pattern of the $W(s)$ function follows immediately from (1.3) and player’s 2 threshold participation strategy. However, for illustrative purposes we proceed in a more intuitive way.

Divide the signal domain in Fig. 1 into two signal sets; one that denotes signals $(s, s_2)$ where player 2 is the winner, the lightly shaded set, and the darker one that corresponds to signals where player 1 is the winner; see Fig. 2a and b. We explain how the split was obtained for $t < \rho$. The case $t > \rho$ is obtained similarly. Note that Fig. 2a and b induce Fig. 2c and d.

Let $t < \rho$ and freely interchange $s$ and $s_1$ to denote signals of player 1. Note that by (1.2) if $s \in (0, \rho)$ the highest signal player 2 can have is $s/\rho$. Hence, if $s < t/\rho$ player 2 does not participate and therefore, trivially, player 1 is the winner with probability 1; consequently, the subset of signals $(s, s_2)$ that correspond to $s < t/\rho$ is all dark, see Fig. 2a. If $s \in (t/\rho, t)$, player 2 starts receiving signals higher than $t$ and if this happens he (player 2) is the winner since $s < t$. Hence, back to Fig. 2a, the lightly shaded triangular area above $t$ consists of signals $(s_1, s_2)$ at which player 2 is the winner, and since $S_2$ is distributed uniformly on the interval $(s/\rho, \rho s)$, $P(S_2 > t)$ is given by the vertical distance between $s/\rho$ and $t$ divided by the vertical distance between $s/\rho$ and $s\rho$. Consequently, $W(s) = 1 - P(S_2 > t) = 1 - \frac{s/\rho - t}{s/\rho - ps} = \frac{t - ps}{s/\rho - ps}$. This is how Fig. 2a induces Fig. 2c.

If $s \in (t, \rho)$, player 2 can win only if $S_2 > \max(t, s)$ which boils down to $S_2 > s$. This is how the dark and lightly shaded areas were obtained for $s \in (t, \rho)$. Consequently, player 2
can win with probability $\frac{s - s}{s + t - s} = \frac{s}{t + s}$, see Fig. 2c. When $s > \rho$ the procedure (but not the results), is the same, implying $W(s) = s - \rho$. □

To summarize, participation is motivated by two different considerations; when $s$ is small, player 1 should participate hoping that player 2 does not participate. This hope is a certainty when $s < t\rho$. High signals dictate participation because the chances are that player 1 has the higher signal; the standard argument. These two different considerations dictate participation when signals are low or high, and non-participation otherwise.
In the next section we augment and formalize the notion that higher signals can be both good and bad news in the sense of lowering the probability of winning, which is another reason why a monotone equilibrium is not an ‘educated guess’ once participation is part of equilibrium.4

2. A pure participation game and the probability of winning

2.1. The game

Augmenting the setting considered in the previous section, we consider here a game of n risk neutral and, initially, identical players who observe their private one-dimensional signals $S_1, \ldots, S_n$. Player $k$ is allowed to participate if and only if his signal $S_k$ exceeds a reserve level $r$. In that case the player chooses an action $A_k$, either 1 (participate) or 0 (quit). All players allowed to participate choose their actions simultaneously and independently.5 If no one participates, all get nothing. Otherwise, each participant pays an entry cost $c$, and the one with the highest signal $S_k = \max \{ S_l \mid S_l > r, A_l = 1 \}$ is the winner and gets the prize, $C$.6 Parameters $C, c, r$ and the joint distribution of signals are common knowledge. Naturally, $0 < c < C < \infty$.

The joint distribution of signals is assumed to be of the form

$$F_n(s_1, \ldots, s_n) = \int F(s_1|\theta) \cdots F(s_n|\theta) G(d\theta), \quad (2.1)$$

where $F_n(s_1, \ldots, s_n) = \mathbb{P}(S_1 \leq s_1, \ldots, S_n \leq s_n)$ is the n-dimensional (cumulative) distribution function, $(F(\cdot|\theta))$ is a family of one-dimensional distributions parametrized by $\theta$, and the parameter $\theta$ is drawn from its distribution $G$.7 In other words, we have $n+1$ random variables $\Theta, S_1, \ldots, S_n$ such that
- $\Theta$ is distributed $G$,
- every $S_k$ is distributed $F_1$,
- $S_1, \ldots, S_n$ are conditionally independent given $\Theta$,
- given $\Theta = \theta$, each $S_k$ is distributed $F(\cdot|\theta)$,

here $F_1(s) = \int F(s|\theta) G(d\theta)$, which is just (2.1) for $n = 1$.

A strategy (possibly mixed) of player $k$ is described by a function $p_k(\cdot)$ such that

$$p_k(s_k) = \mathbb{P}(A_k = 1 \mid S_k = s_k) \quad (2.2)$$

for $s_k \in (r, \infty)$; that is, $p_k(s_k)$ is the participation probability for player $k$ possessing signal $s_k$. A pure strategy corresponds to the case where $p_k(\cdot)$ takes on two values only, 0 and 1.

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4 Milgrom [8] noted that “The analysis of equilibrium in symmetric bidding models is usually guided by the ‘educated guess’ that there will be a symmetric pure strategy equilibrium, ... and that the equilibrium strategy will be increasing”, see [8, p. 928].

5 We do not consider correlated equilibria.

6 Ties will be excluded by assuming non-atomic distributions of signals.

7 The parameter $\theta$ need not be one-dimensional, this is why we prefer the notation $G(d\theta)$ to $dG(\theta)$. 

In general, $0 \leq p_k(s_k) \leq 1$. Nothing like monotonicity, continuity etc. is assumed; $p_k(\cdot)$ is just a function, defined on $(r, \infty)$ almost everywhere w.r.t. the distribution $F_1$ of a signal, measurable w.r.t. the same distribution, and taking on values on $[0, 1]$. We leave $p_k(\cdot)$ undefined on $(-\infty, r)$, since such signals exclude participation.

Actions are conditionally independent given signals, that is,

$$\Pr(A_1 = a_1, \ldots, A_n = a_n \mid S_1 = s_1, \ldots, S_n = s_n) = \Pr(A_1 = a_1 \mid S_1 = s_1) \cdots \Pr(A_n = a_n \mid S_n = s_n) = (a_1 p_1(s_1) + (1 - a_1)(1 - p_1(s_1))) \cdots (a_n p_n(s_n) + (1 - a_n)(1 - p_n(s_n))),$$

here and henceforth $s_1, \ldots, s_n$ run over $(r, \infty)$.\(^8\)

2.2. Probability of winning and the equilibrium equation

The payoff $\Pi_k$ of player $k$ is a random variable (in fact, a function of $S_1, \ldots, S_n, A_1, \ldots, A_n$) taking on three values: 0 in the case of no participation, $-c$, participation but no win, and $C - c$, participation and win. In order to calculate the expected payoff given the signal, $\mathbb{E}(\Pi_k \mid S_k)$, we need first the probability of winning, given participation and signal. To this effect, we start by conditioning also on the common (random) factor $\Theta$. To exclude ties, we assume that each $F(\cdot \mid \theta)$ is continuous. Just for notational convenience, assume for a while that $k = n$ and all signals are positive. Then, the desired probability is given by

$$\Pr(\Pi_n = C - c \mid A_n = 1, S_n = s_n, \Theta) = \Pr(A_1 S_1 < s_n, \ldots, A_{n-1} S_{n-1} < s_n \mid A_n = 1, S_n = s_n, \Theta)$$

$$= \prod_{k=1}^{n-1} \Pr(A_k S_k < s_n \mid A_n = 1, S_n = s_n, \Theta) \quad (2.3)$$

We do not restrict strategies to be monotone, and not even pure, and therefore the probability of winning involves not just signals, $S_i$, but also actions $A_i$ of other players. Consequently, the relevant random variables are products of signals and actions, $A_i S_i$, see (2.3). This complication is a natural result in a strategic situation; it was avoided in most symmetric auction models because if participation cost are ignored, monotone equilibria take place and the probability of winning is a simple expression, independent of strategies.

\(^8\) Or rather, an equivalence class of such functions, where equivalence is treated as equality almost everywhere w.r.t. $F_1$. It is nothing but another form of a distributional strategy for the case of an action space of two points only. Uniqueness of equilibrium is treated up to equivalence.

\(^9\) However, $S_1, \ldots, S_n$ are not conditioned to stay in $(r, \infty)$, unless it follows from explicit conditions $S_k = s_k$. 


Going back to (2.3), we can replace the actions \( A_i \) by strategies that induced them, \( p(\cdot) \), to obtain the conditional probability of winning against a single opponent:

\[
P \left( A_1 S_1 < s_n \mid \Theta = \theta \right) = 1 - P \left( A_1 = 1, S_1 > s_n \mid \Theta = \theta \right) = 1 - \int_{s_n}^{\infty} p_1(s_1) \, dF(s_1|\theta).
\]

Note that \( \int_{s_n}^{\infty} p_1(s_1) \, dF(s_1|\theta) \) stands for the conditional probability that the opponent participates and has a higher signal than \( s_n \). Since we restrict attention to symmetric equilibria, generalizing to \( n-1 \) players is immediate by conditional independence;

\[
P \left( \Pi_k = C - c \mid A_k = 1, S_k = s_k, \Theta = \theta \right) = \left( 1 - \int_{s_k}^{\infty} p(s) \, dF(s|\theta) \right)^{n-1}, \quad (2.4)
\]

which was derived for \( k = n \), but holds equally well for all \( k = 1, \ldots, n \).

Assuming that distributions \( F(\cdot|\theta) \) have densities \( F'(\cdot|\theta) \) we use Bayes formula for densities. The conditional distribution of \( \Theta \) given \( S_k = s_k \) is

\[
\frac{F'(s_k|\theta)}{F'_1(s_k)} \, G(d\theta),
\]

where \( F'_1 \) is the density of the unconditional distribution of a signal and \( F'(s) = \int F'(s|\theta) \, G(d\theta) \). Finally, the unconditional winning probability of a player who decided to participate and whose signal is \( s_k \) is given by

\[
W(s_k) = P \left( \Pi_k = C - c \mid A_k = 1, S_k = s_k \right), \quad (2.5)
\]

which can be rewritten as

\[
W(s_k) = \int P \left( \Pi_k = C - c \mid A_k = 1, S_k = s_k, \Theta = \theta \right) \frac{F'(s_k|\theta)}{F'_1(s_k)} \, G(d\theta)
\]

\[
= \int \left( 1 - \int_{s_k}^{\infty} p(s) \, dF(s|\theta) \right)^{n-1} \frac{F'(s_k|\theta)}{F'_1(s_k)} \, G(d\theta). \quad (2.6)
\]

The expected profit of a participant with signal \( s_k \) is given by

\[
\mathbb{E} \left( \Pi_k \mid A_k = 1, S_k = s_k \right) = -c + CW(s_k). \quad (2.7)
\]

We are in a position to describe equilibria;

2.8. Lemma. A participation strategy \( p(\cdot) \) supports a symmetric equilibrium if and only if \( F_1 \)-almost all \( s \in (r, \infty) \) belong to one of the following three cases: \(^{10}\)

(a) \( 0 < p(s) < 1 \) and \( W(s) = c/C \);
(b) \( p(s) = 0 \) and \( W(s) \leq c/C \);
(c) \( p(s) = 1 \) and \( W(s) \geq c/C \).

\(^{10}\) Different \( s \) may belong to different cases.
Proof. Follows immediately from (2.7) and the definition of a Nash equilibrium. □

From Lemma 2.8 and (2.6) we obtain the equation of a symmetric equilibrium;

$$
\int \left( 1 - \int_{S_k}^\infty p(s) dF(s|\theta) \right)^{n-1} \frac{F'(s_k|\theta)}{F'_1(s_k)} G(d\theta) = c / C,
$$

which is a non-standard integral equation. Note that (b) and (c) are relevant as well.

To formally illustrate the split of a higher signal into good and bad news, we assume a multinormal distribution with parameters $E = 0$, $\text{Var}(S_k) = 1$, $\text{Cov}(S_k, S_l) = \rho$ whenever $1 \leq k \leq n$, $1 \leq l \leq n$, $k \neq l$.

One may take $n + 1$ independent $N(0, 1)$ random variables $\Theta, \xi_1, \ldots, \xi_n$ and form

$$
S_k = \sqrt{\rho}\Theta + \sqrt{1 - \rho}\xi_k
$$

to obtain a multinormal joint distribution of $S_1, \ldots, S_n$. 11 Thus, $F(s|\theta) = \Phi\left( \frac{s - \sqrt{\rho}\theta}{\sqrt{1 - \rho}} \right)$.

$G(\theta) = \Phi(\theta)$, $F_1(s) = \Phi(s)$, where $\Phi(x) = (2\pi)^{-1/2} \int_{-\infty}^x e^{-u^2/2} du$ is the standard normal distribution function. The relation $S_k = \sqrt{\rho}\Theta + \sqrt{1 - \rho}\xi_k$ shows that the conditional distribution of $S_k$ given $\Theta = \theta$ is $N(\sqrt{\rho}\theta, 1 - \rho)$. Similarly, 12 the conditional distribution of $\Theta$ given $S_k = s$ is $N(\sqrt{\rho}s, 1 - \rho)$. In other words, 13, 14

$$
\frac{F'(s|\theta)}{F'_1(s)} dG(\theta) = d_\theta \Phi\left( \frac{\theta - \sqrt{\rho}s}{\sqrt{1 - \rho}} \right).
$$

Now, the probability of winning function (2.6) can be written as

$$
W(s) = \int \left( 1 - \int_{S_1}^\infty p(s_1) d_{s_1} \Phi\left( \frac{s_1 - \sqrt{\rho}\theta}{\sqrt{1 - \rho}} \right) \right)^{n-1} d_\theta \Phi\left( \frac{\theta - \sqrt{\rho}s}{\sqrt{1 - \rho}} \right)
$$

for $s \in [r, \infty)$.

It is important to observe that $s$ occurs twice on the right-hand side of (2.12), at the lower limit of the internal integral and in $\Phi\left( \frac{\theta - \sqrt{\rho}s}{\sqrt{1 - \rho}} \right)$. This decomposition allows to identify the dual role of a signal; ‘internal’ and ‘external’. On the one hand, a signal tells the player his strength (the internal role) independently of the signals of his opponents. On the other hand, due to interdependence, a signal informs a player (to some extent) about other players’ strength, (the external role). To assess these factors, we split the signal in (2.12) into two,

11 A linear transformation always sends a multinormal distribution into a multinormal distribution.

12 The two-dimensional distribution of the pair $(\Theta, S_k)$ is equal to that of the pair $(S_k, \Theta)$; both are the two-dimensional normal distribution with the mean vector $(0, 0)$ and the covariance matrix $\begin{pmatrix} 1 & \sqrt{\rho} \\ \sqrt{\rho} & 1 \end{pmatrix}$.

13 Now $\Theta$ is one-dimensional, and we may write $dG(\theta)$ rather than $G(d\theta)$.

14 The expression $d\Phi\left( \frac{s - \sqrt{\rho}\theta}{\sqrt{1 - \rho}} \right)$ would be ambiguous. In order to distinguish between $\frac{d}{ds} \Phi(\cdots)$, $d_\theta \Phi(\cdots)$, and $\frac{d}{d\theta} \Phi(\cdots)$.

\( s^{\text{int}} \) and \( s^{\text{ext}} \), and introduce a function of two variables \( s^{\text{int}}, s^{\text{ext}} \), denoting it by \( W_2 \):

\[
W_2(s^{\text{int}}, s^{\text{ext}}) = \int \left( 1 - \int_{s^{\text{int}}}^{\infty} p(s) \, ds \right) \Phi \left( \frac{s - \sqrt{\mu} \theta}{\sqrt{1 - \rho}} \right) ^{n-1} \, d\theta \Phi \left( \frac{\theta - \sqrt{\mu} s^{\text{ext}}}{\sqrt{1 - \rho}} \right),
\]

\[
W(s) = W_2(s, s).
\]

Although \( W(s) = W_2(s^{\text{int}}, s^{\text{ext}}) \) only along the main diagonal, \( s^{\text{int}} = s^{\text{ext}} \), we are free to analyse the effects of \( s^{\text{int}} \) and \( s^{\text{ext}} \) off the main diagonal as well. This is legitimate, and as we show it is also useful since the separate effects of \( s^{\text{int}} \) and \( s^{\text{ext}} \) illustrate the dual effects of higher signals. Evaluating these effects off the main diagonal gives an indication what we may anticipate to be their combined effect along the diagonal.

Clearly, \( W_2(s^{\text{int}}, s^{\text{ext}}) \) is increasing in \( s^{\text{int}} \), which is not surprising since an increase in \( s^{\text{int}} \) can be given the interpretation that the player can obtain a higher signal independently of other players’ signals; note that the inner integral is taken with respect to a given \( \theta \). Hence, a higher \( s^{\text{int}} \) obviously increases the probability of winning. In contrast, the effect of \( s^{\text{ext}} \) on \( W_2(s^{\text{int}}, s^{\text{ext}}) \) is not monotone, in general. In particular, consider the case when \( p(s) \) is an increasing function. The inner integral in (2.2) is an increasing function of \( \theta \) since a higher \( \theta \) represents a shift to the right of the conditional distribution of \( s \) given \( d \theta \), \( \Phi \left( \frac{s - \sqrt{\mu} \theta}{\sqrt{1 - \rho}} \right) \).

Therefore, the integrand of the outer integral is a decreasing function of \( \theta \). A higher \( s^{\text{ext}} \) shifts to the right the conditional distribution of \( \Theta \), \( d\theta \Phi \left( \frac{\theta - \sqrt{\mu} s^{\text{ext}}}{\sqrt{1 - \rho}} \right) \), that lowers the value of that integral. Consequently, a higher \( s^{\text{ext}} \) lowers the probability of winning, that is bad news. The combined effect cannot be uniquely established and therefore there is no reason to be surprised by non-existence of monotone equilibria.

A verbal argument, that is on the verge of a proof, that provides further intuition for the non-existence of monotone equilibria in pure participation games with correlated signals runs as follows. Assume that all players with signals \( s < s_{\text{high}} \) do not participate and players with \( s > s_{\text{high}} \) do. For this to be an equilibrium it must be that \( W(s_{\text{high}}) = c / C \), and \( W(s) \leq c / C \) for all \( s < s_{\text{high}} \), see Lemma 2.8(b) (also the condition for \( s > s_{\text{high}} \) must hold, but does not matter now). Consider player 1 with signal \( s < s_{\text{high}} \) who considers to deviate from this strategy and participate; such a player does not care about players with higher signals, as long as they are below \( s_{\text{high}} \), since they do not participate. Therefore, the only thing that matters for this player is the probability of the event \( \max(S_2, \ldots, S_n) < s_{\text{high}} \). This probability is a strictly decreasing function of \( s \), say, by affiliation, and therefore for all \( s < s_{\text{high}} \) \( W(s) > W(s_{\text{high}}) \). This, however, contradicts the equilibrium conditions stated in Lemma 2.8 that claims that for all \( s < s_{\text{high}} \), \( W(s) \leq W(s_{\text{high}}) \).

Finally, it is interesting to see why the sufficient condition for the existence of monotone equilibria in games of incomplete information, established in the literature are not satisfied in our model. It is convenient to refer to Athey’s paper since it refers to the symmetric case, in similarity to this paper. Athey proved that if certain single crossing condition is satisfied in a game of incomplete information, then this game has a monotone equilibrium. Applied in our context, where there are only two actions and one of them (non-participation) yields a trivial result (zero profit), Athey’s condition requires that when other players use monotone strategies (and player 1 participates always), player 1 probability of winning function \( W(\cdot) \) crosses \( c / C \) only once, and the crossing is from below. To see that this condition is not
satisfied in our model, assume that player 1 has a very low signal \( s_1 \). Then (if the correlation coefficient is not too small), signals \( s_2, \ldots, s_n \) of other players, all of whom use threshold strategies, are also small and probably below their respective thresholds; hence, these players do not participate. It means that \( W(-\infty) = 1 \). On the other hand, let \( s_1 \) be very high; so are the signals of other players, but player 1 probably wins, that is, \( W(+\infty) = 1 \). Hence, at the two ‘ends’ of the signal distribution, the probability of winning of player 1 tends to 1; see Fig. 2c and d in Section 2. Clearly, such \( W(\cdot) \) crosses \( c/C \) at least twice, unless \( c/C < \min W(\cdot) \). The latter can happen for some thresholds \( t_2, \ldots, t_n \) of players 2, \ldots, \( n \).

However, Athey’s condition requires it for all \( t_2, \ldots, t_n \), which is too much to ask. Indeed, consider the case \( t_1 = \cdots = t_n = s_1 < 0 \), assuming that \( |s_1| \) is large. Such a low signal of the first player is probably majorized by some other \( s_k \), the latter being above the threshold \( t_k \). Here, \( \min W(\cdot) \) is close to 0. We see that every \( c/C \) is crossed (at least) twice by the probability of winning function \( W(\cdot) \), for some monotone strategies of other players.

3. Second price auction

To illustrate that the non-existence of monotone equilibria goes well beyond pure participation games, we consider in this section a second price auction with \( n \) players. For a monotone equilibrium to take place, it must be that the participation strategy is a threshold; say, \( s^* \). Such an \( s^* \) corresponds to some participation cost, \( c \). Denote by \( \pi(s) \) the expected profit of a player whose signal is \( s \). Then, it must be that

\[
\begin{align*}
\text{if } s &\leq s^* \rightarrow \pi(s) \leq c, \\
\text{if } s &\geq s^* \rightarrow \pi(s) \geq c,
\end{align*}
\]

which implies \( c = \pi(s^*) \).

Assume that all players with signals \( s_2, \ldots, s_n \) follow the threshold strategy except player 1 with signal \( s_1 \) who is considering to deviate from this strategy and always participate; what could be his gain as a function of \( s_1 \). Denote this gain by \( \pi_{s^*}(s_1) \) and note that a sufficient condition that contradicts \( s^* \) being a threshold equilibrium is that the \( \pi_{s^*}(s_1) \) function has a declining segment and \( s^* \) is located on this segment. This is so since if \( s^* \) is located on the declining segment then if player 1 has signal \( s_1 \) located on this segment and \( s_1 < s^* \), it pays to deviate from the threshold rule and participate, which means that \( s^* \) is not an equilibrium, see (3.1). By the same token, if \( s_1 > s^* \) participation causes losses, which again shows that \( s^* \) is not an equilibrium. Hence, to show that there is no monotone equilibrium we must derive an explicit expression for \( \pi_{s^*}(s_1) \), which is not a trivial matter. A distinction is made between \( s_1 \geq s^* \) and \( s_1 \leq s^* \).

If \( s_1 \leq s^* \), player 1 can win only if all other players’ signals are smaller than \( s^* \) and therefore they do not participate. The probability of this event, which we denote by \( H_{s_1}(s^*) \), is defined by (3.3). The expected gain induced by participating in this situation is

\[
\pi_{s^*}(s_1) = (s_1 - r)_+ H_{s_1}(s^*),
\]
where

\[ H_{s_1}(s) = \mathbb{P}\left(\max(S_2, \ldots, S_n) \leq s \mid S_1 = s_1\right). \]  

(3.3)

Being the only participant, player 1 bids the reserve price \( r \).

If \( s_1 \geq s^* \), player 1 can win not only when he is the only participant, but also again participants who have signals lower than \( s_1 \). However, when winning against an opponent with a signal higher than \( s^* \), player 1 pays the largest losing bid, which is, typically, higher than the reserve price. Consequently, the expected gain from participation is

\[
\pi_{s^*}(s_1) = (s_1 - r)H_{s_1}(s^*) + \int_{s^*}^{s_1} (s_1 - s) dH_{s_1}(s) = (s^* - r)H_{s_1}(s^*) + \int_{s^*}^{s_1} H_{s_1}(s) ds. 
\]  

(3.4)

The first term in (3.4) stands for the expected gain obtained in the event where player 1 was the only participant (and therefore paid \( r \)). The second term is the expected gain induced by winning again opponents with signals in \((s^*, s_1)\) and paying the second highest bid. Note that \( H_{s_1}(s) \) is the function defined in (3.3) evaluated at \( s \).

To obtain explicit expressions for (3.2) and (3.4), we must have an explicit expression for (3.3). To this effect, we invoke exchangeability (see (2.1)), Bayes formula for densities and Corollary 2 from [3, Section 7.3] to obtain

\[
H_{s_1}(s) = \mathbb{E}\left(\mathbb{P}\left(\max(S_2, \ldots, S_n) \leq s \mid \Theta, S_1\right) \mid S_1 = s_1\right) \\
= \mathbb{E}\left(\mathbb{P}\left(\max(S_2, \ldots, S_n) \leq s \mid \Theta\right) \mid S_1 = s_1\right) = \mathbb{E}\left(G_n^{-1}(s) \mid S_1 = s_1\right) \\
= \int F_{\theta}^{-1}(s) F_{\theta}^0(s_1)G'(\theta) F_{\theta}^1(s_1) d\theta, 
\]  

(3.5)

provided that distributions \( G \) and \( F_{\theta} \) have densities \( G' \) and \( F_{\theta}' \). Invoking normality, \( N(0, 1) \) with correlation \( \rho \) we obtain, after some effort (see Appendix A), for \( s_1 \leq s^* \),

\[
\pi_{s^*}(s_1) = (s_1 - r)\mathbb{E} F_{\rho s_1 + \sqrt{\rho(1-\rho)}\tilde{z}_0}^{n-1}(s^*) 
\]  

(3.6)

and for \( s_1 \geq s^* \),

\[
\pi_{s^*}(s_1) = (s^* - r)\mathbb{E} F_{\rho s_1 + \sqrt{\rho(1-\rho)}\tilde{z}_0}^{n-1}(s^*) + \int_{s^*}^{s_1} \mathbb{E} F_{\rho s_1 + \sqrt{\rho(1-\rho)}\tilde{z}_0}^{n-1}(s) ds, 
\]  

(3.7)

where, \( F_{\rho s_1 + \sqrt{\rho(1-\rho)}\tilde{z}_0} \) is the conditional distribution \( F_{\theta} \) evaluated at \( \theta = \rho s_1 + \sqrt{\rho(1-\rho)}\tilde{z}_0 \).

Note that if \( S_k \sim N(0, 1) \), \( \mathbb{E}(S_k S_l) = \rho \in [0, 1] \) for \( k < l \), \( S_k \) has the representation:

\[
S_k = \sqrt{\rho}\tilde{z}_0 + \sqrt{1-\rho}\tilde{z}_k, 
\]  

(3.8)

where \( \tilde{z}_0, \tilde{z}_1, \tilde{z}_2, \ldots \) are i.i.d. \( N(0, 1) \) random variables.
Having explicit expressions for (3.6) and (3.7), \( \pi^* (s_1) \) can be evaluated numerically, given the reserve price, \( r \), the number of players, \( n \), \( s^* \) and \( \rho \). In the model \( s^* \) is endogenous, but knowing that every \( s^* \) corresponds to some \( c \), assigning an arbitrary value to \( s^* \) is equivalent to choosing an arbitrary value of \( c \).\(^{15}\) We chose \( n = 5, \rho = 0.9 \), the reserve price \( r = 0 \) and \( s^* = 1 \). Given this numerical specification, (3.6) and (3.7) were evaluated numerically, and the resulting \( \pi^* (s_1) \) function was plotted in Fig. 3(a). As we can see, \( \pi^* (s_1) \) is not monotone and \( s^* \) is on the declining segment, which proves that a monotone equilibrium cannot take place for the chosen set of parameters. Since Fig. 3(a) covers about 99 percent of the signals it follows that deviating from a threshold strategy is profitable for almost all types of player 1.

The question is whether this result holds for all values of the parameters. We show that had we chosen smaller values of \( s^* \), that correspond to lower values of \( c \), and preserving \( n = 5 \), a monotone equilibrium cannot be ruled out.\(^{16}\) To verify it, we conducted the same analysis as before but choose lower values of \( s^* \), that correspond to lower values of \( c \). Obviously, each value of \( s^* \) generates a new \( \pi^* (s_1) \) function; see Fig. 4. Choosing \( s^* = 0.1 \) or 0.2, we obtain a monotone increasing \( \pi^* (s_1) \) functions which means that a monotone equilibrium takes place. However, higher values of \( s^* \), say, 0.5, 1, or 2 induce non-monotone \( \pi^* (s_1) \) functions with \( s^* \) values on the declining segments of \( \pi^* (s_1) \). Hence, the picture that emerges is less unequivocal then in pure participation games discussed in Sections 1 and 2. As shown in the next section, the reasons for that are income effects that are absent from pure participation games.

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\(^{15}\) This argument uses the property that \( s^* \) is monotone in \( c \). It can be shown that this is so; the question was posed by J.P. Benoit at a theory seminar at University College London.

\(^{16}\) We proved in another paper that if the number of players is sufficiently large, monotone equilibria do not exist in a large family of games, including auction games regardless of the values of the parameters; see [7].
4. Good news bad news

Unlike pure participation games, auctions have income effects which mitigate the ‘bad news’ element of a higher signal. This argument is illustrated in this section.

Inspecting the explicit expressions of the $\pi(s^*)$ function in (3.6) and (3.7), we see that one can decompose the gains of player 1, for every signal that he might have, into those that took place when player 1 was the only participant and therefore paid a low price, $r$, and the other events when player 1 was not the only participant and paid a higher price. This separation is useful since winning is important, but so is the price paid by the winner.

Start with $s_1 \in (r, s^*)$. Player 1 can win only if he is the only participant; in this case he pays the reserve price, $r$. As shown in (3.6) the probability of this event is $E_{\rho_1 + \sqrt{\rho(1-\rho)\xi_0}}(s^*)$, which is decreasing with $s_1$. Hence, a higher signal is bad news; this, however, is only part of the story. A higher signal is also good news since winning with a higher signal implies a larger gain; $s_1 - r$. These arguments are illustrated graphically in Fig. 3b and c; the rectangular shaded areas are the expected gains of player 1 from bidding. The heights of the rectangular areas are the probabilities to win, $E_{\rho_1 + \sqrt{\rho(1-\rho)\xi_0}}(s^*)$, and the horizontal segment, $s_1 - r$, are the gains. As can be seen in Fig. 3b and c, the overall effect of a higher signal on the expected gain is negative; the second rectangular area is smaller than the first, which is why the second heavy dot in Fig. 3a is lower than the first. But note that on this segment $\pi(s_1)$ is first increasing, due to the dominating ‘income’ effect and then, after the decrease in the probability of winning is taking over, it is decreasing. This duality is absent in pure participation games since there are no income effects in these games, which may be the reason why the results obtained for auctions are less unequivocal.

Considering $s_1 > s^*$, we see that $E_{\rho_1 + \sqrt{\rho(1-\rho)\xi_0}}(s^*)$, the probability of winning, is further decreasing. Namely, the probability of being the only participant is decreasing until it finally disappears, see Fig. 3e and f, and the gains obtained while paying low prices disappear (no rectangular shaded areas in Fig. 3e and f; remember that the latter correspond to the first expression on the RHS of (3.7). At the same time, the element that makes a higher signal good news becomes more important; the second expression on the RHS of (3.7), which is illustrated by the, almost, triangular shaded areas in Fig. 3d–f, represent the gains obtained while competing with other participants and paying higher prices. The upper boundaries of the shaded areas in Fig. 3e and f indicate the probability distributions of the price player 1 is paying when he is the winner.
These results illustrate the reasons why the pure participation games considered in Sections 1 and 2 excluded monotone equilibria whereas the results obtained for auctions are less unequivocal.

5. ‘Wild’ distributions of participants

If participation decisions are part of equilibrium, then so is the distribution of participants. Therefore, to get an intuition why certain types of participation strategies are, or are not, reasonable, it is constructive to investigate what are the properties of distribution of participants that they induce. Given the context of this paper, it means that we should explore the properties of the distribution of participants induced by threshold participation strategies. To this effect, we assume the following distribution of signals.

Let

\[ G_\Theta = U(0.9, 1), \] and each \( S \sim U(0, \theta). \]

Then,

\[
\mathbb{P}(S_1 \leq s_1, \ldots, S_n \leq s_n \mid \Theta = \theta) = F_{\theta}(s_1) \ldots F_{\theta}(s_n)
\]

\[
= \min \left( \frac{s_1}{\theta}, 1 \right) \ldots \min \left( \frac{s_n}{\theta}, 1 \right).
\]

Under a threshold strategy \( t_n \in (0, 1) \) for each \( n \), the number of participants is a random variable

\[ K_n = 1(t_n, \infty)(S_1) + \cdots + 1(t_n, \infty)(S_n). \]

The mean number of participants is

\[ \mathbb{E}K_n = np_n, \]

where \( p_n \) is the participation probability;

\[
p_n = \mathbb{P}(S_1 > t_n) = \mathbb{E}[\mathbb{P}(S_1 > t_n \mid \Theta)] = 1 - F_{S_1}(t_n).
\]

(5.1)

Since participation is costly and only one unit of the object is up for sale, we may expect that \( p_n \xrightarrow{n \to \infty} 0 \), and moreover, \( p_n = O(1/n) \), that is, boundedness of \( np_n \). Indeed, the total entry cost paid by all players is \( np_n \cdot c \) in the mean, while the total gain of all players never exceeds \( \max(S_1, \ldots, S_n) \). Voluntary participation implies

\[ np_n c \leq \mathbb{E} \max(S_1, \ldots, S_n). \]

If signals have a compact support, \( \mathbb{P}(S_1 \leq s^{\max}) = 1, \) \( \mathbb{E} \max(S_1, \ldots, S_n) \leq s^{\max} \) and therefore, \( np_n c \leq s^{\max} \). Thus

\[
p_n \leq \frac{s^{\max}}{c} \xrightarrow{n \to \infty} 0
\]

(5.2)

and by (5.1)

\[ t_n \xrightarrow{n \to \infty} 1. \]

(5.3)
If signals have a non-compact support the same result takes place but some more technical considerations must be invoked.

Since \( \Theta \sim U(0.9, 1) \) and \( t_n \to 1 \), \( \Pr(\Theta < t_n) \xrightarrow{n \to \infty} 1 \). However, \( S_1, \ldots, S_n \leq \Theta \), hence

\[
\Pr(K_n = 0) = \Pr(S_1 < t_n, \ldots, S_n < t_n) \geq \Pr(\Theta < t_n) \xrightarrow{n \to \infty} 1. \quad (5.4)
\]

Such a property of the distribution of participants cannot support an equilibrium since it was shown by Landsberger and Tsirelson [7] that in a large class of symmetric games, including auctions, in any symmetric equilibrium \( \liminf_n \Pr(K_n \geq 1) > 0 \), which violates (5.4). In fact, the result that in any equilibrium \( \liminf_n \Pr(K_n \geq 1) > 0 \) is obvious since otherwise a single player with \( s > c \) could always make a positive profit by participating.

To complete the picture that illustrates the effects of threshold participation strategies on the distribution of participants, it can be shown for large \( n \) that, roughly,

\[
K_n = 0 \quad \text{with probability } 1 - \text{const}/\sqrt{n},
\]

and

\[
K_n \sim \text{const} \cdot \sqrt{n} \quad \text{with probability } \text{const}/\sqrt{n}.
\]

Thus, \( \mathbb{E}K_n \sim \text{const} \), but \( K_n \) is mostly zero and otherwise it is very large; a burst in the space of outcomes, \( K_n \). Since we already know that such properties of outcomes are incompatible with equilibrium, a natural question is what are the properties of signals that induce such outcomes (under threshold strategies). This question is addressed in [7].

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\section*{Appendix A. Derivation of the \( \pi_{s^*}(s) \) function}

Invoking normality, \( N(0, 1) \) with correlation \( \rho \), we obtain

\[
\frac{F'_{\theta}(s_1)G'_{\phi}(\theta)}{F'_{\phi}(s_1)} = \frac{1}{\Phi'(s_1)} \cdot \frac{1}{\sqrt{1 - \rho}} \Phi'\left(\frac{s_1 - \theta}{\sqrt{1 - \rho}}\right) \cdot \frac{1}{\sqrt{\rho}} \Phi'\left(\frac{\theta}{\sqrt{\rho}}\right)
\]

\[
= \frac{1}{\sqrt{2\pi\rho(1 - \rho)}} \exp\left(\frac{1}{2} s_1^2 - \frac{1}{2(1 - \rho)}(s_1 - \theta)^2 - \frac{1}{2\rho} \theta^2\right)
\]
\[
\begin{align*}
\frac{1}{\sqrt{2\pi\rho(1-\rho)}} & \exp\left(-\frac{1}{2\rho(1-\rho)}(\theta - \rho s_1)^2\right) \\
\frac{1}{\sqrt{\rho(1-\rho)}} \Phi'\left(\frac{\theta - \rho s_1}{\sqrt{\rho(1-\rho)}}\right),
\end{align*}
\]

where

\[
\Phi(s) = (2\pi)^{-1/2} \int_{-\infty}^{s} e^{-u^2/2} \, du.
\]

Inserting (A.1) into (3.5) we obtain

\[
H_{s_1}(s) = \int F^{-1}_\theta(s_1) \, d\Phi\left(\frac{\theta - \rho s_1}{\sqrt{\rho(1-\rho)}}\right) = \mathbb{E} F^{-1}_{\rho s_1 + \sqrt{\rho(1-\rho)}\xi_0}(s).
\]

Inserting (A.3) into (3.2) we obtain an explicit equation for \(\pi(s_1)\) that holds for \(s_1 \leq s^*\):

\[
\pi_{s^*}(s_1) = (s_1 - r) + \mathbb{E} F^{-1}_{\rho s_1 + \sqrt{\rho(1-\rho)}\xi_0}(s^*).
\]

Doing the same for (3.4) we obtain

\[
\pi_{s^*}(s_1) = (s^* - r) + \mathbb{E} F^{-1}_{\rho s_1 + \sqrt{\rho(1-\rho)}\xi_0}(s^*) + \int_{s^*}^{s_1} \mathbb{E} F^{-1}_{\rho s_1 + \sqrt{\rho(1-\rho)}\xi_0}(s) \, ds,
\]

that holds for \(s_1 \geq s^*\).

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