Asymptotic behaviour of the solutions of inverse problems for pseudo-parabolic equations

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Abstract

We study asymptotic proximity as $t \to \infty$ of the solutions of the inverse problem for a pseudo-parabolic equation with an unknown source function. The overdetermination condition is given in integral form.

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1. Introduction

We study the problem of asymptotic behaviour as $t \to \infty$ of the solutions $\{u(x,t), f(t)\}$ of the inverse problem

$$u_t(x,t) - \Delta u_t(x,t) - \Delta u(x,t) + \alpha u(x,t) = f(t)g(x,t), \quad Q_{\infty} = \Omega \times (0, \infty),$$

(1)

$$u(x,0) = u_0(x), \quad x \in \Omega,$$

(2)

$$u(x,t) = 0, \quad (x,t) \in \partial \Omega \times [0, \infty),$$

(3)

$$\int_{\Omega} u(x,t)(w - \Delta w)(x) \, dx = \rho(t), \quad t \in [0, \infty).$$

(4)

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Here \( Q_\infty \equiv \Omega \times (0, \infty) \), where \( \Omega \) is a bounded domain in \( IR^n \) with smooth boundary. The functions \( g(x, t), w(x), \varphi(t), u_0(x) \), and the constant \( \alpha \) are given, while \( \{ u(x, t), f(t) \} \) is unknown. Additional information about the solution to the inverse problem is given in the form of integral overdetermination condition (4). From the physical point of view, this condition may be interpreted as measurements of the temperature \( u(x, t) \) by a device averaging over the domain of spatial variables \( \Omega \).

There are some studies about inverse problem for a parabolic and pseudo-parabolic equations with integral overdetermination (4) and its unique solubility \([1,2,4–8]\). Existence and uniqueness of the problem (1)–(4) is studied earlier \([3]\).

Let us introduce certain notations used below. We set

\[
G(t) = \int_\Omega g(x, t)w(x) \, dx, \quad Q_T = \Omega \times (0, T], \quad Q_{t_1 t_2}^0 = \Omega \times (t_1, t_2]
\]

\[
\|u\| \equiv \|u\|_{L^2(\Omega)} \text{ for } u(x) \in L^2(\Omega) \text{ and we denote by } \theta \text{ the constant in the Poincare inequality}
\]

\[
\|u\| \leq \theta \|\nabla u\|
\]

which is valid for each \( u(x) \in W^1_2(\Omega) \) and \( \theta = \theta(\Omega, n) > 0 \). We note that the Cauchy’s inequality

\[
2|\xi \eta| \leq \varepsilon \xi^2 + \varepsilon^{-1} \eta^2
\]

for \( \varepsilon > 0 \). Here

\[
\|\nabla u\| = \left( \int_\Omega \sum_{i=1}^n u_{x_i}^2 \, dx \right)^{1/2}, \quad \|\Delta u\| = \left( \int_\Omega \sum_{i,j=1}^n u_{x_ix_j}^2 \, dx \right)^{1/2}.
\]

We use the ordinary Banach spaces \( L^2(\Omega), L^2(0, T), W^1_2(\Omega), W^2_2(\Omega) \). By \( W^2_2(\Omega) \), we denote the Banach function spaces obtained by the closure of \( C^\infty(\Omega) \) with respect to the norm of \( W^1_2(\Omega) \). Let us also introduce the Sobolev space \( W^{2,1}_2(Q_T) \) of functions \( u(x, t) \) with finite norm

\[
\|u\|_{W^{2,1}_2(Q_T)} = \left( \|u\|_{L^2_2(Q_T)}^2 + \|D_x u\|_{L^2_2(Q_T)}^2 + \sum_{j=1}^n \|D_{x_j} u\|_{L^2_2(Q_T)}^2 \right)^{1/2}.
\]

2. Statement of the problem

We assume that the functions involved in the problem (1)–(4) and data of the problem are measurable and satisfy the following conditions;
(i) \( w \in W^2_t(\Omega) \cap W^1_2(\Omega), \|w\| \leq K_w, \)
(ii) \( \|g(. , t)\| \leq K_g, g \in L^\infty(0, T; L_2(\Omega)), \int_{\Omega} g(x, t) w(x) \, dx \geq g_0 > 0, \ t \in [0, \infty), \)
(iii) \( u_0(x) \in L_2(\Omega), \ \varphi(t) \in W^1_2(0, T), \ \forall T > 0, \ \text{and} \ \int_{\Omega} u_0(x)(w - \Delta w)(x) \, dx = \varphi(0). \)

\( K_w, K_g, g_0 \) are positive constants.

We multiply the Eq. (1) by \( w(x) \) and integrate over \( \Omega \). Then, with regard to the overdetermination condition (4), we obtain the relation

\[
f(t) = \frac{1}{G(t)} \left\{ \varphi'(t) + \varphi(t) + (z - 1) \int_{\Omega} u(x, t) w(x) \, dx \right\},
\]

where both sides are treated as elements of \( L^2_2(0, T) \). For \( z = 1 \), the problem turns to direct problem.

**Definition 1.** A pair of functions \( \{u(x, t), f(t)\} \) is called a generalized solution of the inverse problem (1)–(4) if

\[
u \in C\left( [0, T]; W^2_2(\Omega) \right) \cap W^{2,1}_2(Q_T), \ f \in L_2(0, T)
\]

and satisfy Eq. (1) almost everywhere in \( Q_T = \Omega \times (0, T) \), and \( u(x, t) \) satisfies conditions (2)–(4) in the usual sense.

3. Estimates and results

**Theorem 1.** Suppose that the conditions (i)–(iii) are satisfied. Assume the following inequality holds;

\[
\beta = z + \frac{1}{\theta} - \frac{K_w K_g}{g_0} \left( 1 + |z - 1| \right) > 0
\]

and assume also that

\[
\lim_{k \to \infty} \int_k^{k+1} |\varphi'(t)|^2 \, dt = 0, \quad \lim_{k \to \infty} \int_k^{k+1} |\varphi(t)|^2 \, dt = 0.
\]

Then for every initial function \( u_0 \in L_2(\Omega) \) the solution \( \{u(x, t), f(t)\} \) to the inverse problem (1)–(4) meets the following limit relations;

\[
\lim_{t \to \infty} \|u(. , t)\| = 0, \quad \lim_{k \to \infty} \int_k^{k+1} \|\nabla u(. , t)\|^2 \, dt = 0, \quad \lim_{k \to \infty} \int_k^{k+1} |f(t)|^2 \, dt = 0.
\]
Proof. Let us multiply Eq. (1) by \( u \) and integrate over \( \Omega \), and using (4) we get the relation

\[
\frac{1}{2} \frac{d}{dt} \left\{ \|u(., t)\|^2 + \|\nabla u(., t)\|^2 \right\} + \|\nabla u(., t)\|^2 + \alpha \|u(., t)\|^2 = f(t) \int_{\Omega} g(x, t) u(x, t) \, dx.
\]

(9)

From (5) and (9), we obtain the inequality

\[
\frac{1}{2} \frac{d}{dt} \left\{ \|u(., t)\|^2 + \|\nabla u(., t)\|^2 \right\} + \|\nabla u(., t)\|^2 + \alpha \|u(., t)\|^2 \\
\leq \frac{K_r}{g_0} \|u(., t)\| \left\{ |\varphi'(t)| + |\varphi(t)| + K_u |\alpha - 1\|u(., t)\| \right\}.
\]

(10)

Using the Poincare’s inequality left-hand side and the Cauchy’s inequality right-hand side of the inequality (10), we get

\[
\frac{1}{2} \left( 1 + \frac{1}{\delta^2} \right) \frac{d}{dt} \|u(., t)\|^2 + \left( \alpha + \frac{1}{\delta^2} \right) \|u(., t)\|^2 \\
\leq \frac{1}{2} \left( \delta_1 + \delta_2 + |\alpha - 1| \frac{K_u K_g}{g_0} \right) \|u(., t)\|^2 \\
+ \left( \frac{K_u}{g_0} \right)^2 \left\{ \frac{1}{2\delta_1} |\varphi'(t)|^2 + \frac{1}{2\delta_2} |\varphi(t)|^2 \right\}.
\]

(11)

Choose the constants \( \delta_1, \delta_2 > 0 \) such that \( \delta_1 = \delta_2 = (K_u K_g/2g_0) \), then from (11) and (6) we obtain

\[
\frac{d}{dt} \|u(., t)\|^2 + \beta \|u(., t)\|^2 \leq M \left\{ |\varphi'(t)|^2 + |\varphi(t)|^2 \right\},
\]

(12)

where \( M = (2K_u K_g/2g_0) \) positive constant. Integrating (12) with respect to \( t \) from 0 to \( t \), one has the following estimate.

\[
\|u(., t)\|^2 \leq \|u_0\|^2 e^{-\beta t} + M \int_0^t \left( |\varphi'(\tau)|^2 + |\varphi(\tau)|^2 \right) e^{-\beta(t-\tau)} \, d\tau.
\]

(13)

Estimate (13) can be written as

\[
\|u(., t)\|^2 \leq \|u_0\|^2 e^{-\beta t} + c \left\{ \int_k^{k+1} |\varphi'(t)|^2 \, dt + \int_k^{k+1} |\varphi(t)|^2 \, dt \right\} \sum_{0 \leq k < t} e^{-\beta(t-k-1)},
\]

(14)

where constant \( c \) depends on given values. From (14), (6) and (7), we obtain the result

\[
\lim_{t \to \infty} \|u(., t)\| = 0.
\]

(15)
By integrating inequality (10) on the interval \((k, k+1]\) with respect to \(t\) and using Poincare and Cauchy inequalities, we get the following inequality.

\[
\frac{1}{2} \left( 1 + \frac{1}{\theta^2} \right) \| u(\cdot, k+1) \|^2 + \int_{k}^{k+1} \| \nabla u(\cdot, t) \|^2 \, dt + \frac{x \| u \|_{L_2(Q_{k+1}^t)}}{1 + \theta^2} \]

\[
\leq \frac{1}{2} \left( 1 + \frac{1}{\theta^2} \right) \| u(\cdot, k) \|^2 + \frac{K_x}{g_0} \| u \|_{L_2(Q_{k+1}^t)} \left( \int_{k}^{k+1} |\varphi'(t)|^2 \, dt \right)^{1/2}
\]

\[
+ \frac{K_x}{g_0} \| u \|_{L_2(Q_{k+1}^t)} \left( \int_{k}^{k+1} |\varphi(t)|^2 \, dt \right)^{1/2} + \frac{|x-1|K_xK_g}{g_0} \| u \|_{L_2(Q_{k+1}^t)}^2.
\]

(16)

Omitting the positive terms (first and third terms) on the left-hand side of (16) and applying the Cauchy’s inequality, then we obtain the following relation.

\[
\int_{k}^{k+1} \| \nabla u(\cdot, t) \|^2 \, dt
\]

\[
\leq \left( 1 + \frac{1}{\theta^2} \right) \| u(\cdot, k) \|^2 + \frac{K_x}{g_0} \left( K_w |x-1| + \frac{K_g}{g_0} \right) \int_{k}^{k+1} \| u(\cdot, t) \|^2 \, dt
\]

\[
+ \left\{ \int_{k}^{k+1} |\varphi'(t)|^2 \, dt + \int_{k}^{k+1} |\varphi(t)|^2 \, dt \right\}.
\]

(17)

The inequality (17) can be written as follows:

\[
\int_{k}^{k+1} \| \nabla u(\cdot, t) \|^2 \, dt \leq c_1 \left( \| u(\cdot, k) \|^2 + \int_{k}^{k+1} \| u(\cdot, t) \|^2 \, dt \right)
\]

\[
+ c_2 \left( \int_{k}^{k+1} |\varphi'(t)|^2 \, dt + \int_{k}^{k+1} |\varphi(t)|^2 \, dt \right),
\]

where constants \(c_1\) and \(c_2\) are known values. By taking \(\delta(t) = (|\varphi'(t)|^2 + |\varphi(t)|^2)\), from inequality (13), we write

\[
\| u(\cdot, k) \|^2 + \int_{k}^{k+1} \| u(\cdot, t) \|^2 \, dt \leq \mu(k) = e^{-\beta k} \| u_0 \|^2 + \int_{0}^{k+1} e^{-\beta(k+1-\tau)} \delta(\tau) \, d\tau.
\]

So we get the following estimate

\[
\int_{k}^{k+1} \| \nabla u(\cdot, t) \|^2 \, dt \leq c_2 \mu(k) + c_3 \left( \int_{k}^{k+1} |\varphi'(t)|^2 \, dt + \int_{k}^{k+1} |\varphi(t)|^2 \, dt \right).
\]

(18)

Inequality (18), \(\lim_{k \to \infty} \mu(k) = 0\) and assumption (7) leads to following result

\[
\lim_{k \to \infty} \int_{k}^{k+1} \| \nabla u(t) \|^2 \, dt = 0.
\]

(19)
By applying the Poincare’s inequality to relation (10) and employing the same calculations as above we find the following limit relation.

\[
\lim_{k \to \infty} \int_{k}^{k+1} \| u(., t) \|^2 \, dt = 0. \tag{20}
\]

From the relation (5), we obtain the inequality

\[
\int_{k}^{k+1} |f(t)|^2 \, dt \\
\leq \frac{3}{8^2} \left( \int_{k}^{k+1} |\varphi'(t)|^2 \, dt + \int_{k}^{k+1} |\varphi(t)|^2 \, dt + (z - 1)^2 K_w^2 \int_{k}^{k+1} \| u(., t) \|^2 \, dt \right). \tag{21}
\]

Finally from the relations (21), (7) and (20) we get the last result

\[
\lim_{k \to \infty} \int_{k}^{k+1} |f(t)|^2 \, dt = 0.
\]

So, the solutions of the problem (1)–(4) are stable asymptotically. \( \square \)

References


