Triple positive solutions for a class of \( m \)-point dynamic equations on time scales with \( p \)-Laplacian

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Abstract

This paper deals with a class of second-order nonlinear \( m \)-point dynamic equation on time scales with one-dimensional \( p \)-Laplacian. Using a fixed point theorem for operators on a cone, we provide sufficient conditions for the existence of at least three positive solutions to the above boundary value problem. The interesting point is that the nonlinear term \( f \) is involved with the first-order delta derivative explicitly. Meanwhile, an example is worked out to demonstrate the main results.

Keywords: Time scales; Delta and nabla derivatives; Positive solution; Fixed point theorem; Existence

1. Introduction

Recently, there have been many papers working on the existence of positive solutions to boundary value problems for differential equations on time scales, see, for example, [1–7,9–12,14,15,17–23]. This has been mainly due to the unification of the theory of differential and difference equations. An introduction to this unification is given in [10,11,21,22]. Now, this study is still a new area of fairly theoretical exploration in mathematics. However, it has led to several important applications, for example, in the study of insect population models, neural networks, heat transfer, and epidemic models, see, for example, [1,10].

However, to the best of our knowledge, there are not much concerning the \( p \)-Laplacian problems on time scales, see [4,17]. The nonlinear term \( f \) in [4,17] is not involved with the first-order delta derivative. Many difficulties occur when the nonlinear term \( f \) is involved with the first-order delta derivative. For example, basic tools from calculus such as Fermat’s theorem, Rolle’s theorem and the intermediate value theorem may not necessarily hold.

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Motivated by works mentioned above, in this paper, we study the existence of multiple positive solutions for the second-order nonlinear m-point dynamic equation on time scales with one-dimensional p-Laplacian

\[(\phi_p(x^{\Delta}(t)))^p + w(t)f(t, x(t), x^{\Delta}(t)) = 0, \quad t_1 < t < t_m,\]  

(subject to one of the following boundary conditions:

\[x(t_1) = \sum_{i=2}^{m-1} \alpha_ix(t_i), \quad x^{\Delta}(t_m) = 0,\]  

(1.2)

or

\[x^{\Delta}(t_1) = 0, \quad x(t_m) = \sum_{i=2}^{m-1} \alpha_ix(t_i),\]  

(1.3)

where \(\phi_p(s) = |s|^{p-2}s,\ p > 1,\ (\phi_p)^{-1} = \phi_q,\ \frac{1}{p} + \frac{1}{q} = 1,\) and the points \(t_i \in \mathbb{T}_k^m\) for \(i \in \{1, 2, \ldots, m\}\) with \(0 = t_1 < t_2 < \cdots < t_m = 1,\ \mathbb{T}\) is a time scale;

\[w(t) \in C_c([t_1, t_m], [0, +\infty))\]  

(1.5)

and does not vanish identically on any closed subinterval of \([t_1, t_m],\) where \(C_c([t_1, t_m], [0, +\infty))\) denotes the set of all left dense continuous functions from \(\mathbb{T}\) to \([0, +\infty);\)

the function \(f : [t_1, t_m] \times [0, +\infty) \times \mathbb{R} \to [0, +\infty)\) is continuous.  

(1.6)

We remark that by a solution \(x\) of (1.1) and (1.2) (respectively (1.1) and (1.3)) we mean \(x : \mathbb{T} \to \mathbb{R}\) which is delta differentiable, \(x^{\Delta}\) and \((|x^{\Delta}|^{p-2}x^{\Delta})^P\) are both continuous on \(\mathbb{T}_k^m\), and \(x\) satisfies (1.1) and (1.2) (respectively (1.1) and (1.3)). If \(x^{\Delta}v(t) \leq 0\) on \([t_1, t_m]_{\mathbb{T}_k^m}\), then we say \(x\) is concave on \([t_1, t_m]\).

Our main results will depend on an application of a fixed point theorem due to Avery and Peterson which deals with fixed points of a cone-preserving operator defined on an ordered Banach space. The emphasis here is that the nonlinear term is involved explicitly with the first-order derivative.

The time scale related notations adopted in this paper can be found, if not explained specifically, in almost all literature related to time scales. The readers who are unfamiliar with this area can consult for example [10,11,21,22] for details.

**Definition 1.1.** A time scale \(\mathbb{T}\) is a nonempty closed subset of \(\mathbb{R}\).

**Definition 1.2.** Define the forward (backward) jump operator \(\sigma(t)\) at \(t\) for \(t < \sup \mathbb{T}(\rho(t) \leq t \leq \inf \mathbb{T})\) by

\[\sigma(t) = \inf\{\tau > t : \tau \in \mathbb{T}\}(\rho(t) = \sup\{\tau < t : \tau \in \mathbb{T}\})\]

for all \(t \in \mathbb{T}\). We assume throughout that \(\mathbb{T}\) has the topology that it inherits from the standard topology on \(\mathbb{R}\) and say \(t\) is right-scattered, left-scattered, right-dense and left-dense if \(\sigma(t) > t, \rho(t) < t, \sigma(t) = t\) and \(\rho(t) = t\), respectively. Finally, we introduce the sets \(\mathbb{T}_k^m\) and \(\mathbb{T}_k\), which are derived from the time scale \(\mathbb{T}\) as follows. If \(\mathbb{T}\) has a left-scattered maximum \(t_1\), then \(\mathbb{T}_k^m = \mathbb{T} - t_1,\) otherwise \(\mathbb{T}_k^m = \mathbb{T}\). If \(\mathbb{T}\) has a right-scattered minimum \(t_2\), then \(\mathbb{T}_k = \mathbb{T} - t_2,\) otherwise \(\mathbb{T}_k = \mathbb{T}\).

**Definition 1.3.** Fix \(t \in \mathbb{T}_k^m\) and let \(y : \mathbb{T} \to \mathbb{R}\). Define \(y^\Delta(t)\) to be the number (if it exists) with the property that given \(\varepsilon > 0\) there is a neighborhood \(U\) of \(t\) with

\[|y(\sigma(t)) - y(s)| - y^\Delta(t)|\sigma(t) - s| < \varepsilon|\sigma(t) - s|\]

for all \(s \in U\), where \(y^\Delta\) denotes the (delta) derivative of \(y\) at \(t\).
**Definition 1.4.** Fix \( t \in \mathbb{T}_k \) and let \( y : \mathbb{T} \to \mathbb{R} \). Define \( y^\nabla (t) \) to be the number (if it exists) with the property that given \( \varepsilon > 0 \) there is a neighborhood \( U' \) of \( t \) with

\[
\| (y(\rho(t)) - y(s)) - y^\nabla(t)(\rho(t) - s) \| < \varepsilon |\rho(t) - s|
\]

for all \( s \in U' \). Call \( y^\nabla(t) \) the (nabla) derivative of \( y \) at the point \( t \).

If \( \mathbb{T} = \mathbb{R} \) then \( f^\Delta(t) = f^\nabla(t) = f'(t) \). If \( \mathbb{T} = \mathbb{Z} \) then \( f^\Delta(t) = f(t+1) - f(t) \) is the forward difference operator while \( f^\nabla(t) = f(t) - f(t-1) \) is the backward difference operator.

**Definition 1.5.** A function \( F : \mathbb{T}^k \to \mathbb{R} \) is called a delta-antiderivative of \( f : \mathbb{T}^k \to \mathbb{R} \) provided \( F^\Delta(t) = f(t) \) holds for all \( t \in \mathbb{T}^k \). In this case we define the delta integral of \( f \) by

\[
\int_a^t f(s) \Delta s = F(t) - F(a),
\]

for all \( a, t \in \mathbb{T} \).

**Definition 1.6.** A function \( \Phi : \mathbb{T}^k \to \mathbb{R} \) is called a nabla-antiderivative of \( f : \mathbb{T}^k \to \mathbb{R} \) provided \( \Phi^\nabla(t) = f(t) \) holds for all \( t \in \mathbb{T}^k \). In this case we define the nabla integral of \( f \) by

\[
\int_a^t f(s) \nabla s = \Phi(t) - \Phi(a),
\]

for all \( a, t \in \mathbb{T} \).

**Remark 1.1.** All right-dense continuous bounded functions on \([0, 1]\) are delta integrable from 0 to 1, and all left-dense continuous bounded functions on \((0, 1]\) are nabla integrable from 0 to 1.

2. Preliminaries

In this section, we provide some background materials from the theory of cones in Banach spaces, and we then state the triple fixed point theorem for a cone-preserving operator. The following definitions can be found in the book by Deimling [13] as well as in the book by Guo and Lakshmikantham [16].

**Definition 2.1.** Let \( E \) be a real Banach space over \( \mathbb{R} \). A nonempty closed set \( P \subset E \) is said to be a cone, provided that

1. \( au + bv \in P \) for all \( u, v \in P \) and all \( a \geq 0, b \geq 0 \) and
2. \( u, -u \in P \) implies \( u = 0 \).

Every cone \( P \subset E \) induces an ordering in \( E \) given by \( x \leq y \) if and only if \( y - x \in P \).

**Definition 2.2.** An operator is called completely continuous if it is continuous and maps bounded sets into precompact sets.

**Definition 2.3.** The map \( \alpha \) is said to be a nonnegative continuous concave functional on a cone \( P \) of a real Banach space \( E \) provided that \( \alpha : P \to [0, \infty) \) is continuous and

\[
\alpha(tx + (1-t)y) \geq t\alpha(x) + (1-t)\alpha(y)
\]

for all \( x, y \in P \) and \( 0 \leq t \leq 1 \). Similarly, we say the map \( \gamma \) is a nonnegative continuous convex functional on a cone \( P \) of a real Banach space \( E \) provided that \( \gamma : P \to [0, \infty) \) is continuous and

\[
\gamma(tx + (1-t)y) \leq t\gamma(x) + (1-t)\gamma(y)
\]

for all \( x, y \in P \) and \( 0 \leq t \leq 1 \).
Let γ and θ be nonnegative continuous convex functionals on \( P \), \( \alpha \) be a nonnegative continuous concave functional on \( P \), and \( \psi \) be a nonnegative continuous functional on \( P \). Then for positive real numbers \( a, b, c \), and \( d \), we define the following convex sets:

\[
P(\gamma, d) = \{ x \in P | \gamma(x) < d \},
\]

\[
P(\gamma, \alpha, b, d) = \{ x \in P | b \leq \alpha(x), \gamma(x) \leq d \},
\]

\[
P(\gamma, \theta, \alpha, b, c, d) = \{ x \in P | b \leq \alpha(x), \theta(x) \leq c, \gamma(x) \leq d \},
\]

and a closed set

\[
R(\gamma, \psi, a, d) = \{ x \in P | a \leq \psi(x), \gamma(x) \leq d \}.
\]

The following fixed point theorem due to Avery and Peterson is fundamental in the proofs of our main results.

**Theorem 2.1** ([8]). Let \( P \) be a cone in a real Banach space \( E \). Let \( \gamma \) and \( \theta \) be nonnegative continuous convex functionals on \( P \), \( \alpha \) be a nonnegative continuous concave functional on \( P \), and \( \psi \) be a nonnegative continuous functional on \( P \) satisfying \( \psi(\lambda x) \leq \lambda \psi(x) \) for \( 0 \leq \lambda \leq 1 \), such that for some positive numbers \( M \) and \( d \),

\[
\alpha(x) \leq \psi(x), \| x \| \leq M \gamma(x)
\]

for all \( x \in \overline{P(\gamma, d)} \). Suppose \( T : \overline{P(\gamma, d)} \rightarrow \overline{P(\gamma, d)} \) is completely continuous and there exist positive numbers \( a, b, c \) with \( a < b \) such that

(S1) \( \{ x \in P(\gamma, \theta, b, c, d) | \alpha(x) > b \} \neq 0 \) and \( \alpha(Tx) > b \) for \( x \in P(\gamma, \theta, b, c, d) \);

(S2) \( \alpha(Tx) > b \) for \( x \in P(\gamma, \alpha, b, d) \) with \( \theta(Tx) > c \);

(S3) \( 0 \notin R(\gamma, \psi, a, d) \) and \( \psi(Tx) < a \) for \( x \in R(\gamma, \psi, a, d) \) with \( \psi(x) = a \).

Then \( T \) has at least three positive solutions \( x_1, x_2, x_3 \in \overline{P(\gamma, d)} \), such that

\[
\gamma(x_i) \leq d \quad \text{for } i = 1, 2, 3,
\]

\[
b < \alpha(x_1),
\]

\[
a < \psi(x_2), \quad \text{with } \alpha(x_2) < b,
\]

and

\[
\psi(x_3) < a.
\]

### 3. Existence of triple positive solutions to (1.1) and (1.2)

In this section, we impose growth conditions on \( f \) which allow us to apply **Theorem 2.1** to establish the existence of triple positive solutions of (1.1) and (1.2).

Now we define the real Banach space \( E = C^\Delta[[t_1, \sigma(t_m)] \) to be the set of all \( \Delta \)-differentiable functions with continuous \( \Delta \)-derivative on \( [t_1, \sigma(t_m)] \) with the norm

\[
\| x \|_{1, T} = \max \{ \| x \|_0, T, \| x^\Delta \|_{0, T} \}, \quad x \in E,
\]

where \( \| x \|_0, T := \sup \{ |x(t)| : t \in [t_1, t_m] \} \), \( \| x \|_{0, T} := \sup \{ |x^\Delta(t)| : t \in [t_1, t_m] \} \), \( x \in E \).

From the fact \( (\phi_p(x^\Delta(t)))^\vee = -w(t) f(t, x(t), x^\Delta(t)) \leq 0 \) for \( t \in [t_1, t_m] \), we know that \( x \) is concave on \( [t_1, t_m] \). So, define a cone \( P \subset E \) by

\[
P = \left\{ x \in E : x(t) \geq 0, x(t_1) = \sum_{i=2}^{n-1} \alpha_i x(t_i), \text{ } x \text{ is concave on } [t_1, t_m] \right\} \subset E.
\]

For the sake of applying **Theorem 2.1**, we let the nonnegative continuous concave functional \( \alpha_1 \), the nonnegative continuous convex functional \( \gamma_1 \), \( \psi_1 \) be defined on the cone \( P \) by

\[
\gamma_1(x) = \max_{t \in [t_1, t_m]} |x^\Delta(t)|, \quad \psi_1(x) = \theta_1(x) = \max_{t \in [t_1, t_m]} |x(t)|, \quad \alpha_1(x) = \min_{t \in [1/n, (n-1)/n]} |x(t)|, \quad x \in P.
\]
where \( n > \max \{1, \frac{t_2 - t_{m-1}}{t_m - t_{m-1}}\} \).

In our main results, we will make use of the following lemmas.

**Lemma 3.1.** If (1.4) holds, then for \( x \in P \), there exists a constant \( M > 0 \) such that
\[
\max_{t \in [t_1, t_m]} |x(t)| \leq M \max_{t \in [t_1, t_m]_{\gamma k}} |x^\Delta(t)|.
\]

**Proof.** By the concavity of \( x \), there is
\[
x(t) - x(t_1) \leq x^\Delta(t_1) \leq \max_{\xi \in [t_1, t_m]_{\gamma k}} |x^\Delta(\xi)|.
\]

(3.1)

Similarly, we have
\[
\left(1 - \sum_{i=2}^{m-1} \alpha_i\right)x(t_1) = \sum_{i=2}^{m-1} \alpha_i [x(t_i) - x(t_1)] \leq \sum_{i=2}^{m-1} \alpha_i \max_{\xi \in [t_1, t_m]_{\gamma k}} |x^\Delta(\xi)|.
\]

Thus, we have
\[
\max_{t \in [t_1, t_m]} |x(t)| \leq \frac{1}{1 - \sum_{i=2}^{m-1} \alpha_i} \max_{t \in [t_1, t_m]_{\gamma k}} |x^\Delta(t)|.
\]

Therefore, setting
\[
M = \frac{1}{1 - \sum_{i=2}^{m-1} \alpha_i},
\]
the proof is complete. \( \square \)

With Lemma 3.1 and the concavity of \( x \), for all \( x \in P \), the functionals defined above hold the relations
\[
\frac{1}{n} \theta_1(x) \leq \alpha_1(x) \leq \theta_1(x), \quad \|x\|_{1, \mathbb{T}} = \max \{\theta_1(x), \gamma_1(x)\} \leq M \gamma_1(x).
\]

(3.3)

Therefore, the condition (2.1) of Theorem 2.1 is satisfied.

**Lemma 3.2.** Suppose that (1.4) is satisfied. If \( y \in C_{ld}[t_1, t_m] \), then BVP
\[
\begin{cases}
(\phi_p(x^\Delta(t)))^\nabla + y(t) = 0, & t_1 < t < t_m, \\
x(t_1) = \sum_{i=2}^{m-1} \alpha_i x(t_i), & x^\Delta(t_m) = 0,
\end{cases}
\]

(3.4)

has a unique solution
\[
x(t) = \int_{t_1}^{t} \phi_q \left( \int_{\xi}^{t_m} y(r) \nabla r \right) \Delta s + \frac{1}{1 - \sum_{i=2}^{m-1} \alpha_i} \sum_{i=2}^{m-1} \phi_q \left( \int_{t_1}^{t} \int_{s}^{t_m} y(r) \nabla r \right) \Delta s.
\]

(3.5)

**Proof.** Integrating the differential equation from \( t \) to \( t_m \) one has
\[
\phi_p(x^\Delta(t)) = \int_{t}^{t_m} y(r) \nabla r, \quad \text{i.e.,} \quad x^\Delta(t) = \phi_q \left( \int_{t}^{t_m} y(r) \nabla r \right).
\]

Integrate the differential equation above from \( t_1 \) to \( t \) one has
\[
x(t) - x(t_1) = \int_{t_1}^{t} \phi_q \left( \int_{s}^{t_m} y(r) \nabla r \right) \Delta s,
\]
i.e.,

\[ x(t) = x(t_1) + \int_{t_1}^{t} \phi_q \left( \int_{s}^{t_m} y(r) \nabla r \right) \Delta s. \]  

(3.6)

Applying the second boundary condition and (3.6) one has

\[ x(t_1) = \frac{1}{1 - \sum_{i=2}^{m-1} \alpha_i} \sum_{i=2}^{m-1} \alpha_i \int_{t_1}^{t_i} \phi_q \left( \int_{s}^{t_m} y(r) \nabla r \right) \Delta s. \]  

(3.7)

Therefore, from (3.6) and (3.7), one has that (3.5) holds.

Otherwise, let \( x \) be as in (3.5). Taking the delta derivative of \( x(t) \) gives

\[ x^\Delta(t) = \phi_q \left( \int_{t}^{t_m} y(r) \nabla r \right), \quad \text{i.e.,} \quad \phi_p(x^\Delta(t)) = \int_{t}^{t_m} y(r) \nabla r. \]  

(3.8)

It is that \( x^\Delta(t_m) = 0 \). Substituting the boundary points \( t_i \) in (3.5) we see that

\[
\sum_{i=2}^{m-1} \alpha_i x(t_i) = \sum_{i=2}^{m-1} \alpha_i \int_{t_1}^{t_i} \phi_q \left( \int_{s}^{t_m} y(r) \nabla r \right) \Delta s \left( 1 + \frac{\sum_{i=2}^{m-1} \alpha_i}{1 - \sum_{i=2}^{m-1} \alpha_i} \right)
\]

\[
= \frac{\sum_{i=2}^{m-1} \alpha_i}{1 - \sum_{i=2}^{m-1} \alpha_i} \int_{t_1}^{t_m} \phi_q \left( \int_{s}^{t_m} y(r) \nabla r \right) \Delta s
\]

\[
= x(t_1).
\]

So the boundary conditions of BVP (3.4) are satisfied. Taking the nabla derivative of (3.8) gives

\[ (\phi_p(x^\Delta(t)))^\nabla = -y(t). \]

The proof is complete. □

Lemma 3.3. Suppose that (1.4) is satisfied. If \( y \in C_{ld}[t_1, t_m] \) and \( y \geq 0 \), then the unique solution \( x \) of BVP (3.4) satisfies the condition that \( x(t) \geq 0 \).

Proof. By Lemma 3.2, for \( t \in [t_1, t_m] \), we have

\[ x(t) = \int_{t_1}^{t} \phi_q \left( \int_{s}^{t_m} y(r) \nabla r \right) \Delta s + \frac{1}{1 - \sum_{i=2}^{m-1} \alpha_i} \sum_{i=2}^{m-1} \alpha_i \int_{t_1}^{t_i} \phi_q \left( \int_{s}^{t_m} y(r) \nabla r \right) \Delta s. \]

This together with (1.4) implies that \( x(t) \geq 0 \). The proof is complete. □

Define an operator \( A : P \rightarrow P \) by

\[
(Ax)(t) := \int_{t_1}^{t} \phi_q \left( \int_{s}^{t_m} w(r) f(r, x(r), x^\Delta(r)) \nabla r \right) \Delta s
\]

\[
+ \frac{1}{1 - \sum_{i=2}^{m-1} \alpha_i} \sum_{i=2}^{m-1} \alpha_i \int_{t_1}^{t_i} \phi_q \left( \int_{s}^{t_m} w(r) f(r, x(r), x^\Delta(r)) \nabla r \right) \Delta s.
\]  

(3.9)
By (3.9), it is well known that (1.1) and (1.2) have a positive solution \( x \) if and only if \( x \in P \) is a fixed point of \( A \) and
\[
(Ax)^\Delta (t) := \phi_q \left( \int_t^{s_m} w(r) f(r, x(r), x^\Delta (r)) \nabla r \right), \text{ for } x \in P, \quad t \in [t_1, t_m]. \tag{3.10}
\]

**Lemma 3.4.** Let (1.4)–(1.6) hold. Then \( AP \subset P \) and \( A : P \to P \) is completely continuous.

**Proof.** For \( x \in P \), by (3.9), we deduce that there is \( Ax \in C^\Delta [t_1, \sigma(t_m)] \) which is nonnegative and \( (Ax)(t_1) = \sum_{i=2}^{m-1} \alpha_i x(t_i) \).

On the other hand, we have by using Theorem 8.39 in [10]
\[
(\phi_p(x^\Delta (t)))^\nabla = -w(t) f(t, x(t), x^\Delta (t)) \leq 0, \quad t \in [t_1, t_m],
\]
which implies that \( Ax \) is concave on \([t_1, t_m]\). Therefore, \( A(P) \subset P \).

Next we shall prove that operator \( A \) is completely continuous. We break the proof into several steps.

**Step 1.** Operator \( A \) is continuous. Since the function \( f \) is continuous, this conclusion can be easily obtained.

**Step 2.** For each constant \( l > 0 \), let \( B_l = \{ x \in P : \|x\|_{1, T} \leq l \} \). Then \( B_l \) is a bounded closed convex set in \( P \). \( \forall x \in B_l \), from (3.9) and (3.10), we have
\[
\| (Ax)(t_0) \|_{0, T} \leq (MN)^{q-1}(t_1 - t_0) + \frac{1}{1 - \sum_{i=2}^{m-1} \alpha_i (MN)^{q-1}} \sum_{i=2}^{m-1} \alpha_i (MN)^{q-1}(t_i - t_0)
\]
\[
\leq (MN)^{q-1}(1 - t_0) + \frac{1}{1 - \sum_{i=2}^{m-1} \alpha_i (MN)^{q-1}} \sum_{i=2}^{m-1} \alpha_i (MN)^{q-1}(1 - t_0)
\]
\[
= (MN)^{q-1} + \frac{1}{1 - \sum_{i=2}^{m-1} \alpha_i (MN)^{q-1}} \sum_{i=2}^{m-1} \alpha_i (MN)^{q-1}
\]
\[
= (MN)^{q-1},
\]
and
\[
\| (Ax)^\Delta (t_0) \|_{0, T} \leq (MD)^{q-1},
\]
where
\[
M = \sup_{r \in [s, t_m]} f(t, x(t), x^\Delta (t)), \quad N = \int_{s}^{t_m} w(r) \nabla r,
\]
\[
D = \int_{s}^{t_m} w(r) \nabla r, \quad s \in [t_1, t_m], \quad t \in [t_1, t_m].
\]

Therefore, \( A(B_l) \) is uniformly bounded.

**Step 3.** The family \( \{ Ax : x \in B_l \} \) is a family of equicontinuous functions.

Let \( \bar{t}, t^* \in [t_1, t_m], \bar{t} < t^* \) and \( B_l = \{ x \in P : \|x\|_{1, T} \leq l \} \) be a bounded set of \( P \).

Hence
\[
| (Ax)(\bar{t}) - (Ax)(t^*) | = \left| \int_{\bar{t}}^{t^*} \phi_q \left( \int_{s}^{t_m} w(r) f(r, x(r), x^\Delta (r)) \nabla r \right) \Delta s \right|
\]
\[
- \left| \int_{\bar{t}}^{t^*} \phi_q \left( \int_{s}^{t_m} w(r) f(r, x(r), x^\Delta (r)) \nabla r \right) \Delta s \right|
\]
Assume \( \text{holds.} \) Let \( f \) and \( \lim_{t \to t^*} f(t) = l \) for the existence of at least three positive solutions to problem \( (\text{A}_1) \).

In fact, the concavity of \( f \) implies that \( A \) is completely continuous.

Thus, the set \( \{ Ax : x \in B_t \} \) is equicontinuous.

As a consequence of Step 1 to Step 3 together with the Ascoli–Arzela theorem we can prove \( A : P \to P \) is completely continuous. \( \square \)

In addition, we can prove the following result.

\[
\min_{t \in [1/n, (n-1)/n]} (Ax)(t) \geq \frac{1}{n} \| Ax \|_{0, \mathbb{T}} = \frac{1}{n} (Ax)(t_m). \tag{3.11}
\]

In fact, the concavity of \( Ax \) on \([t_1, t_m]\), \( t_m = 1 \) and (3.9) imply

\[
\frac{(Ax)(t)}{t} \geq \frac{(Ax)(t_m)}{t_m} = \| Ax \|_{0, \mathbb{T}}, \quad \text{for } t \in [1/n, (n-1)/n].
\]

which implies that (3.11) holds.

Let

\[
\delta_i = \int_{t_i}^{t_i'} \phi_q \left( \int_s^{t_i} w(r) \Delta r \right) \Delta s, \quad \delta = \min_{i \in [1, m-1]} \{ \delta_i \}, \quad \rho = \phi_q \left( \int_{t_1}^{t_m} w(r) \Delta r \right),
\]

\[
L_i = \int_{t_1}^{t_m} \phi_q \left( \int_s^{t_m} w(r) \Delta r \right) \Delta s + \frac{1}{1 - \sum_{i=2}^{m-1} \alpha_i} \sum_{i=2}^{m-1} \alpha_i \int_{t_1}^{t_i} \phi_q \left( \int_s^{t_i} w(r) \Delta r \right) \Delta s,
\]

\[
L = \max_{i \in [1, m-1]} \{ L_i \}, \quad t_i^* = \frac{t_i + t_{i+1}}{2} \quad (i = 1, 2, \ldots, m-1),
\]

\[
N = \frac{n}{2} \left[ 1 + \left( 1 - \sum_{i=2}^{m-1} \alpha_i \right) \right], \quad 0 = t_1 < \frac{1}{n} < t_2 < \cdots < t_{m-1} = t_m - \frac{2}{n} < t_m = 1.
\]

We are now ready to apply the Avery–Peterson fixed point theorem to the operator \( A \) to give sufficient conditions for the existence of at least three positive solutions to problem (1.1) and (1.2).

**Theorem 3.1.** Assume (1.4)–(1.6) hold. Let \( 0 < a < b \leq \frac{2Md}{N} \), and suppose that \( f \) satisfies the following conditions:

\( (A_1) \) \( f(t, u, v) \leq \phi_p(d/\rho) \), for \((t, u, v) \in [t_1, t_m] \times [0, Md] \times [-d, d]; \)
(A2) \( f(t, u, v) > \phi_p(nb/\delta) \), for \((t, u, v) \in \left[ \frac{1}{n}, \frac{n-1}{n} \right] \times [b, Nb] \times [-d, d] \);

(A3) \( f(t, u, v) < \phi_p(a/L) \), for \((t, u, v) \in [t_1, t_m] \times [0, a] \times [-d, d] \).

Then boundary value problem (1.1) and (1.2) has at least three positive solutions \( x_1, x_2, \) and \( x_3 \) such that

\[
\max_{t \in [t_1, t_m]} |x_i^\Delta(t)| \leq d, \quad i = 1, 2, 3;
\]

\[
b < \min_{t \in [1/n, (n-1)/n]} |x_1(t)|, \quad \max_{t \in [1/n, (n-1)/n]} |x_1(t)| \leq Md,
\]

\[
a < \max_{t \in [t_1, t_m]} |x_2(t)|, \quad \min_{t \in [1/n, (n-1)/n]} |x_2(t)| < b,
\]

and

\[
\max_{t \in [t_1, t_m]} |x_3(t)| < a.
\]

**Proof.** Problem (1.1) and (1.2) has a solution \( x = x(t) \) if and only if \( x \) solves the operator equation \( x = Ax \). Thus we set out to verify that the operator \( A \) satisfies the Avery--Peterson fixed point theorem which will prove the existence of three fixed points of \( A \) which satisfy the conclusion of the theorem.

For \( x \in \overline{P}^{(1)}(\gamma_1, d) \), there is \( \gamma_1(x) = \max_{t \in [t_1, t_m]} |x^\Delta(t)| \leq d \). For \( t \in [t_1, t_m] \), With Lemma 3.1, there is \( \max_{t \in [t_1, t_m]} |x^\Delta(t)| \leq Md \), then condition (A1) implies that \( f(t, x(t), x^\Delta(t)) \leq \phi_p(d/\rho) \). On the other hand, for \( x \in P \), there is \( Ax \in P \), then \( Ax \) is concave on \([t_1, t_m] \), and \( \max_{t \in [1/n, (n-1)/n]} |(Ax)^\Delta(t)| = (Ax)^\Delta(t) \). So

\[
\gamma_1(Ax) = \max_{t \in [t_1, t_m]} |(Ax)^\Delta(t)|
\]

\[
= \phi_p^{-1} \left( \int_{t_1}^{t_m} w(r) f(r, x(r), x^\Delta(r)) \Delta r \right)
\]

\[
\leq \frac{d}{\rho} \phi_p^{-1} \left( \int_{t_1}^{t_m} w(r) \Delta r \right) = \frac{d}{\rho} \rho = d.
\]

Therefore, \( A : \overline{P}(\gamma_1, d) \to \overline{P}(\gamma_1, d) \).

To check condition (S1) of Theorem 2.1, we choose

\[
x_0(t) = \frac{1}{2} n b (1 - \sum_{i=2}^{n-1} \alpha_i x(t_i)) t + \frac{nb}{2} = \frac{Mnb}{2} t + \frac{nb}{2} = Nb t, \quad t \in [t_1, t_m].
\]

It is easy to see that \( x_0 \in P(\gamma_1, \theta_1, \alpha_1, b, Nb, d) \) and \( \alpha_1(x_0) > b \), and so

\[
\{ x \in P(\gamma_1, \theta_1, \alpha_1, b, Nb, d) | \alpha_1(x) > b \} \neq \emptyset.
\]

Hence, for \( t \in [1/n, (n-1)/n] \), \( x(t) \in P(\gamma_1, \theta_1, \alpha_1, b, Nb, d) \), there is

\[
b \leq x(t) \leq Nb, \quad |x^\Delta(t)| \leq d.
\]

Thus, for \( t \in [1/n, (n-1)/n] \), by condition (A2) of this theorem, we have

\[
f(t, x(t), x^\Delta(t)) > \phi_p(nb/\delta),
\]

and combining the conditions of \( \alpha_1 \) and \( P \), we have by (3.11)

\[
\alpha_1(Ax) = \min_{t \in [1/n, (n-1)/n]} |(Ax)(t)| \geq \frac{1}{n} \|Ax\|_{0, \mathbb{T}} = \frac{1}{n} (Ax)(t_m)
\]

\[
= \frac{1}{n} \left[ \int_{t_1}^{t_m} \phi_t \left( \int_{t_1}^{t_m} w(r) f(r, x(r), x^\Delta(r)) \Delta r \right) \right] \Delta s.
\]
is satisfied. On the other hand, for \( t \in [1, t_m] \), we have at least three positive solutions \( x_1, x_2, x_3 \) such that

\[
\frac{1}{L} \left[ \int_{t_1}^{t_m} \phi_q \left( \int_{s}^{t_m} w(r, x(r), x^\Delta(r)) \Delta s \right) \right]
\]

\[
\leq a.
\]

(3.12)

Hence, from (3.12), we have

\[
\psi_1(Ax) = \max_{t \in [t_1, t_m]} |Ax(t)| < a.
\]

So, the condition (S_3) of Theorem 2.1 is satisfied. On the other hand, for \( x \in P \), (3.3) holds. Therefore, an application of Theorem 2.1 implies that problems (1.1) and (1.2) have at least three positive solutions \( x_1, x_2, x_3 \) such that

\[
\max_{t \in [t_1, t_m]} |x_i^\Delta(t)| \leq d, \quad i = 1, 2, 3;
\]

\[
b < \min_{t \in [1/n, (n-1)/n] \mathbb{T}} |x_1(t)|, \quad \max_{t \in [t_1, t_m]} |x_1(t)| \leq Md,
\]

\[
a < \max_{t \in [t_1, t_m]} |x_2(t)|, \quad \min_{t \in [1/n, (n-1)/n] \mathbb{T}} |x_2(t)| < b,
\]

and

\[
\max_{t \in [t_1, t_m]} |x_3(t)| < a.
\]

The proof is complete. □

i.e., \( \alpha_1(Ax) > b \) for all \( x \in P(\gamma_1, \theta_1, \alpha_1, b, Nb, d) \).

This shows that condition (S_1) of Theorem 2.1 is satisfied.

Secondly, with (3.2), we have for all \( x \in P(\gamma_1, \alpha_1, b, d) \) with \( \theta_1(Ax) > nb \)

\[
\alpha_1(Ax) \geq \frac{1}{n} \theta_1(Ax) > \frac{1}{n} nb = b.
\]

Thus, condition (S_2) of Theorem 2.1 is satisfied.

Finally, we show that condition (S_3) of Theorem 2.1 also holds. Clearly, as \( \psi_1(0) = 0 < a \), there holds \( 0 \notin R(\gamma_1, \psi_1, a, d) \). Suppose that \( x \in R(\gamma_1, \psi_1, a, d) \) with \( \psi_1(x) = a \). Then, by the condition (A_3) of this theorem, we obtain

\[
\psi_1(Ax) = \max_{t \in [t_1, t_m]} |(Ax)(t)| = \|Ax\|_{0, \mathbb{T}} = (Ax)(t_m)
\]

\[
= \int_{t_1}^{t_m} \phi_q \left( \int_{s}^{t_m} w(r, x(r), x^\Delta(r)) \Delta s \right) \Delta s
\]

\[
+ \frac{1}{L} \left[ \int_{t_1}^{t_m} \phi_q \left( \int_{s}^{t_m} w(r, x(r), x^\Delta(r)) \Delta s \right) \right]
\]

\[
< \frac{a}{L} \left[ \int_{t_1}^{t_m} \phi_q \left( \int_{s}^{t_m} w(r, x(r)) \Delta s \right) \right]
\]

\[
\leq a.
\]
4. Existence of triple positive solutions to (1.1) and (1.3)

Now we deal with problem (1.1) and (1.3). The method is just similar to what we have done in Section 3, so we omit the proof of the main result of this section.

Consider the Banach space $E$ defined as in Section 3 and define a cone $P_1 \subset E$ by

$$P_1 = \left\{ x \in E : x(t) \geq 0, x(t_m) = \sum_{i=2}^{m-1} \alpha_i x(t_i), \text{ x is concave on } [t_1, t_m] \right\} \subset E.$$ 

For the sake of applying Theorem 2.1, we let the nonnegative continuous concave functional $\alpha^*$, the nonnegative continuous convex functional $\theta^*$, $\gamma^*$, and the nonnegative continuous functional $\psi^*$ be defined on the cone $P_1$ by

$$\gamma^*(x) = \max_{t \in [t_1, t_m]} |x^\Delta(t)|, \psi^*(x) = \theta^*(x) = \max_{t \in [t_1, t_m]} |x(t)|, \quad \alpha^*(x) = \min_{t \in [1/n, (n-1)/n] \mathbb{T}} |x(t)|, \quad x \in P_1,$$

where $n$ is defined in Section 3.

In our main results, we will make use of the following lemmas.

**Lemma 4.1.** If (1.4) holds, then for $x \in P_1$, there exists a constant $M > 0$ such that

$$\max_{t \in [t_1, t_m]} |x(t)| \leq M \max_{t \in [t_1, t_m]} |x^\Delta(t)|.$$

With Lemma 4.1 and the concavity of $x$, for all $x \in P_1$, the functionals defined above hold the relations

$$\frac{1}{n} \theta^*(x) \leq \alpha^*(x) \leq \theta^*(x), \quad \|x\|_{1, \mathbb{T}} = \max\{\theta^*(x), \gamma^*(x)\} \leq M \gamma^*(x). \quad (4.1)$$

Therefore, the condition (2.1) of Theorem 2.1 is satisfied.

**Lemma 4.2.** Suppose that (1.4) is satisfied. If $y \in C_{ld}[t_1, t_m]$, then BVP

$$\begin{cases}
(\phi_p(x^\Delta(t)))^\Delta + y(t) = 0, & t_1 < t < t_m, \\
x^\Delta(t_1) = 0, & x(t_m) = \sum_{i=2}^{m-1} \alpha_i x(t_i),
\end{cases} \quad (4.2)$$

has a unique solution

$$x(t) = \int_{t_1}^{t_m} \phi_q \left( \int_{t_1}^{s} w(r) f(r, x(r), x^\Delta(r)) \, dr \right) \Delta s$$

$$+ \frac{1}{1 - \sum_{i=2}^{m-1} \alpha_i} \sum_{i=2}^{m-1} \alpha_i \int_{t_i}^{t_m} \phi_q \left( \int_{t_1}^{s} w(r) f(r, x(r), x^\Delta(r)) \, dr \right) \Delta s. \quad (4.3)$$

**Lemma 4.3.** Suppose that (1.4) is satisfied. If $y \in C_{ld}[t_1, t_m]$ and $y \geq 0$, then the unique solution $x$ of BVP (4.2) satisfies that $x(t) \geq 0$.

Define an operator $A^*: P_1 \to P_1$ by

$$(A^*x)(t) := \int_{t_1}^{t_m} \phi_q \left( \int_{t_1}^{s} w(r) f(r, x(r)) \, dr \right) \Delta s + \frac{1}{1 - \sum_{i=2}^{m-1} \alpha_i} \sum_{i=2}^{m-1} \alpha_i \int_{t_i}^{t_m} \phi_q \left( \int_{t_1}^{s} w(r) f(r, x(r)) \, dr \right) \Delta s. \quad (4.4)$$
By (4.4), we obtain
\[
(A^*x)^\Delta(t) := -\phi_q \left( \int_{t_1}^{t} w(r) f(r, x(r), x^\Delta(r)) \, \mathrm{d}r \right), \quad \text{for } x \in P_1, \ t \in [t_1, t_m]_T, \tag{4.5}
\]

It is well known that problem (1.1) and (1.3) has a positive solution \( x \) if and only if \( x \in P_1 \) is a fixed point of \( A^* \).

**Lemma 4.4.** Let (1.4)–(1.6) hold. Then \( A^*P_1 \subset P_1 \) and \( A^* : P_1 \to P_1 \) is completely continuous.

In addition, we can prove the following result.

\[
\min_{t \in [1/n,(n-1)/n]_T} \|A^*x(0,T)\| = \frac{1}{n} \|A^*x\|_{0,T} = \frac{1}{n} A^*x(t_1). \tag{4.6}
\]

We are now ready to apply the Avery–Peterson fixed point theorem to the operator \( A^* \) to give sufficient conditions for the existence of at least three positive solutions to problem (1.1) and (1.3).

Let
\[
\delta_i = \int_{t_1}^{t_i} \phi_q \left( \int_{t_1}^{s} w(r) \, \mathrm{d}r \right) \, \mathrm{d}s, \quad \delta = \min_{i \in [1, t_{m-1}]} \{ \delta_i \}, \quad \rho = \phi_q \left( \int_{t_1}^{t_m} w(r) \, \mathrm{d}r \right),
\]
\[
\bar{L}_i = \int_{t_1}^{t_m} \phi_q \left( \int_{t_1}^{s} w(r) \, \mathrm{d}r \right) \, \mathrm{d}s + \frac{1}{1 - \sum_{i=2}^{m-1} \alpha_i} \int_{t_1}^{t_m} \phi_q \left( \int_{t_1}^{s} w(r) \, \mathrm{d}r \right) \, \mathrm{d}s,
\]
\[
\bar{L} = \max_{i \in [1, t_{m-1}]} \{ \bar{L}_i \}, \quad t_i^* = \frac{t_i + t_{i+1}}{2} \ (i = 1, 2, \ldots, m-1),
\]
\[
N = \frac{n}{2} \left[ 1 + \left( 1 - \sum_{i=2}^{m-1} \alpha_i \right) \right], \quad 0 = t_1 < 1 < t_2 < \cdots < t_{m-1} = t_m = \frac{2}{n} < t_m = 1.
\]

We are now ready to apply the Avery–Peterson fixed point theorem to the operator \( A \) to give sufficient conditions for the existence of at least three positive solutions to problem (1.1) and (1.2).

**Theorem 4.1.** Assume (1.4)–(1.6) hold. Let \( 0 < a < b \leq \frac{2Md}{N} \), and suppose that \( f \) satisfies the following conditions:

(A_4) \( f(t, u, v) \leq \phi_p(d/\rho), \) for \( (t, u, v) \in [t_1, t_m] \times [0, Md] \times [-d, d] \);

(A_5) \( f(t, u, v) > \phi_p(nb/\delta), \) for \( (t, u, v) \in \left[ \frac{1}{n}, \frac{(n-1)}{n} \right]_T \times [b, Nb] \times [-d, d] \);

(A_6) \( f(t, u, v) < \phi_p(a/\bar{L}), \) for \( (t, u, v) \in [t_1, t_m] \times [0, a] \times [-d, d] \).

Then boundary value problem (1.1) and (1.3) has at least three positive solutions \( x_1, x_2, \) and \( x_3 \) such that

\[
\max_{t \in [1/n,(n-1)/n]_T} |x_i^\Delta(t)| \leq d, \quad i = 1, 2, 3;
\]

\[
b < \min_{t \in [1/n,(n-1)/n]_T} |x_1(t)|, \quad \max_{t \in [1/n,(n-1)/n]_T} |x_1(t)| \leq Md,
\]

\[
a < \max_{t \in [1/n,(n-1)/n]_T} |x_2(t)|, \quad \min_{t \in [1/n,(n-1)/n]_T} |x_2(t)| < b,
\]

and

\[
\max_{t \in [1/n,(n-1)/n]_T} |x_3(t)| < a.
\]

5. Example

To illustrate how our main results can be used in practice we present an example.
Let $T = \{0, \frac{1}{n} \} \cup \{1/5^n : n \in \mathbb{N}_0\}$, where $\mathbb{N}_0$ denotes the set of all nonnegative integers. Take $p = 3$, $\alpha_2 = \frac{1}{2}$, $t_1 = 0$, $t_2 = \frac{1}{2}$, $t_m = 1$ in (1.1) and (1.2). Now we consider the following three point boundary value problem

$$
\begin{cases}
\left(\phi_p(x^\Delta(t))\right)^\nu + f(t, x(t), x^\Delta(t)) = 0, & t_1 < t < t_m, \\
x(0) = \frac{1}{2}x\left(\frac{1}{2}\right), & x^\Delta(1) = 0,
\end{cases}
$$

(5.1)

where

$$
f(t, u, v) = \begin{cases}
\frac{1}{5}t + \frac{60^{14}}{73^6}u^{13} + \frac{1}{1000}\left(\frac{73^6}{60^{15}}\right)^2, & \text{if } u \leq \frac{1}{16} \times \frac{74^6}{60^\pi}, \\
\frac{1}{5}t + \frac{60^{14}}{73^6} \times \left[\frac{74^6}{60^\pi}\right]^{13} + \frac{1}{1000}\left(\frac{73^6}{60^{15}}\right)^2, & \text{if } u > \frac{1}{16} \times \frac{74^6}{60^\pi}.
\end{cases}
$$

Thus it is easy to see by calculating that $\rho = 1, \delta = \frac{1}{12}, L = 2$.

Choose $a = \frac{22^7 \pi}{4^4 \times 60^\pi}, b = \frac{24^7 \pi}{60^\pi}, d = \left(\frac{15}{4}\right)^{13} \times \frac{24^7}{60^\pi}$, then by

$$
M = 2, \quad n = 5 > \max \left\{ \frac{1}{t_2}, \frac{1}{1-t_2} \right\}, \quad N = \frac{n}{2} \left[ 1 + (1 - \alpha_2) \right] = \frac{15}{4}
$$

we have $0 < a < b < \frac{2Md}{N}$, then $f(t, u, v)$ satisfies

(1) $f(t, u, v) \leq \frac{1}{5} + \frac{60^{14}}{73^6} \times \frac{1}{16} \times \frac{74^6}{60^\pi} + \frac{1}{1000} < \left(\frac{15}{4}\right)^{26} \times \frac{74^6}{60^\pi} = \phi_3\left(\frac{d}{\rho}\right),$

for $(t, u, v) \in [0, 1] \times \left[0, 2 \times \left(\frac{15}{4}\right)^{13} \times \frac{74^6}{60^\pi}\right] \times \left[-\left(\frac{15}{4}\right)^{13} \times \frac{74^6}{60^\pi}, \left(\frac{15}{4}\right)^{13} \times \frac{74^6}{60^\pi}\right]$;

(2) $f(t, u, v) \geq \frac{1}{5} + \frac{60^{14}}{73^6} \times \frac{74^{12}}{60^\pi} + \frac{1}{1000} > \frac{74^6}{60^\pi} = \phi_3\left(\frac{nb}{\delta}\right),$

for $(t, u, v) \in \left[\frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right] \times \left[\frac{74^6}{60^\pi}, \frac{74^6}{60^\pi} \times \frac{15^4}{60^\pi}\right] \times \left[-\left(\frac{15}{4}\right)^{13} \times \frac{74^6}{60^\pi}, \left(\frac{15}{4}\right)^{13} \times \frac{74^6}{60^\pi}\right]$;

(3) $f(t, u, v) \leq \frac{1}{5} + \frac{60^{14}}{73^6} \times \frac{72^6}{4^3 \times 60^3^\pi_2} + \frac{1}{1000} < \frac{72^6}{4^3 \times 60^\pi} = \phi_3\left(\frac{a}{L}\right),$

for $(t, u, v) \in [0, 1] \times \left[0, \frac{72^6}{4^3 \times 60^3^\pi_2}\right] \times \left[-\left(\frac{15}{4}\right)^{13} \times \frac{74^6}{60^\pi}, \left(\frac{15}{4}\right)^{13} \times \frac{74^6}{60^\pi}\right]$.

Thus by Theorem 3.1, problem (5.1) has at least three positive solutions $x_1, x_2, x_3$ satisfying

$$
\max_{t \in [0, 1]} |x^\Delta_i(t)| \leq \left(\frac{15}{4}\right)^{13} \times \frac{74^6}{60^\pi}, \quad i = 1, 2, 3;
$$

$$
\frac{74^6}{60^\pi} < \min_{t \in [1/3, 1/2]} |x_1(t)|, \quad \max_{t \in [1/3, 1/2]} |x_1(t)| \leq 2 \times \left(\frac{15}{4}\right)^{13} \times \frac{74^6}{60^\pi},
$$

$$
\frac{72^6}{4^3 \times 60^\pi} < \max_{t \in [0, 1]} |x_2(t)|, \quad \min_{t \in [1/3, 1/2]} |x_2(t)| < \frac{74^6}{60^\pi}.
$$
and

\[
\max_{t \in \left[ \frac{1}{5}, \frac{4}{5} \right]} |x_3(t)| = \frac{72 \pi}{4 \times 60 \pi} = \frac{6}{4} \times 60 \pi
\].

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