Abstract—This article addresses the problem of output feedback control design for a class of multi-input multi-output (MIMO) nonlinear systems where the number of inputs is less than that of outputs. There are two difficulties in this design problem: 1) too few control inputs will not generally allow independent control over all outputs, and 2) that the state of the system is not available for measurements, and only the outputs are available through measurements. To address the first issue, a practical output feedback control problem is formulated aiming to regulate only part of the outputs, and a controller structure with two design components in all or some chosen control inputs is proposed. To cope with the second difficulty, the recently developed high-order sliding mode differentiators (HOSMD) are used to estimate the derivatives of the outputs needed in the controller design. With the derivatives estimated using high-order sliding mode differentiators, an output feedback controller is designed using backstepping approach. Stability results are established for the designed controller under certain conditions. In order to test the applicability of the proposed output feedback controller in practical industrial problems, experiments are carried out through implementing the controller on a laboratory scale 3D crane. The experimental results are presented and reveal the advantage of the proposed controller structure, as well as the effect of controller gain and sampling periods.

Index Terms—Nonlinear systems, high-order sliding mode, output feedback control, backstepping.

I. INTRODUCTION

Because of its robustness to system uncertainties and external disturbances, sliding mode control (SMC) has received a great deal of attention from the research community [1]-[4], and has found many industrial applications [5]-[8]. The main idea in sliding mode control is to design a proper sliding surface first and then design a switching control law that can force the closed loop system dynamics to reach and remain on the sliding surface that is designed in such a way to meet the control objectives.

Sliding mode control enjoys several features such as robustness to disturbances, perfect accuracy after reaching the sliding surface, and perhaps finite time transient period—at least theoretically. However, standard SMC also suffers from certain shortcomings. One of the inherent problems in SMC is the chattering effect caused by high frequency switching control. Another shortcoming of the traditional SMC methodology is the requirement that the relative degree between the control and the sliding surface needs to be one [9]. This restriction makes it difficult for SMC to be applicable to systems with high relative degrees such as mechanical systems [10] or vehicle control systems [9].

In order to remove some or even all of the restrictions associated with the first order sliding mode techniques, high-order sliding mode design approaches have recently been proposed, see [9], [10], [11], [15], [16] and the related references therein. According to [9], the idea of high-order sliding mode control (HOSMC) is to first choose a proper $r$th order sliding mode determined by $\sigma = \dot{\sigma} = \ddots = \ddots = \sigma^{(r-1)} = 0$, and then design a switching control to drive the controlled system to reach and remain on the $r$th order sliding mode. According to the definition of $r$th order sliding mode, standard SMC aims to achieve first order sliding mode. Some examples of second-order sliding mode controllers and observers can be found in [10], [11], [15], [16] and the related references therein. Examples of HOSMC were proposed in [9] and in [17], [18], [19], [20], [21], [22], to name only a few.

In order to implement HOSMC with $r$th order sliding mode, the derivatives of $\sigma$, that is, $\dot{\sigma}, \ddots, \sigma^{(r-1)}$ have to be used. Because they are not usually measured, they have to be estimated based on the measurement of $\sigma$. One approach for estimating these derivatives is to use high-gain observers [23], [24], [25], [26]. However, high-gain observers only provide asymptotic estimation of the derivatives, and they cannot provide exact reconstruction of the derivatives. Additionally, they do not enjoy finite-time convergence even for the ideal situation, and, in industrial applications, high-gain observers can be quite sensitive to measurement noises [17]. Another approach for estimating the derivatives is to use the recently developed HOSMD, see [9] and the related references therein. Ideally, high-order sliding mode differentiators presented in [9] can provide exact reconstruction of the derivatives after a finite-time transient period. Moreover, they possess robustness to measurement noises.
The main purpose of this paper is to study the output feedback control design problem for a class of multi-input multi-output (MIMO) nonlinear systems where the number of inputs is less than that of outputs. Not only this is a challenging problem from a theoretical standpoint, more importantly, there are many systems in practice that belong to this class of nonlinear systems. Under-actuated systems, such as overhead cranes [12], [13], [14] belong to this class of systems. Because of the advantages of the HOSMD over the high-gain observers, in terms of their ability for finite-time exact derivative reconstruction, the output feedback controller uses HOSMDs to estimate the derivatives of the outputs. In the controller design, because there are fewer inputs than outputs, a practical output feedback control problem is formulated aiming to regulate only part of the outputs. Further, a controller structure with two design components in all or some chosen control inputs is proposed. With the derivatives estimated using HOSMDs, an output feedback controller is designed using the backstepping approach. One difference between the controller in this paper with those HOSMCs is that it is designed using backstepping approach to ensure output tracking and closed-loop stability in an asymptotic way rather than to achieve an rth order sliding mode in finite time. One reason for not pursuing to reach an rth order sliding mode in finite time is that the ideal rth order sliding mode and finite convergence property will be lost in practical industrial applications due to limitations on sampling rate and the speed of actuation.

Another aim of this paper is to test the practicality of designing an output feedback controller using the estimated output derivatives provided by HOSMDs. This is achieved through experiments on a laboratory scale 3D crane system. The experimental results are presented to reveal the advantages of the proposed controller structure, as well as the effect of controller gain and sampling period.

The remainder of the paper is arranged as follows: In Section II, the considered MIMO nonlinear system is introduced, and a realistic output feedback control problem of interest is formulated. In Section III, the sliding mode differentiators given in [9] are presented along with their important properties. In Section IV, a controller structure with two design components in all or some chosen control inputs is proposed. Each design component is designed using backstepping approach based on the estimation of the derivatives of the outputs provided by HOSMDs. Section V, provides experimental results on a laboratory scale 3D crane system to illustrate the practicality and performance of the designed output feedback controller using HOSMDs. The experimental results reveal the advantages of the proposed controller structure, and the effect of controller gain and sampling period. Finally, concluding remarks are made in the last section.

II. SYSTEM OF INTEREST AND PROBLEM FORMULATION

A. System description

Consider a class of MIMO nonlinear systems which is of or can be changed through state transformation into the following form:

\[
\begin{align*}
\dot{q}_{i,j} &= q_{i,j+1}, 1 \leq j \leq r_i - 1 \\
\dot{q}_{i,r_i} &= f_i(x) + G_i(x)u, \\
y_i &= q_{i,1}, 1 \leq i \leq p
\end{align*}
\]

where \(u = (u_1 \cdots u_m)^T \in \mathbb{R}^m\) is the input vector, \(x = (q_{1,1} \cdots q_{1,r_1} \cdots q_{p,1} \cdots q_{p,r_p})^T \in \mathbb{R}^n\) is the state vector with \(n = r_1 + \cdots + r_p\), \(y \in \mathbb{R}^p\) is the measured output vector with \(y = (y_1 y_2 \cdots y_p)^T = (q_{1,1} q_{2,1} \cdots q_{p,1})^T\), \(f_i(x)\) is a scalar function, and \(G_i(x) = (g_{i,1} g_{i,2} \cdots g_{i,m}), 1 \leq i \leq p\) are row vectors consisting of nonlinear functions of \(x\).

It is easy to see that (1) consists of \(p\) subsystems of the same form and each subsystem is related to a particular output. For later use, the subsystem related to \(y_i\) is called the \(i\)th subsystem.

In the remaining part of this paper, variables of functions will be dropped for the sake of simplicity wherever appropriate.

Because the controller design for \(m \geq p\) is much easier than that for \(m < p\), only the case for \(m < p\) will be considered in this paper. Let \(G(x) = (G_1^T \cdots G_p^T)^T = (g_{i,j})_{p \times m}\) and \(F(x) = (f_1 \cdots f_p)^T\), and the following assumption is made.

A1– All the elements in \(F(x)\) and \(G(x)\) are bounded for bounded \(x\), and \(\text{rank}(G) = m\) for all \(x\) in the region of interest.

B. Problem formulation

When \(m < p\), it is in general very difficult if not impossible to design an output feedback controller to regulate all the outputs (that is, to make the outputs independently track desired independent reference signals). Actually, in this case, even the stabilization problem may become very hard to solve. Because of this restriction, a realistic output feedback control problem is formulated as follows.

Output feedback control problem – Under the assumptions A1 and \(m < p\), the objective is to design if possible an output feedback controller such that it can regulate a chosen group of outputs (say \(m_\mu\) outputs, where \(1 \leq m_\mu \leq m\)) and ensure all the signals in the closed-loop system are bounded at the same time.

Without loss of generality, assume that the chosen group of \(m_\mu\) outputs are the first \(m_\mu\) outputs, i.e., \(y_1, y_2, \cdots, y_{m_\mu}\). The problem becomes how to design an output feedback controller such that it can make \(y_1, y_2, \cdots, y_{m_\mu}\) track desired reference signals and ensure all the signals in the closed-loop system are bounded at the same time.

Denote the desired reference signals as \(y_{1,r_1}(t), y_{2,r_2}(t), \cdots, y_{m_\mu,r_{m_\mu}}(t)\), and the following assumption is made on them.

A2– For each \(1 \leq i \leq m_\mu\), \(y_{i,r_i}(t)\) is \(r_i\) times differentiable, and \(y_{i,r_i}^{(r_i)}(t)\) is bounded.

III. A HIGH-ORDER SLIDING MODE DIFFERENTIATOR AND ITS PROPERTIES

Since HOSMDs will be used to provide the estimates of the derivatives of the outputs for the purpose of controller design,
A HOSMD presented in [9] will be introduced along with its properties in this section before we move on to controller design.

**A. A high-order sliding mode differentiator**

Let \( s(t) = s_0(t) + n(t) \) be a function on \([0, \infty)\), where \( s_0(t) \) is an unknown base function with the \( n \)-th derivatives having a Lipschitz constant \( L \), and \( n(t) \) is a bounded Lebesgue-measurable noise with unknown features. The problem of high-order sliding-mode robust differentiator design is to find real-time robust estimations of \( \dot{s}_0(t), \ddot{s}_0(t), \ldots, \dot{s}^{(n)}_0(t) \) being exact when \( n(t) = 0 \). A HOSMD proposed in [9] takes on the following form.

\[
\begin{align*}
\dot{z}_0 &= v_0 \\
v_0 &= -\lambda_0|z_0 - s(t)^{n/(n+1)}\text{sign}(z_0 - s(t)) + z_1 \\
\dot{z}_1 &= v_1 \\
v_1 &= -\lambda_1|z_1 - v_0|^{(n-1)/n}\text{sign}(z_1 - v_0) + z_2 \\
\vdots \\
\dot{z}_{n-1} &= v_{n-1} \\
v_{n-1} &= -\lambda_{n-1}|z_{n-1} - v_{n-2}|^{1/2}\text{sign}(z_{n-1} - v_{n-2}) + z_n \\
\dot{z}_n &= -\lambda_n\text{sign}(z_n - v_{n-1}) \\
\end{align*}
\]

where \( \lambda_0, \lambda_1, \ldots, \lambda_n \) are positive design parameters.

**B. Properties of the high-order sliding mode differentiator**

Regarding the HOSMD given by (2), the following three results have been proved [9].

**Theorem 1:** If \( n(t) = 0 \) and all the parameters are chosen properly, then after a finite transient, the following equalities are obtained

\[
\begin{align*}
z_0 &= s_0(t); z_i = v_{i-1} = s_0^{(i)}(t), i = 1, 2, \ldots, n \\
\end{align*}
\]

**Theorem 2:** If \( |n(t)| = |s(t) - s_0(t)| \leq \epsilon \) and all the parameters are chosen properly, then after a finite transient, the following inequalities are obtained

\[
\begin{align*}
|z_i - s_0^{(i)}(t)| &\leq \mu_i \epsilon^{(n-i+1)/(n+1)}, i = 0, 1, \ldots, n \\
|v_i - s_0^{(i+1)}(t)| &\leq \nu_i \epsilon^{(n-i)/(n+1)}, i = 0, 1, \ldots, n-1 \\
\end{align*}
\]

where \( \mu_i, i = 0, 1, \ldots, n \) and \( \nu_i, i = 0, 1, \ldots, n-1 \) are some positive constants depending only on the parameters of the differentiator.

Consider the discrete-sampling case, when \( z_0(t) - s(t) \) is replaced by \( z_0(t_j) - s(t_j) \) on \([t_j, t_{j+1}]\) with \( \tau = t_{j+1} - t_j \).

**Theorem 3:** Let \( \tau \) be the constant sampling time. If \( n(t) = 0 \) and all the parameters are chosen properly, then after a finite transient, the following inequalities are obtained

\[
\begin{align*}
|z_i - s_0^{(i)}(t_j)| &\leq \mu_i \tau^{n-i+1}, i = 0, 1, \ldots, n \\
|v_i - s_0^{(i+1)}(t_j)| &\leq \nu_i \tau^{n-i}, i = 0, 1, \ldots, n-1 \\
\end{align*}
\]

**Remark 1:** Theorem 1 shows that finite-time exact reconstruction of the derivatives of \( s(t) \) can be achieved—at least ideally. Theorem 2 says the HOSMD can provide good estimates for the derivatives only if the measurement noises are small enough. Theorem 3 reveals that the estimation accuracy of the derivatives depends also heavily on the sampling period even for the noise free case, and the faster the sampling is, the better the estimation accuracy of the derivatives.

**IV. OUTPUT FEEDBACK CONTROLLER DESIGN**

In this section, first a controller structure with two design components in all or some chosen control inputs will be proposed. Then, each component will be designed using backstepping approach assuming all states are available, and the stability of the designed state feedback controller is analyzed. Finally, by the use of the estimated derivatives of the outputs offered by HOSMDs, an output feedback controller is provided, and under certain conditions, stability results will be given.

Define \( G_{tr}(x) = (G_{tr1}^T \cdots G_{trm}^T \cdots G_{trn}^T)^T \), \( F_{tr}(x) = (f_1 \cdots f_{m_1} \cdots f_m)^T \), \( \bar{G}_{tr}(x) = (G_{tr1}^T \cdots G_{trm}^T)^T \), and \( \bar{F}_{tr}(x) = (f_{m_1+1} \cdots f_p)^T \). The following assumption is needed for the controller design.

**A3—** For all \( x \) in the region of interest, \( G_{tr} \) is bounded and invertible, and \( \text{rank}(\bar{G}_{tr}) = p - m \).

**A. A controller structure**

In order to solve the formulated output feedback control problem, the following controller structure is proposed.

\[
u = u_{tr} + u_{sta},
\]

where \( u_{tr} = (u_{tr1} \cdots u_{tr,m})^T \) is a design component that is introduced to make the outputs \( y_1, y_2, \ldots, y_{m_1} \) track the desired reference signals \( y_{1,r}(t), y_{2,r}(t), \ldots, y_{m_1,r}(t) \) asymptotically when \( u_{sta} = 0 \), and \( u_{sta} = (u_{sta1} \cdots u_{sta,m})^T \) is another design component that is introduced to ensure that all the closed-loop signals are bounded.

The detailed design of \( u_{tr} \) and \( u_{sta} \) is given in Subsection IV-B.

**B. State feedback controller**

In this subsection, the design of \( u_{tr} \) and \( u_{sta} \) using state feedback will be proposed, and the stability of the designed controller is analyzed under certain conditions.

Define \( U(x) = (U_1(x) \ U_2(x) \cdots U_p(x))^T = F + Gu \), then (1) can be rewritten as

\[
\begin{align*}
\dot{q}_{i,j} &= q_{i,j+1}, 1 \leq j \leq r_i - 1 \\
\dot{q}_{i,r_i} &= U_{i,s} \\
y_i &= q_{i,1}, 1 \leq i \leq p \\
\end{align*}
\]

For the design purpose, let us introduce a group of reference signals for \( y_{m_1+1}, \ldots, y_p \) as \( y_{m_1+1,r}, \ldots, y_p,r \). Unlike \( y_{1,r}(t), y_{2,r}(t), \ldots, y_{m_1,r}(t) \) which are predetermined by control objectives, \( y_{m_1+1,r}, \ldots, y_p,r \) are left to be chosen freely.
in order to provide the designers with more freedom. The designers might want to choose some particular groups of \(y_{m+1,r}, \ldots, y_{p,r}\) that will make the controller design easier to ensure all the signals in the closed-loop system are bounded. In order to illustrate this point clearly, consider a linear system described as

\[
\begin{align*}
\dot{x}_1 &= u_1, \\
\dot{x}_2 &= u_2, \\
\dot{x}_3 &= x_4, \\
\dot{x}_4 &= u_2, \\
y_1 &= x_1, y_2 = x_2, y_3 = x_3.
\end{align*}
\] (8)

For this system, choose \(m_\mu = 1\), which means one would like only to make \(y_1\) to track any reference signal \(y_{1,r}\). If \(y_{2,r} = 1\) is also fixed and \(u_2 = -(x_2 - 1)\) is chosen in order to track \(y_{2,r}\), it is easy to check that \(x_3\) will go unbounded for any \(x_4(0) - x_2(0) + 1 \neq 0\) and for any \(u_1\). However, if \(y_{2,r}\) and \(y_{3,r}\) are allowed to be chosen freely and \(u_2 = -2x_3 - 3x_4\) is chosen, it is easy to show that \(x_3\) and \(x_4\) both tend to zero exponentially with \(x_2\) being bounded. All signals in the closed-loop system can be made bounded if \(u_1\) is designed properly to make \(y_1\) track \(y_{1,r}\).

**Remark 2:** The above example showed that, for some systems, the proper choice of \(y_{m+1,r}, \ldots, y_{p,r}\) together with proper controller design will help to stabilize the whole system. In order to do this, it is generally required that \(m_\mu < m\). If \(m_\mu = m\), all the controls are fixed by the proposed controller design. The stability of the overall closed-loop system would be system dependent and there is generally no guarantee that the system is not unstable. That is the reason that some control design freedom is required to be left in this paper. However, a systematic design approach for those free chosen reference signals is unable to be provided at the present time, which is quite interesting and requires further research.

**Remark 3:** It might be possible to define a new output as a proper combination of the actual outputs to make the guaranteed stabilization for some systems. However, whether this can be done for all systems of the form (1), and how to define such a new output are not trivial and require further investigations.

For the time being, let us assume that \(x = (q_{1,1} \cdots q_{1,r_1} \cdots q_{p,1} \cdots q_{p,r_p})^T\) is available and \(U_i\) is free to design for all \(i\), then for each \(i\), \(U_i\) can be designed as follows using backstepping approach (since backstepping approach is now quite standard, for the detailed design procedure the interested readers are referred to [27]):

\[
U_i = -(c_{i,r_i} + 1)\xi_{i,r_i} - \xi_{i,r_i} + \frac{r_i - 1}{r_i} \sum_{j=1}^{r_i-1} \frac{\partial \alpha_{i,r_i-1}}{\partial y_{i,j}} q_{i,j+1} + \sum_{j=0}^{r_i-1} \frac{\partial \alpha_{i,r_i-1}(j+1)}{\partial y_{i,r}} y_{i,r},
\] (9)

where \(\xi_{i,k}, 1 \leq k \leq r_i\) and \(\alpha_{i,k}, 1 \leq k \leq r_i - 1\) are defined as

\[
\begin{align*}
\xi_{i,1} &= q_{i,1} - y_{i,r}, \alpha_{i,1} = -c_{i,1}\xi_{i,1} + \dot{y}_{i,r}, \\
\xi_{i,k} &= q_{i,k} - \alpha_{i,k-1}, 2 \leq k \leq r_i, \\
\alpha_{i,k} &= -c_{i,k}\xi_{i,k} - \xi_{i,k-1} + \sum_{j=1}^{k-1} \frac{\partial \alpha_{i,k-1}}{\partial q_{i,j}} q_{i,j+1} + \sum_{j=0}^{k-1} \frac{\partial \alpha_{i,k-1}(j+1)}{\partial y_{i,j}} y_{i,j+1},
\end{align*}
\] (10)

and \(y_{i,r}\) is the \(j\)th order derivative of \(y_{i,r}\) with \(y_{i,r}^{(0)} = y_{i,r}, c_{i,j}, 1 \leq i \leq p, 1 \leq j \leq r_i\) are positive constants that can be chosen freely.

For each \(i\), if \(U_i\) was to be applied to the \(i\)th subsystem, according to the backstepping approach, one would be able to prove that \(\lim_{t \to \infty}\xi_{i,1} = 0\) and establish the boundness of \(\xi_{i,k}, 1 \leq k \leq r_i\) or equivalently \(q_{i,k}, 1 \leq k \leq r_i\). The formulated output feedback control problem would thus be solved.

However, because \(m < p\), it is generally not possible to achieve all \(U_i, 1 \leq i \leq p\) through the design of \(m\) controls, i.e., \(u_i, 1 \leq i \leq m\). This makes the controller design much more complicated.

In order to solve the formulated output feedback control problem, one needs to design \(u_{tr}\) to achieve \(U_i\), where \(U_i = (U_1 \cdots U_m \cdots U_p)^T\).

To achieve \(U_i\), \(u_{tr}\) is designed as

\[
u_{tr} = G_{tr}^{-1}(U_i - F_i),
\] (11)

where \(G_{tr}^{-1}\) is the inverse of \(G_{tr}\).

Now, all that remains in the controller design is to design \(u_{sta}\).

Since \(\text{rank}(G_{tr}) = p - m\), it can be assumed without loss of generality that the first \(p - m\) columns of \(G_{tr}\) are independent. Denote \(\bar{G}_{tr} = (\bar{G}_{tr,1} \bar{G}_{tr,2})\), where \(\bar{G}_{tr,1}\) consists of the first \(p - m\) columns of \(G_{tr}\) and \(\bar{G}_{tr,2}\) consists of the remaining columns.

Denote \(\bar{U}_{tr} = (U_{m+1} \cdots U_p)^T\) and define \(\bar{u}_{p-m} = \bar{G}_{tr,1}^{-1}\bar{U}_{tr} = (\bar{u}_{m+1} \cdots \bar{u}_p)^T\), then, because the primary purpose of designing \(u_{sta}\) is to ensure the signals in the closed loop are bounded, the principle of the design of \(u_{sta}\) is to make the linearized parts of the last \(p - m\) subsystems stable. To this end, \(u_{sta}\) is designed as

\[
u_{sta} = (u_{p-m}^T 0)^T,
\] (12)

where \(0\) is zero vector with compatible dimension, and \(u_{p-m} = (\text{sat}_M(\bar{u}_{m+1}) \cdots \text{sat}_M(\bar{u}_p))^T\) with function \(\text{sat}_M(\bar{u}_j)\) being defined as

\[
\begin{align*}
\text{sat}_M(\bar{u}_j) &= \bar{u}_j, \text{ if } |\bar{u}_j| \leq M \\
&= M, \text{ if } \bar{u}_j > M \\
&= -M, \text{ if } \bar{u}_j < -M
\end{align*}
\] (13)

where \(M\) is a positive design constant. The saturation functions are introduced to prove the stability of the closed-loop system, and \(M\) can be chosen as large as possible.
Define $\tilde{x}^1 = (q_{1,1} \cdots q_{1,r_1} \cdots q_{m,1} \cdots q_{m,r_m})^T$ and $\tilde{x}^2 = (q_{m+1,1} \cdots q_{m+1,r_{m+1}} \cdots q_{p,1} \cdots q_{p,r_p})^T$. Based on these definitions, one more assumption is made below.

**A4**—When the state feedback controller given by (6) and (11) is applied to (1),

$$
\bar{x}^2 = \left( q_{m+1,1} \cdots q_{m+1,r_{m+1}} \cdots q_{p,1} \cdots q_{p,r_p} \right)^T
$$

is bounded if $\tilde{x}^1 = (q_{1,1} \cdots q_{1,r_1} \cdots q_{m,1} \cdots q_{m,r_m})^T$ is bounded.

At this juncture we are ready to state stability results related to the state feedback controller.

**Theorem 4:** Given A1–A4, and assuming that the state vector $x$ is available, if the state feedback controller given by (6) and (11) is applied to (1), then

- the output tracking error, that is, $|y_i - y_{i,r}|$ will enter a small neighborhood of zero for any $1 \leq i \leq m$; and moreover, $\lim_{t \to \infty} |y_i - y_{i,r}| = 0$ for any $1 \leq i \leq m$ if $\lim_{t \to \infty} u_{sta} = 0$;
- all closed-loop signals are bounded.

**Proof:** Consider only the subsystem related to $\tilde{x}^1 = (q_{1,1} \cdots q_{1,r_1} \cdots q_{m,1} \cdots q_{m,r_m})^T$.

With the definitions of $F(x), F_{tr}(x), G(x), G_{tr}(x)$, and $u_{tr}, U_{tr}(x)$, and using (6), one gets

$$
F_{tr}(x) + G_{tr}(x)u = U_{tr}(x) + G_{tr}(x)u_{sta}. \tag{14}
$$

For the sake of simplicity, denote $N = (N_1 \cdots N_m)^T = G_{tr}u_{sta}$. Because $G_{tr}(x)$ is bounded for all $x$ by assumption, and $u_{sta}$ is designed to be bounded, $N_i, 1 \leq i \leq m$ are bounded.

By the definitions of $u_{tr}$ and $U_{tr}(x)$, then, after applying (6) and (11) to (10) into (15), the following equations can be derived

$$
\dot{\xi}_{i,1} = -c_{i,1} \xi_{i,1} + \xi_{i,2},
$$

$$
\dot{\xi}_{i,j} = -c_{i,j} \xi_{i,j} + \xi_{i,j+1} - \xi_{i,j-1} - 2 \leq j \leq r_i - 1,
$$

$$
\dot{\xi}_{i,r_i} = -(c_{i,r_i} + 1) \xi_{i,r_i} - N_i. \tag{16}
$$

Choosing a Lyapunov function as $V_i = \frac{1}{2} \sum_{j=1}^{r_i} \xi_{i,j}^2$, and differentiating it along (16), one obtains

$$
\dot{V}_i = - \sum_{j=1}^{r_i} c_{i,j} \xi_{i,j}^2 - \frac{N_i}{c_{i,min}} \xi_{i,r_i}. \tag{17}
$$

Denote $c_{i,min} = \min\{c_{i,j}|1 \leq j \leq r_i\}$ and $N_i = N_i^2/4$, then it follows from (17) that

$$
\dot{V}_i \leq -2c_{i,min} V_i + \frac{N_i}{c_{i,min}}. \tag{18}
$$

Since $N_i$ is bounded, $c_{i,j}, 1 \leq j \leq r_i$ can be chosen such that

$$
\frac{|N_i|}{c_{i,min}} \leq \epsilon_i, \tag{19}
$$

where $\epsilon_i$ is a small constant.

Then, it can be derived

$$
V_i(t) \leq e^{-2c_{i,min}t} (V_i(0) + \int_0^{t} \frac{N_i}{c_{i,min}} e^{2c_{i,min}t} d\tau)
$$

$$
\leq e^{-2c_{i,min}t} (V_i(0) + \epsilon_i c_{i,min} \int_0^{t} e^{2c_{i,min}t} d\tau)
$$

$$
= e^{-2c_{i,min}t} (V_i(0) - \frac{\epsilon_i}{2}) + \frac{\epsilon_i}{2}. \tag{20}
$$

This implies that

$$
|\xi_{i,1}| \leq \sqrt{2(V_i(0) - \frac{\epsilon_i}{2}) e^{-2c_{i,min}t} + \epsilon_i}. \tag{21}
$$

(21) means the tracking error $|y_i - y_{i,r}| = |\xi_{i,1}|$ will eventually enter the neighborhood of zero, namely, $|y_i - y_{i,r}| \leq \sqrt{\epsilon_i}$.

Moreover, if $\lim_{t \to \infty} u_{sta} = 0$, it follows from assumptions A1–A3 that $\lim_{t \to \infty} N_i = 0$ and thus $\lim_{t \to \infty} \epsilon_i = 0$. This together with (21) implies that $\lim_{t \to \infty} |y_i - y_{i,r}| = 0$.

Based on (20), it can also be concluded that $\xi_{i,j}, 1 \leq j \leq r_i$ and $q_{i,j}, 1 \leq j \leq r_i$ are bounded. Since this conclusion is true for any $1 \leq i \leq m$, it is proved that $\tilde{x}^1$ is also bounded. This fact together with Assumption A4 implies that $\tilde{x}^2$ is also bounded. Therefore, the state vector $x$ of the closed loop system is bounded. The boundness of $x$ together with Assumptions A1–A3 implies that $u_{tr}$ is bounded, which proves that the control $u$ is also bounded. This completes the proof.

Based on the results in Theorem 4, it is easy to prove the following results.

**Theorem 5:** Assume that the second design component is chosen as zero, i.e. $u_{sta} = 0$, and that the assumptions of Theorem 4 are all satisfied. If the state feedback controller given by (6) and (11) is applied to (1), then

- $\lim_{t \to \infty} |y_i - y_{i,r}| = 0$ for any $1 \leq i \leq m$,
- all closed-loop signals are bounded.

**Proof:** Because $u_{sta} = 0$, following the arguments in the proof of Theorem 4, for any $1 \leq i \leq m$, one can reach

$$
\dot{V}_i \leq - \sum_{j=1}^{r_i} c_{i,j} \xi_{i,j}^2. \tag{22}
$$

This implies that $\lim_{t \to \infty} V_i = 0$ and thus $\lim_{t \to \infty} |y_i - y_{i,r}| = 0$. The arguments for proving the boundness of closed-loop signals are exactly the same those in the proof of Theorem 4.

**Theorem 6:** Under Assumptions A1 and A2, assuming that $p = m$, if $u = u_{tr}$ defined by (11) is applied to (1), then

- $\lim_{t \to \infty} |y_i - y_{i,r}| = 0$ for any $1 \leq i \leq m$,
- all closed-loop signals are bounded.

**Proof:** Because $p = m$ and according to assumption A1, one has $G_{tr} = G$, and thus $u_{tr} = u$. Following the arguments in the proof of Theorem 4, for any $1 \leq i \leq m$, one can reach

$$
\dot{V}_i \leq - \sum_{j=1}^{r_i} c_{i,j} \xi_{i,j}^2. \tag{23}
$$

This implies that $\lim_{t \to \infty} V_i = 0$ and thus $\lim_{t \to \infty} |y_i - y_{i,r}| = 0$, and also that $\xi_{i,j}, 1 \leq j \leq r_i$ and $q_{i,j}, 1 \leq j \leq r_i$ are bounded. Because $q_{i,j}, 1 \leq j \leq r_i, 1 \leq i \leq m$ are bounded, $x$
is bounded. Using $A1$ and $A2$ and the definition of $u_{tr}$, it is easy to see that $u$ is bounded. The theorem is thus proved.

Remark 4: Note that for $m = p$. Assumptions $A3$ and $A4$ are not needed and much stronger results are obtained than the case $m < p$. If $m > p$ and $\text{rank}(G(x)) \geq p$, one can always choose $p$ control inputs in $v$ to form $u_p$. By letting the other inputs be zero, it is easy to see the problem can be dealt with as in the case for $m = p$.

C. Output feedback controller

Note in the design of $u_{tr}$ and $u_{sta}$, it is assumed that $x = (q_{1,1} \cdots q_{1,r} \cdots q_{p,1} \cdots q_{p,p})^T$ is available. However, in this paper, only the outputs, that is, $y_1 = q_{1,1}, y_2 = q_{2,1}, \cdots y_p = q_{p,1}$ are measured. Therefore, the controller designed cannot be directly implemented.

According to (1), it is easy to see that

\[ q_{i,j+1} = y_1^{(j)}y_i, \quad 1 \leq i \leq p, \quad 1 \leq j \leq r_i - 1, \]

where $y_1^{(j)}$ is the $j$th order derivative of $f$.

This implies that

\[ x = (y_1 y_1 \cdots y_1^{(r_1-1)} \cdots y_p y_p \cdots y_p^{(r_p-1)})^T. \]

Because of the derivatives of the outputs are not measured, they have to be estimated in order to implement $u_{tr}$ and $u_{sta}$. For this, the HOSMDs presented in Section III will be used to estimate the derivatives of the outputs. For $y_i$, design an $r_i$th order sliding mode differentiator of the form (2), and denote the estimates of $y_i$ and the derivatives of $y_i$ as $z_{i,0}, z_{i,1}, \cdots, z_{i,r_i-1}$, then one obtains an estimate of $x$ as

\[ \hat{x} = (y_1 z_{i,1} \cdots z_{i,r_i-1} \cdots y_p z_{p,1} \cdots z_{p,r_p-1})^T. \]

Replace $x$ with $\hat{x}$, an output feedback controller using HOSMDs is given as

\[ u = u_{tr} + u_{sta}, \]

where $u_{tr}$ and $u_{sta}$ are designed as follows

\[ u_{tr} = G_{tr}^{-1}(\hat{x})(U_{tr}(\hat{x}) - F_{tr}(\hat{x})), \]

\[ u_{sta} = ((\hat{u}_{p-m})^T 0)^T, \]

where $\hat{u}_{p-m} = (\text{sat}_{M}(\hat{u}_{m+1}) \cdots \text{sat}_{M}(\hat{u}_p))^T$ with $\hat{u}_{p-m} = \hat{G}_{tr}^{-1}(\hat{x})\hat{U}_{tr}(\hat{x})$, and the elements in $U_{tr}(x)$ and $\hat{U}_{tr}(x)$ are replaced by $U_1(\hat{x}), \cdots, U_p(\hat{x})$ and

\[ U_1(\bar{x}) = -(c_{i,r_i+1})\xi_{i,r_i} - \xi_{i,r_i-1} + \frac{\sum_{j=0}^{r_i-1} \frac{\partial c_{i,r_i}}{\partial z_{i,r_i}} z_{i,j}}{\partial z_{i,r_i}}, \]

\[ \xi_{i,j} = q_{1,i} - y_i, \quad \alpha_{i,j} = -c_{i,j}\xi_{i,j} + \hat{y}_i, \]

\[ \alpha_{i,j} = c_{i,j}\xi_{i,j} - \xi_{i,j-1} + \frac{\sum_{j=0}^{k-1} \frac{\partial \xi_{i,j}}{\partial z_{i,j}} z_{i,j}}{\partial z_{i,j}}, \]

where $z_{i,0} = q_{1,i}$.

In the remainder of this subsection, the output tracking error and closed-loop stability using the output feedback controller will be analyzed. To this end, define $\sigma_{i,0} = y_i - z_{i,0}, \sigma_{i,1} = y_i - z_{i,1}, \cdots, \sigma_{i,r_i-1} = y_i^{(r_i-1)} - z_{i,r_i-1}, 1 \leq i \leq p$ and $\sigma = (\sigma_{1,0} \cdots \sigma_{1,r_1-1} \cdots \sigma_{p,0} \cdots \sigma_{p,r_p-1})^T$, and denote $\Omega = \{x : \|x\| \leq \delta\}$ and $\Psi = \{\sigma : \|\sigma\| \leq \delta\}$ for any $\delta > 0$.

The main stability result is presented in Theorem 7.

Theorem 7: Given $A1 - A4$, and assuming that there is no measurement noise and $c_{i,j}, 1 \leq i \leq p, 1 \leq j \leq r_i$ are chosen large enough, if the controller given by (27) $\sim$ (30) is applied to (1), then, for any $x(0) \times \sigma(0) \in \Omega \times \Psi$,

- the output tracking errors, that is, $\|y_i - y_i\|$ will enter a small neighborhood of zero for any $1 \leq i \leq m$, and moreover, $\lim_{t \to \infty} |y_i - y_i| = 0$ for any $1 \leq i \leq m$ if $\lim_{t \to \infty} u_{sta} = 0$.

- all closed-loop signals are bounded.

Proof: Define $\Omega = \{\bar{x}1^T V_1 \leq \frac{N_1}{\sigma c_{min,1}}, \cdots, V_m \leq \frac{N_m}{\sigma c_{min,m}}\}$.

Now choose $\delta_2 > \delta_1 > \delta$ and define for $i = 1, 2, \Omega_i = \{x : \|x\| \leq \delta_i\}$ and $\Psi_i = \{\sigma : \|\sigma\| \leq \delta_i\}$ such that $\cup_{i=1}^N \bar{V}_i(\Omega) \subseteq \bar{V}_m(\Omega) \subseteq \bar{V}_2(\Omega)$.

With the controller given by (27) $\sim$ (30) being applied to (1) under the Assumptions $A1 - A4$, there exists a positive constant $M_3$ such that, for any $(x, \sigma) \in \Omega \times \Psi, \|\bar{x}\| \leq M_3$. Similarly, there exist positive constants $M_{31}, M_{32}$ such that for any $(x, \sigma) \in \Omega_1 \times \Psi_1, \|\bar{x}\| \leq M_{31}, i = 1, 2$.

For any $x(0) \times \sigma(0) \in \Omega \times \Psi$, because $\|\bar{x}\| \leq M_{31}$ for any $(x, \sigma) \in \Omega_1 \times \Psi_1$, there exists $T_1 > 0$ such that for any $t \in [0, T_1], \|\bar{x}\| \leq \delta_1$.

Because there is no measurement noise, according to Theorem 1, exact derivative reconstruction can be achieved after some transient time. Moreover, according to [9], the transient time can be made arbitrarily small by choosing the design constants in the differentiators sufficiently large. Therefore, by choosing the design constants in the differentiators large enough (which is possible because $\|\bar{x}\| \leq M_{31}$ within $[0, T_1]$), exact derivative reconstruction can be achieved within $[0, T_1]$.

Using (18), it can be shown that $\Omega$ is a positive invariant attractive set of $\bar{x}$. Therefore, $\bar{x}(t)$ is bounded. This together with Assumption 4 guarantees that $\bar{x}(t)$ is bounded. Hence, $\|\bar{x}\|$ is bounded and there exists $M_{32}$ such that $\|\bar{x}\| \leq M_{32}$ for all $t \geq 0$. Therefore, by choosing the design constants in the differentiators sufficiently large (according to $M_{32}$), exact derivative reconstruction can be achieved for all $t \geq T_1$.

Because $\bar{x} = x$ for all $t \geq T_1$, the remaining part of the conclusions can be proved the same way as in the proof of Theorem 4.

Remark 5: Using similar arguments as in the proof of Theorem 7, it is possible to give results corresponding to those provided in Theorem 5 and Theorem 6.

Remark 6: It should be pointed out that all the results are proved under the assumption that exact derivative reconstruction can be achieved after some transient time. In real applications, exact derivative reconstruction is not achievable because of noise and sampling rate restrictions. In the presence of noise and sampling rate restrictions, the derivative estimation
controller design takes the following form

In this subsection, a brief introduction of the 3D crane and its nonlinear mathematical model is presented. The experimental setup of the 3D crane system provided by InTeCo Ltd is shown in Figure 1.

In experimental setup shown in Figure 1, the 3D crane consists of a construction frame, a rail attached to the frame that moves along the frame, a cart that moves on the rail, and a payload that is shifted up and down.

The mathematical model used for the output feedback controller design takes the following form

\[
\dot{x}_1 = x_2 \\
\dot{x}_2 = \Phi_1 + \mu_1 \cos(x_5)\Phi_3 \\
\dot{x}_3 = x_4 \\
\dot{x}_4 = \Phi_2 + \mu_2 \sin(x_5) \sin(x_7)\Phi_3 \\
\dot{x}_5 = x_6 \\
\dot{x}_6 = \left(\sin(x_5)\Phi_1 - \cos(x_5)\sin(x_7)\Phi_2 / x_9 + ((\mu_1 - \mu_2 \sin^2(x_7)) \sin(x_5) \cos(x_5)\Phi_3 + \Psi_5) / x_9\right) \\
\dot{x}_7 = x_8 \\
\dot{x}_8 = - (\cos(x_7)\Phi_2 + \Psi_6) / (\sin(x_5) x_9) - \mu_2 \sin(x_5) \cos(x_7) \sin(x_7)\Phi_3 / (\sin(x_5) x_9) \\
\dot{x}_9 = x_{10} \\
\dot{x}_{10} = - \cos(x_5) \Phi_1 - \sin(x_5) \sin(x_7)\Phi_2 + \Psi_7 \\
- (1 + \mu_1 \cos^2(x_5) + \mu_2 \sin^2(x_5) \sin^2(x_7)) \Phi_3 \\
y_i = x_{2i-1} + 1 \leq i \leq 5, \\
\Phi_1, i = 1, 2, 3 \text{ are defined as} \\
\Phi_1 = k_1 u_1 - T_{1x} x_2 - T_{sy} \text{sign}(x_2) \\
\Phi_2 = k_2 u_2 - T_{2x} x_4 - T_{sx} \text{sign}(x_4) \\
\Phi_3 = k_3 u_3 + T_{3x} x_{10} + T_{sz} \text{sign}(x_{10}), \\
\Psi_1, i = 5, 6, 7 \text{ are defined as} \\
\Psi_5 = \sin(x_5) \cos(x_5) x_9^2 x_9 + g \cos(x_5) \cos(x_7) - 2 x_6 x_{10} \\
\Psi_6 = -g \sin(x_7) - 2 x_6 x_2 x_9 \cos(x_5) - 2 x_8 x_{10} \sin(x_5) \\
\Psi_7 = x_5^2 x_9 \sin^2(x_5) + g \sin(x_5) \cos(x_7) + x_9^2 x_9. \\
\text{The definition of the variables in the above model are provided in Table I and Table II.} \\
\text{All parameters in the model given by (31)~(33) are listed below}
\]

\[
\begin{align*}
\mu_1 & = 0.4156, \mu_2 = 0.1431, \\
T_{sy} & = 6.4935, T_{sx} = 1.4903, T_{sz} = 20.8333, \\
k_1 & = 49.8636, k_2 = 16.0336, k_3 = -103.8606, \\
T_1 & = 11.5242, T_2 = 26.3263, T_3 = 217.3535.
\end{align*}
\]

Fig. 1. Experimental set-up: A laboratory scale 3D crane system
Hence, the model given by (31) can be rewritten in the form (1) as

\[ \begin{align*}
\dot{q}_i &= q_i, \\
\dot{q}_r &= f_i(x) + G_i(x)u, \\
y_i &= q_{i,1}, 1 \leq i \leq 5
\end{align*} \]

(35)

where

\[ \begin{align*}
f_1 &= \frac{m_w}{m_w + m_c} f_x = \sin(x_1)(T_1 x_2 + T_{sy} \text{sign}(x_2)) + \mu_1 \cos(x_5) \cos(x_5) T_{sz} \text{sign}(x_{10}) / x_9 \sin(x_5) \\
&+ \Psi_6 / (x_9 \sin(x_5)) \\
f_2 &= \frac{m_w}{m_w + m_c} f_x = \cos(x_5)(T_1 x_2 + T_{sy} \text{sign}(x_2)) + \sin(x_5) \sin(x_7) (T_3 x_{10} + T_{sz} \text{sign}(x_{10})) \cdot [(1 + \mu_1 \cos^2(x_5))(T_3 x_{10} + T_{sz} \text{sign}(x_{10}))] \\
&- \mu_2 \sin^2(x_5) \sin^2(x_7)(T_3 x_{10} + T_{sz} \text{sign}(x_{10})) + \Psi_7,
\end{align*} \]

(37)

and \( G_i = (g_{i1}, g_{i2}, g_{i3}), 1 \leq i \leq 5 \) with their elements defined as

\[ \begin{align*}
g_{i1} &= k_1 q_{i,1} = 0, g_{i,3} = \mu_1 k_3 \cos(x_5), \\
g_{i2} &= x_{1,1} = 0, g_{i,3} = k_2 \sin(x_5) \sin(x_7), \\
g_{i4} &= k_1 \sin(x_5) / x_{9,1} = -k_2 \cos(x_5) \sin(x_7) / x_9, \\
g_{i,3} &= k_3 \sin(x_5) \cos(x_5)(\mu_1 - \mu_2 \sin^2(x_7)), \\
g_{i,4} &= 0, g_{i5} = -k_2 \cos(x_7) / (x_9 \sin(x_5)), \\
g_{i,3} &= -\mu_2 k_3 \sin(x_5) \sin(x_7) \cos(x_7) / (x_{9,1} \sin(x_5)), \\
g_{i,1} &= -k_1 \cos(x_5), g_{i,2} = -k_2 \sin(x_5) \sin(x_7), \\
g_{i,3} &= k_3(-\mu_1 \cos^2(x_5) - \mu_2 \sin^2(x_5)) \sin^2(x_7) - 1.
\end{align*} \]

(38)

### B. Output feedback controller design

For the system given by (36) ~ (38), it is easy to see that \( m = 3 \) and \( p = 5 \) and that \( m < p \). The control objective is to design an output feedback controller to make \( y_1, y_2, y_5 \) to track their desired reference signals and ensure the signals in the closed-loop system are bounded. For this problem, one has \( m_\mu = m = 3 \).

According to Section IV, there is one \( G_{tr} = (G_1, G_2, G_3)^T \), \( F_{tr}(x) = (f_1, f_2, f_3)^T \), \( G_{tr}(x) = (G_3, G_4)^T \), \( F_{tr}(x) = (f_3, f_4)^T \), and \( G_{tr,1}(x) \) consists of the first two columns of \( G_{tr}(x) \).

Following the output feedback control design procedure in Section IV, an output feedback controller using HOSMDs can be designed as

\[ u = u_{tr} + u_{sta}, \]

(39)

where \( u_{tr} \) and \( u_{sta} \) are designed as follows

\[ \begin{align*}
u_{tr} &= G_{tr}^{-1}(\dot{x})(U_{tr}(\dot{x}) - F_{tr}(\dot{x})), \\
u_{sta} &= (u_{tr}^T 0)^T,
\end{align*} \]

(40)

where \( u_{2} = (\text{sat}_M(\tilde{u}_4) \text{ sat}_M(\tilde{u}_5))^T \) with \( \tilde{u}_2 = (\tilde{u}_4 \tilde{u}_5)^T = G_{tr}^{-1}(\dot{x}) U_{tr}(\dot{x}) \), and \( U_{tr}(\dot{x}) = (U_1(\dot{x}) U_2(\dot{x}) U_5(\dot{x}))^T \) and \( U_{tr}(\dot{x}) = (U_3(\dot{x}) U_4(\dot{x}))^T \) with \( U_i(\dot{x}) \) define as follows

\[ \begin{align*}
U_1(\dot{x}) &= -c_{i,1} \dot{z}_i - \xi_{i,1} - c_{i,1} \xi_{i,1} + c_{i,1} y_{i,r}, \\
&+ c_{i,1} y_{i,r},
\end{align*} \]

(41)

and

\[ \begin{align*}
\xi_{i,1} &= q_{i,1} - y_{i,r}, \alpha_{i,1} = -c_{i,1} \xi_{i,1} + \dot{y}_{i,r}, \\
\xi_{i,2} &= z_{i,1} - \alpha_{i,1}.
\end{align*} \]

(42)
To provide the estimates for $\hat{y}_i, 1 \leq i \leq 5$, for each $1 \leq i \leq 5$, a second-order sliding mode differentiator with the following form is used.

\[
\begin{align*}
\dot{z}_{i,0} &= v_{i,0} \\
v_{i,0} &= -\lambda_0 |z_{i,0} - y_i|^{2/3} \text{sign}(z_{i,0} - y_i) + z_{i,1} \\
\dot{z}_{i,1} &= v_{i,1} \\
v_{i,1} &= -\lambda_1 |z_{i,1} - v_{i,0}|^{1/2} \text{sign}(z_{i,1} - v_{i,0}) + z_{i,2} \\
\dot{z}_{i,2} &= -\lambda_2 \text{sign}(z_{i,2} - v_{i,1})
\end{align*}
\]

(43)

Where $z_{i,1}$ provides the estimate for $\hat{y}_i$.

Remark 7: Since only the first order derivatives need to be estimated, first-order differentiators is also able to provide estimation for those first order derivatives. The reason for using the second order differentiators is that they may provide a slightly better estimation accuracy in presence of measurement noises and the discrete-sampling (Theorem 2 and Theorem 3).

C. Experimental results on the 3D crane system

In order to test the applicability of the output feedback controller proposed in this paper to practical industrial systems, experiments are carried out on the 3D crane system.

In all experiments, the design constants for second-order sliding mode differentiators are chosen as $\lambda_0 = \lambda_1 = 20$ and $\lambda_2 = 30$, and the design constants in the controller are chosen as $c_{1,1} = c_{1,2} = 30, c_{2,1} = c_{2,2} = 10, c_{3,1} = c_{3,2} = 15, c_{4,1} = c_{4,2} = 15$, and $c_{5,1} = c_{5,2} = 150$ except in one particular experiment where $c_{1,1} = c_{1,2} = 9, c_{2,1} = c_{2,2} = 3, c_{3,1} = c_{3,2} = 3, c_{4,1} = c_{4,2} = 3$, and $c_{5,1} = c_{5,2} = 45$.

The sampling period is chosen as $T_s = 0.01s$ except in one experiment where $T_s = 0.002s$.

The tracking reference signals are chosen as $y_{1,r} = y_{5,r} = 0.2 + 0.1 \sin(0.2\pi t), y_{2,r} = 0.15 + 0.15 \sin(0.2\pi t), y_{3,r} = \pi/2, y_{4,r} = 0$.

Four experiments were performed.

The first experiment illustrates successful application of the controller given by (39)–(43) to the 3D crane system. The results are presented in Figures 2 to 4.

The second experiment was carried out to illustrates what happens if $u_{sta} = 0$, and the results are shown in Figures 5 to 7.

The third experiment was intended to show the effect of the design constants in the proposed controller, and the results are provided in Figures 8 to 10 for the case where $c_{1,1} = c_{1,2} = 9, c_{2,1} = c_{2,2} = 3, c_{3,1} = c_{3,2} = 3, c_{4,1} = c_{4,2} = 3$, and $c_{5,1} = c_{5,2} = 45$.

The last experiment shows the effect of sampling rate, and the results are illustrated in Figures 11 to 14.

D. Discussions based on the experimental results

The results presented in Figures 2 to 4 show that the output tracking performance and angle stabilization performance are
very satisfactory.
If $u_{sta} = 0$, good output tracking performance can also be achieved as can be seen from Figure 5 with similar control effort, see Figure 4. However, the angle $x_7$ in Figure 6 is much larger than that in Figure 3 which is not desirable. This result implies that the introduction of the design component $u_{sta}$ can be beneficial.

Because HOSMD could not produce derivative estimation with enough accuracy according to Theorem 3 for $Ts = 0.1s$, those derivative estimation errors have to be taken care of thorough the design of the controller constants before one can achieve satisfactory control performance in practical applications. For the case that $c_{1,1} = c_{1,2} = 9$, $c_{2,1} = c_{2,2} = 3$, $c_{3,1} = c_{3,2} = 3$, $c_{4,1} = c_{4,2} = 3$, and $c_{5,1} = c_{5,2} = 45$, although the control effort in Figure 10 is smaller, the results in Figure 8 and Figure 9 show a very poor control performance. Therefore, the controller design constants should be chosen suitably large to obtain satisfactory control performance.

Again according to Theorem 3, one should be able to achieve better performance by sampling faster. This conclusion is confirmed by the last experiment with $Ts = 0.002s$. The results in Figure 11 and Fig. 12 show a better performance.
is not fast enough in order to use HOSMD in the controller design.

VI. CONCLUSIONS

This paper addressed a practical output feedback control design problem aiming to regulate only part of the outputs for a class of multi-input multi-output (MIMO) nonlinear systems where the number of inputs is less than that of outputs. To solve the problem, a controller structure with two design components in all or some chosen control inputs was proposed, where the two design components were designed using backstepping approach together with the use of the recently developed HOSMDs to estimate the derivatives of the outputs. Stability results were established for the proposed controller under certain conditions.

In order to show the practicality of the proposed output feedback controller, the controller was implemented and tested on a laboratory scale 3D crane system. The experimental results demonstrated that the proposed output feedback controller is able to achieve very satisfactory control performance. They have also revealed the advantage of the proposed controller structure and the effect of the controller design constants and sampling periods.

Although the experimental results confirm that a carefully designed $u_{pred}$ helpful in the control of the damping of the swing, a strict analysis has not been performed on this yet. This is so, because the current research focused mainly on precise positioning. As such, a future research topic is to explore how to achieve good position tracking with reduced sway.

VII. ACKNOWLEDGMENTS

This research was supported by Natural Sciences and Engineering Research Council (NSERC) of Canada through its Discovery Grant Program. The authors are grateful to anonymous reviewers for their careful reading of the manuscript and their many questions and suggestions that resulted in improvements to the original draft.

REFERENCES


Weitian Chen received the B.S. degree in Mathematics and the M.S. degree in control theory from Ludong University, Shandong, China, in 1989 and Qufu Normal University, Shandong, China, in 1991, respectively. After working at Qufu Normal University for two years, he became a student at the Department of Automation, Shanghai Jiaotong University, Shanghai, China, where he received the Ph.D. degree in 1996. He worked as an associate professor from 1996 to 1998 and a full professor from 1999 to 2000 both at the Institute of Automation, Qufu Normal University, Shandong, China. From 2001 to 2002, he visited Simon Fraser University as a research associate. In 2002, he became a student at the Department of Engineering Science, Simon Fraser University, where he received the Ph.D. degree in 2007. He is currently working as a research fellow at the Department of Information Engineering, the Australian National University, Canberra, Australia. He has published over forty papers in refereed international journals and conferences. His research interests include model based fault diagnosis in control systems, multiple model based observation and control of uncertain and nonlinear systems, fuzzy systems and control, and nonlinear systems and control.

Mehrdad Saif received B.S. in 1982, M.S. in 1984, and PhD in 1987 all in Electrical Engineering. During his graduate studies he worked on research projects sponsored by NASA Lewis (now Glenn) Research Center, as well as Cleveland Advanced Manufacturing Program (CAMP). In 1987 he joined the School of Engineering Science at Simon Fraser University as an Assistant Professor. He is currently a Full Professor and has been the Director of the School since 2002. From 1993-1994, Dr. Saif was a Visiting Scholar at General Motors North American Operation (NAO) R & D Center in Warren, MI. At GM he was a member of the Powertrain Control Group in the Electrical and Electronics Research Department where he worked on engine control and on-board engine diagnostic problems.

Dr. Saif’s research interests are in systems and control, in particular estimation and observer theory, model based fault diagnostics in control systems, and application of these to automotive, power, and other complex engineering systems. He has published over 150 refereed journal and conference papers plus an edited book in these areas. Dr. Saif has been a consultant to a number of industries and agencies such as GM, NASA, B.C. Hydro, Ontario Council of Graduate Studies, etc. He served two terms (1995,1997) as the Chairman of the Vancouver Section of the IEEE Control Systems Society. He is the Associate Editor of IEEE Systems Journal, and International Journal of Control and Computers. He is also on the editorial board of the American Control Conference, as well as the IEEE Conference on Decision and Control. Dr. Saif is a Senior Member of the IEEE and is a Registered Professional Engineer in British Columbia.