MIMO Cooperative Diversity in a Transmit Power Limited Environment

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Abstract—This paper considers a fading relay channel where the total transmit power used is constrained to be equal to that of the standard single-hop channel. The relay channel used operates in what is termed as MIMO cooperative diversity mode, where the source transmits to both relay and destination terminals in the first instance. Both the source and relay then transmit to the destination in the second instance. Initially the cooperative diversity framework is introduced to consider system constraints so a direct and fair comparison with the single-hop case can be made. In particular a power constraint is placed on the system and the optimal transmit power levels are derived and presented. The derived technique for finding the optimal transmit power levels is then used to demonstrate the advantages of using cooperative diversity in a wireless network. The results presented show that MIMO cooperative diversity offers a 3.4dB increase in spectral efficiency at 5% outage, with no additional cost incurred in transmit time, power or bandwidth.

I. INTRODUCTION

One of the biggest problems faced by designers of mobile telecommunication systems is fast-fading of the received signal due to the multi-path nature of the communications channel. The very multi-path that creates difficulties also offers us diversity [1]. This can help mitigate the problem as a result of the transmission of redundant information over (ideally) independent fading paths (in time/frequency/space) in conjunction with a suitable receiver. Spatial diversity techniques are particularly attractive as they provide diversity gain while incurring no penalty of extra transmission time or bandwidth. Spatial diversity techniques such as Multiple-Input Multiple-Output (MIMO) wireless systems [2] have been shown to significantly increase the spectral efficiency of point-to-point wireless links, including cooperative diversity relay channels [3] which are designed to take advantage of spatial diversity where Mobile Terminals (MTs) are limited to a single transmit/receive antenna.

Contributions and relation to previous work.

Cooperative diversity is very popular due to its flexibility, which was highlighted by Nabar et al. [4] developing three basic cooperative communication protocols where a source, relay and destination are considered. This paper considers a single protocol, Protocol I, where the source MT can communicate with the destination for the full transmission time available, and is supported by the relay midway through the transmission.

The advantage of using cooperative diversity in an environment where the total transmit power is limited to that of a single-hop communication channel is considered [5]. This allows a fair and direct comparison between the cooperative diversity relay channel and the single-hop channel to be made. The power constraint placed on the system is an important factor to be considered for any practical deployment of a cooperative diversity network, both due to interference concerns [6] and to ensure battery life prolongation of the relay terminal. Furthermore, the DF mechanism introduced by Laneman et al. [7] is utilised.

Laneman et al. considered an access protocol where by multiple relays may transmit at a single time, in the same channel [7]. Although only a single relay is used in this paper, the signal from the source and relay will cause collision at the destination, and this problem can be viewed as similar to the multiple relay problem. Space-Time Codes (STC) are used to allow the destination to utilise the full diversity available to it from the multiple transmission points. As such, STCs in cooperative diversity have attracted considerable attention [8], [9]. Only STC cooperative diversity is considered in this paper.

The first part of this paper introduces the cooperative diversity protocol and channel model framework, originally developed by Nabar et al. [4] and later extended in [10]. A jointly analytical-numerical method to optimise the transmit power levels of each transmitting terminal within the power constraint, in terms of probability of outage, is then developed. This extends the work presented in [10] to consider an alternative Decode-and-Forward (DF) relaying scheme and introduces an analytical approach to optimising the transmit power levels. Results for the optimal power levels are then introduced. The second part of this paper presents a comparison between the cooperative diversity protocol considered and the single-hop case. Finally conclusions are drawn in the final section.
II. PROTOCOL DESCRIPTIONS AND SYSTEM CONSTRAINTS

A. Protocol Description

Consider the ad-hoc fading relay channel shown in Figure 1 with three MTs. Data is transmitted from the source terminal, S, to the destination terminal, D, potentially with the assistance of the relay, R. All terminals in this paper are considered to have single transmit and single receive antenna elements. Additionally, a terminal cannot simultaneously transmit and receive information. Although it is possible for the relay to assist in either an Amplify-and-Forward (AF) or Decode-and-Forward (DF) manner, only DF is considered in this paper. In DF mode the relay demodulates and decodes the signal before re-encoding the signal and retransmission occurs. The DF mode that is used in this paper requires the relay to receive a required information rate, R, before it can take part in the second phase, which is denoted \( R \in D \), where D is the set of supporting relays. If the relay is unable to decode the first phase signal from the source, it does not transmit any information in the second phase [7], which is denoted \( R \notin D \).

Due to the constraint of MTs not being able to transmit and receive information at the same time, the transmission of information from the source to destination, in a cooperative diversity network, is broken into two time slots, termed phase I and phase II [11]. This paper considers the cooperative diversity scenario where the source transmits in both time phases and the relay only in the second having (potentially) received the first phase transmission from the source. Furthermore the source transmits different information to the destination in the second phase than that which was transmitted during the first phase. The destination can therefore potentially receive two different information streams during the second phase.

This cooperative diversity protocol was originally introduced by Nabar et al. [4] as one member of a family of cooperative diversity protocols. The subject protocol is referred to as Protocol I in literature, however as it is the only protocol to be considered in this paper, any mention of cooperative diversity refers to this protocol unless otherwise specified.

When considering the complete cooperative diversity model, it can be seen that it bears a strong resemblance to a MIMO system with two antenna elements at both the transmitter and receiver. The fading relay channel from Figure 1 is redrawn in Figure 2 to highlight the MIMO characteristics by splitting the destination into the two time phases, and adding an extra subscript to the channel notation to denote the phase of transmission. This is a schematic presentation only and the destination terminal actually uses only one antenna element. However, its access of the channel is split over the two time phases. This scenario is therefore termed MIMO cooperative diversity due to the similarity to MIMO wireless systems.

B. System constraints

To be able to fairly compare the performance several constraints are placed upon the channel model:

- The time used must not exceed that used by the single-hop transmission case (1 unit)
- The bandwidth used must not exceed that used by the single-hop transmission case (1 unit)
- The total transmit power of the complete system (power used by the source and relay for transmission) must not exceed that used by the single-hop case (1 unit)

The effect of the time and bandwidth constraints is that the transmission of the two phase cooperative diversity must take place in the same amount of time as the single-hop system. This typically involves a factor of \( 1/2 \) when considering the spectral efficiency of the system. An important point that arises from this constraint is that the Rayleigh block-fading is now considered to be unchanged over the two time phases.

The effect of the transmit power constraint is particularly important for the cooperative diversity protocol considered in this paper since it uses three transmissions (the source in both the first and second phase and the relay in the second phase only) and therefore must reduce their transmit power to the power constraint. The following transmit power constraint variables are placed upon the system:

- A - first phase transmission from the source
- B - second phase transmission from the source
- C - second phase transmission from the relay

To constrain the transmit power to be the same as the single-hop case, first recall that the transmission in the relaying network has been split into two phases, unity in each phase.
It therefore follows that the sum of the total transmit power from the source in both phases, and the relay in the second phase must not exceed that of a single transmission over the full time period. This can be written as

$$A + B + C = 2$$  

(1)

A transmit power constraint is also placed upon each transmitting terminal such that it cannot broadcast with more power than the single-hop case would (1 unit), i.e.,

$$A \leq 1 \quad B \leq 1 \quad C \leq 1.$$  

(2)

### III. CHANNEL AND SIGNAL MODELS

Throughout this paper the channel fading model is assumed to be frequency-flat, Rayleigh block-fading channels with fading coefficients being circularly symmetric zero mean complex Gaussian random variables. Perfect timing synchronisation is assumed. Furthermore it is assumed that the receivers have perfect channel state information of all the reverse channels. It is also assumed that there is limited feedback to the source and relay from the destination of the average channel Signal to Noise Ratio (SNR) which is required to perform power control for MIMO cooperative diversity.

The signals transmitted by the source during the first and second phases are denoted $x_1[n]$ and $x_2[n]$ respectively. Symbol-by-symbol transmission is considered so the time index $n$ can be dropped to simply give $x_1$ and $x_2$. Note also that for the data symbols transmitted is it assumed that $\mathcal{E}\{x_1\} = 0$ and $\mathcal{E}\{|x_i|^2\} = 1$ for $i = 1, 2$. The signal received at the destination terminal in the first phase is given by

$$y_{D,1} = \sqrt{AE_{SD}}h_{SD}x_1 + n_{D,1}$$  

(3)

where $y_{D,1}$ is the received signal at the destination (Y) in the first phase, $E_{XY}$ is the average signal energy over one symbol period. The scalar $h_{XY}$ is the random, complex-valued, unit-power channel gain between the source and destination terminals, and $n_{Y,1} \sim \mathcal{N}(0,N_0)$ is the additive white noise for transmitting terminal X and received terminal Y, in this case S and D respectively. Note that $E_{XY}$ and $h_{XY}$ does not have a phase subscript due to the earlier assumption of flat fading across the two transmission phases. Similarly the signal received at the relay in the first time slot is given by

$$y_{R,1} = \sqrt{AE_{SR}}h_{SR}x_1 + n_{R,1}$$  

(4)

Finally assuming that the relay can fully decode the signal from the first phase source transmission, the signal received by the destination in the second phase is given by

$$y_{R,2} = \begin{cases} 
\sqrt{BE_{SD}}h_{SD}x_2 + \sqrt{CE_{RD}}h_{RD}x_2 + n_{D,2} & \text{if } R \in D \\
\sqrt{BE_{SD}}h_{SD}x_2 + n_{D,2} & \text{if } R \notin D
\end{cases}$$  

(5)

The input-output relationship can now be summarised as

$$y = Hx + n$$  

(6)

where $y = [y_{D,1} y_{D,2}]^T$ is the received signal vector, $H$ is the MIMO cooperative diversity $2 \times 2$ channel matrix

$$H = \begin{bmatrix} \sqrt{AE_{SD}}h_{SD} & 0 \\
\sqrt{CE_{RD}}h_{RD} & \sqrt{BE_{SD}}h_{SD} \end{bmatrix}$$  

(7)

$x = [x_1, x_2]^T$ is the transmitted signal vector and $n$ is additive white Gaussian noise.

#### A. Information-Theoretic performance

The DF protocol used in this paper states that the rule for the relay to be considered part of the decoding set is that the relay must be able to support the required spectral efficiency, R. Since the source/relay channel mutual information, $I_{SR}$, is a function of the random fading coefficients, the mutual information is also a random variable. The event $I_{SR} < R$ is defined as the source/relay channel being in outage and therefore $\Pr[I_{SR} < R]$ is referred to as the outage probability of the channel. Moreover this definition is extended from the relay inclusion condition to the general channel such that the event $I < R$ is considered channel outage, where $I$ is the total mutual information of the cooperative channel.

The total probability of the channel being in outage is given by the sum of the probability of the channel being in outage when the relay either is available or not, and is given by

$$\Pr[I < R] = \Pr[R \in D] \Pr[I < R|R \in D] + \Pr[R \notin D] \Pr[I < R|R \notin D]$$  

(8)

The mutual information between the source and relay in the first phase, is given by Shannon’s capacity equation, with a factor of 1/2 due to the two equal transmission time phases, as

$$I_{SR} = \frac{1}{2} \log_2 \left(1 + \frac{AE_{SR}}{N_0} |h_{SR}|^2 \right) \text{ b/s/Hz}$$  

(9)

Rearranging (9) to give the condition of the relay being able to decode the first phase transmission, and substituting the mutual information $I_{SR}$ for the required spectral efficiency, R, gives

$$|h_{SR}|^2 > \frac{2^{2R} - 1}{\text{SNR}}$$  

(10)

where $E_{SR}/N_0 = \text{SNR}$. In the remainder of this paper it is assumed that all network channel links have the same SNR, specifically $E_{SR}/N_0 = E_{RD}/N_0 = E_{SD}/N_0 = \text{SNR}$ to simplify presentation and simulation results. This assumption can be readily relaxed for the more general case.

Finally, the mutual information for the MIMO cooperative diversity channel is obtained from (6) and (7) as

$$I = \frac{1}{2} \log_2 \det \left( I_2 + \frac{1}{N_0} HH^H \right) \text{ b/s/Hz}$$  

(11)
which is expanded to

\[
I = \begin{cases} 
\frac{1}{2} \log_2 \left( (\text{ABSNR})^2 |h_{SD}|^4 + \text{ASNR}|h_{SD}|^2 ight) & R \in \mathbb{D} \\
\frac{1}{2} \log_2 \left( (\text{ABSNR})^2 |h_{SD}|^4 + \text{ASNR}|h_{SD}|^2 + C \text{SNR}|h_{SD}|^2 + 1 \right) & R \notin \mathbb{D}
\end{cases}
\]

IV. POWER CONTROL ANALYSIS

Consider now the problem of minimising the probability of outage of the cooperative diversity channel as a function of the power control levels \( A, B \) and \( C \). Since \( A, B \) and \( C \) are limited by (1), by varying \( A \) and \( B \) between 0 and 1 independently, the entire range of results are explored, and subsequently the minimum probability of outage can be found. To begin an analytical approach to this problem, consider the channel probability of outage (8), substitute the required mutual information equations (9) & (12) and rearrange so the random variables are on the Left Hand Side (LHS) of each individual probability inequality. This is written as

\[
\Pr \left[ I_{1, \mu} < R \right] = \left( \Pr \left[ |h_{SR}|^2 < \frac{x}{A} \right] \Pr \left[ \text{ABSNR}|h_{SD}|^4 + (A + B)|h_{SD}|^2 < x \right] \right) + \left( \Pr \left[ |h_{SR}|^2 > \frac{x}{A} \right] \Pr \left[ \text{ABSNR}|h_{SD}|^4 + (A + B)|h_{SD}|^2 + C|h_{RD}|^2 < x \right] \right)
\]

where \( x = (2^{2k} - 1)/\text{SNR} \).

The probability inequalities with a single channel coefficient on the LHS (probability of relaying being able to support the source) can readily be solved analytically as they are single Rayleigh random variables for which the Cumulative Distribution Function (CDF) is well known as \( [1 - \exp(-x/A)] \) when the standard deviation is 1 as it is in this case. Moreover \( x \) for the Rayleigh CDF is given by the Right Hand Side (RHS) for each probability inequality.

A. Derivation of CDF with relay available

The CDF of the more complex joint distribution of the two random variables \( \text{ABSNR}|h_{SD}|^4 + (A + B)|h_{SD}|^2 + C|h_{RD}|^2 \) is not so well known. To find an analytical expression for the probability of outage conditioned on the relay being available, an expression for the CDF of the joint random variables must be found. This is done by convolving the Probability Density Functions (PDF) of each random variable to obtain the joint PDF, which is then integrated to obtain the required CDF.

The probability of outage of the relay channel, operating with the relay available, is given by

\[
\Pr \left[ \text{ABSNR}|h_{SD}|^4 + (A + B)|h_{SD}|^2 + C|h_{RD}|^2 < x \right]
\]

Before a PDF can be derived for the joint \( h_{SD} \) and \( h_{RD} \) terms of (14) a single equation must be found for the \( h_{SD} \) term. Begin by defining

\[
Y = \text{ABSNR}|h_{SD}|^4 + (A + B)|h_{SD}|^2
\]

which shows that \( Y \) is a function of the single random variable \( h_{SD} \). To find the PDF of a function of a random variable employ (5-15) and (5-16) as presented by Papolius and Pillai [12] where the equation \( Y = g(X) \) is solved. Denoting the real roots by \( X_n \),

\[
y = g(X_1) = ... = g(X_n)
\]

Papolius and Pillai show that the PDF of a single random variable is given by

\[
f_Y(Y) = \frac{f_X(1)}{|g'(X_1)|} + \frac{f_X(2)}{|g'(X_2)|} + ... + \frac{f_X(n)}{|g'(X_n)|}
\]

where \( f_X(X) \) is the function of the single random variable in question, and \( g'(X) \) is the derivative of \( g(X) \). For the case in question let \( X = |h_{SD}|^2 \) to give

\[
g(X) = \text{ABSNR}X^2 + (A + B)X - Y
\]

To find the roots of (18) use the standard quadratic equation where \( a = \text{ABSNR} \), \( b = A + B \) and \( c = -Y \).

\[
0 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} - \frac{(A + B) \pm \sqrt{(A + B)^2 + 4ABY\text{SNR}}}{2ABSNR} \frac{2ABSNR}{\sqrt{(A + B)^2 + 4ABY\text{SNR}} - (A + B)}
\]

where in (21) discard the negative term since the interest lies only in probabilities (i.e. between 0 and 1 inclusive) leaving only one root. Furthermore the derivative of \( g(x) \) is found as

\[
g'(x) = \text{ABSNR}X + A + B
\]

By combining (17), (21) and (22) the general PDF is given by

\[
f(Y) = \frac{\sqrt{(A + B)^2 + 4ABY\text{SNR}} - (A + B)}{2ABSNR} \frac{2ABSNR}{\sqrt{(A + B)^2 + 4ABY\text{SNR}}}
\]

Since \( h_{SD}^2 \) is a Rayleigh distributed random variable, the PDF of which is given by \( \Pr[x] = e^{-x} \), the PDF of the joint \( h_{SD} \) term is given by
where the value of interest is denoted $x$.

Before deriving the final combined PDF, note that the PDF of $C|h_{SD}|^2$ is given by the RHS of (14) as

$$\frac{1}{C} e^{-\frac{\sqrt{x}}{C}} $$ \hspace{1cm} (25)

Therefore the joint PDF is found by convolving the two PDFs.

This is a non-trivial integration. However, symbolic computer algebra can be employed to find a suitable integral

$$ (f * g)(x) = \frac{\sqrt{\pi e^{\frac{2x - 2AB^2 + 2xAB^2 + 2CA - 2CB + C^2}{4ABSNR}}}}{2ABCSNR} \left( \text{erf} \left( \frac{\text{csign}(A + B)B - C}{2ABCSNR} \right) + \text{erf} \left( \frac{-\sqrt{4xABSNR + A^2 + 2AB + B^2 + C}}{2ABCSNR} \sqrt{-\frac{1}{ABCSNR}} \right) \right) $$ \hspace{1cm} (26)

where erf($z$) is the Error Function and csgn($z$) is the sign function is used to determine in which half-plane (left or right) the complex-valued number $z$ lies.

It can be seen that the derived PDF for the joint distribution (26) is very complex. To find the CDF of the joint distribution it is necessary to integrate this expression again, however it is not of a standard form and a closed form solution can not be readily found using either standard methods or symbolic computer algebra. Due to this, Runge-Kutta numerical integration methods are used to approximate the integral.

### B. Derivation of CDF with relay not available

To be able to fully calculate the probability of outage for MIMO cooperative diversity, consideration must also be made for the situation where no relays are available to support the source in the second phase, for example when the source to relay signal is itself in outage. The previous section derived the PDF of the $h_{SD}$ term as (23). To find the CDF (23) simply needs to be integrated. Again using symbolic computer algebra the CDF of the probability of outage where the relay is not available is shown to be

$$ \frac{\sqrt{\pi e^{\frac{2x - 2AB^2 + 2xAB^2 + 2CA - 2CB + C^2}{4ABSNR}}}}{2ABCSNR} \left( \text{erf} \left( \frac{\text{csign}(A + B)B - C}{2ABCSNR} \right) + \text{erf} \left( \frac{-\sqrt{4xABSNR + A^2 + 2AB + B^2 + C}}{2ABCSNR} \sqrt{-\frac{1}{ABCSNR}} \right) \right) $$ \hspace{1cm} (26)

The necessary components to fully construct (13) are now available using numerical integration of (26) and the closed form solution (27), however the trivial re-writing this equation in full is omitted in this paper due to space constraints.

### C. Numerical results

Results from the numerical evaluation of (13) are shown in Figure 3 for a spectral efficiency of 1b/s/Hz. Below approximately 6dB SNR only the source is preferred to transmit and all of the available power is dedicated to the source’s transmission over both time phases. This is due to the fact that for SNRs lower than this the source to relay link is not suitable for the required 1b/s/Hz spectral efficiency. Above the 6dB point the relay power allocation increases sharply while the second phase source transmission power level falls off proportionally and both approach an asymptote at high SNRs. Interestingly at high SNR it is optimal in terms of outage for the relay to use a greater power allocation than the source in the second phase. This is due to the extra diversity that the relay offers, once it is able to take part in the second phase.

At lower required spectral efficiencies it would be expected that the relay would be be allocated transmit power in the second phase at lower SNRs than that recorded for 1b/s/Hz. This can be seen to be true by inspection of (10), which suggests that at low spectral efficiencies the SNR for a similar probability of inclusion of the relay in the second phase will also be lower. Similarly at higher spectral efficiencies the SNR level where the relay would actively be used would be higher.
As can be seen from Figure 5 MIMO cooperative diversity offers a considerable increase in the spectral efficiency of the channel, with the extra diversity offered by the relay effectively combating the Rayleigh fading at low SNR. A significant gain of 3.4dB can be observed over the single-hop link at low SNR. At high SNR the spectral efficiency of the MIMO cooperative diversity protocol converges with the single-hop case.

VI. CONCLUSION

This paper has presented a framework for MIMO cooperative diversity which is directly comparable with single-hop transmission. In-particular the power constraint was analytically considered and values for the optimal transmit power control values were presented. It was shown that at low SNR MIMO cooperative diversity can offer a considerable advantage over the single-hop case, due to the extra spatial diversity of the relay, with no additional cost incurred in transmit time, power or bandwidth.

The work presented in this paper may be generalised to the MIMO relay network where MTs are equipped with more than one antenna element in future, [13]. This is important to cooperative diversity as a unified information theory framework can potentially be developed to include support for cooperative diversity networks with multiple transmit and receive antenna elements.

REFERENCES


