Existence and Uniqueness of Solutions for DC Networks Containing Nonlinear Ideal Op-amps

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SUMMARY

In this paper we are concerned with networks obtained by connecting independent sources, linear resistors and nonlinear ideal op amps.

A necessary and sufficient condition for the existence and uniqueness of solutions for every positive output saturation voltage of the op amps and every value of the independent sources is found.

KEY WORDS: DC networks, nonlinear ideal op amps, unique solvability, non-singular $P_0$ matrices.

1. INTRODUCTION

Concerning a network obtained by connecting independent sources, linear resistors, linear controlled sources and nonlinear resistors described by increasing functions, Sandberg and Willson [1]–[4] have found necessary and sufficient algebraic conditions for the existence and uniqueness of solutions for given values of the linear elements, every allowed characteristic of the nonlinear elements and every value of the independent sources.

Concerning a network obtained by connecting independent sources, linear resistors, linear controlled sources and nonlinear resistors described by increasing functions, or by connecting independent sources, nullors and nonlinear resistors described by increasing functions, Nielsen, Willson [5], Nishi, Chua [6], and Hasler [7] respectively, have found necessary and sufficient topological conditions for the existence and uniqueness of solutions for every value of the linear elements, every allowed characteristic of the nonlinear elements and every value of the independent sources.

Concerning a network obtained by connecting independent sources, linear resistors, nonlinear resistors described by increasing functions, and nonlinear op amps—either all ideal or all described by increasing saturating functions—Nishi and Chua [8] have found necessary and sufficient topological conditions for the existence and uniqueness of solutions for every value of the linear resistors, every allowed characteristic of the nonlinear elements, every new connection of independent sources at any location—verifying a natural topological condition—and every value of these independent sources.

Observe that in [1]–[7] the conditions are given for fixed locations of the independent sources. However, while in [6] and [7] the actual location plays a substantial role, in [1]–[5] it doesn’t. Precisely, the conditions found in [1]–[5] either are verified or are not, no matter where the independent sources are actually located—apart from natural topological restrictions.

In this paper, we consider a network obtained by connecting independent sources, linear resistors and nonlinear ideal op amps—i.e. nonlinear two-ports with zero input current, a positive output saturation voltage $\pi$ and the graph $\phi_{\pi}$ of the voltage transfer characteristic of the form shown in Figure 1, where $v_0$ and $v_\infty$ are their input and output voltages respectively (see [9])—and study the problem of the existence and uniqueness of solutions for given values of the linear resistors, every positive output saturation voltage and every value of the independent sources.

In the first part of Section 2 we describe a topological condition necessary to the existence and uniqueness of solutions for every positive output saturation voltage and every value of the independent sources, and prove that under this condition the solution of the network is equivalent to the solution of a system of the form

$$\begin{cases}
v_0 - Tv_\infty = Hs \\
(v_0, v_\infty) \in \phi_{\pi_k}, \quad k = 1, \ldots, n
\end{cases}$$

where $n$ is the number of nonlinear ideal op amps, $v_0$ and $v_\infty$ are their input and output voltages respectively, $\pi_1 > 0, \ldots, \pi_n > 0$ are their output saturation voltages, $s$ are the
values of the independent sources, and \( T \) and \( H \) are suitable matrices which are well known in the theory of linear multiloop feedback systems (see [10]).

The existence and uniqueness of solutions of such a system for every \( \pi_1 > 0, \ldots, \pi_n > 0 \) and every \( s \), has been studied in [11] where we prove that the conditions \( -T \in P_0 \) and \( -T \) non-singular are always necessary, necessary and sufficient when \( H = 0 \); and that the condition \( -T \in P \) is always sufficient, necessary and sufficient when rank\((H) = n \).

In the second part of Section 2 we describe and prove the hidden necessary and sufficient condition on \( -T \) and \( H \) which must hold in the general case. This allows us to state the necessary and sufficient condition for the existence and uniqueness of solutions of the network for every \( \pi_1 > 0, \ldots, \pi_n > 0 \) and every \( s \). The actual location of the independent sources plays a substantial role in this condition.

The mathematical background is developed in Appendix I where for every \( n \times n \) non-singular \( P_0 \) matrix \( A \) the set of the \( h \in \mathbb{R}^n \) such that the system

\[
\begin{cases}
    x + Ay = h \\
    (x_k, y_k)^T \in \phi_{\pi_k}, \quad k = 1, \ldots, n
\end{cases}
\]

has a unique solution for every \( \pi_1 > 0, \ldots, \pi_n > 0 \) is described.

The network-theoretic meaning of the algebraic part of the condition is described in Section 3.

2. EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR IDEAL OP AMPS

Let \( N \) be a network obtained by connecting \( m \) independent voltage and current sources \( S_1, \ldots, S_m, r \) linear resistors of given positive values, and \( n \) nonlinear op amps \( O_1, \ldots, O_n \) (see Figure 2).

For every family \( \pi = (\pi_1, \ldots, \pi_n) \) of positive reals (in what follows this statement will be referred to by: for every \( \pi > 0 \)), let \( \phi_{\pi_k} \), \( k = 1, \ldots, n \), be the subset of \( \mathbb{R}^2 \) defined by

\[
\phi_{\pi_k} = (-\infty, 0) \times \{-\pi_k\} \cup \{0\} \times [-\pi_k, \pi_k] \cup (0, +\infty) \times \{\pi_k\}
\]

(see Figure 1) and let \( \Phi_\pi = \phi_{\pi_1} \times \cdots \times \phi_{\pi_n} \).

For every \( \pi > 0 \) and every \( s \in \mathbb{R}^m \), let \( N(\pi, s) \) be the network obtained from \( N \) when each \( S_j \) has value \( s_j \) and each \( O_k \), with reference to Figure 3, has ideal constitutive equations:

\[
\begin{cases}
    i_{0k} = 0 \\
    (v_{0k}, v_{\infty k})^T \in \phi_{\pi_k}
\end{cases}
\]
Let $\mathcal{R}$ be the network obtained from $\mathcal{N}$ by substituting each $O_k$ with an independent current source of value $0$ in place of its input port, and an independent voltage source $W_k$ in place of its output port (see Figure 4 where the independent current sources of value $0$ are represented as open circuits).

**Remark 1**

If for every $\pi > 0$ and every $s \in \mathbb{R}^m$ the network $\mathcal{N}(\pi, s)$ has a unique solution, then $\mathcal{R}$ has neither a loop of independent voltage sources nor a cut-set of independent current sources.

(The statement is a consequence of Remark 1 of [11] and of the proof of Lemma 1 of [11])

**Remark 2**

If $\mathcal{R}$ has neither a loop of independent voltage sources nor a cut-set of independent current sources, then

(a) for every value $s$ of $S_1, \ldots, S_m$ and every value $w$ of $W_1, \ldots, W_n$, $\mathcal{R}$ has a unique solution; in particular, there are unique matrices $T$ and $H$ such that the vector $v_0$ of the voltages across the open circuits is given by

$$v_0 = Tw + Hs$$

(b) for every $\pi > 0$ and every $s \in \mathbb{R}^m$, there is a straightforward bijection of the
solutions of $\mathcal{N}(\pi, s)$ into the solutions of

$$\begin{cases}
    v_0 - Tv_\infty = Hs \\
    (v_{01}, v_{\infty 1}, \ldots, v_{0n}, v_{\infty n})^T \in \Phi_\pi
\end{cases}$$

(1)

where $v_0$ and $v_\infty$ are the vectors of the input and output voltages of the op amps respectively.

(c) for every $\pi > 0$ and every $s \in \mathbb{R}^m$, the network $\mathcal{N}(\pi, s)$ has at least one solution.

(Statement (a) is well known, (b) follows from (a); for the proof of (c), see Appendix II)

The following definitions allow us to describe the necessary and sufficient condition for the existence and uniqueness of solutions for a system of the form (1) for every $\pi > 0$ and every $s \in \mathbb{R}^m$.

**Definition 1**

Let $C$ be a $q \times q$ matrix. For every non-empty subset $I$ of $Q = \{1, \ldots, q\}$, let $C(I)$ denote the principal submatrix of $C$ obtained by omitting rows and columns with indices outside $I$.

A principal submatrix $C(I)$ of $C$ will be called $C$-isolated if $c_{ij} = 0$ for $i \in I$ and $j \in Q \setminus I$. Observe that $C = C(Q)$ is a $C$-isolated principal submatrix of $C$.

The set of all $C$-isolated principal submatrices of $C$ is partially ordered by the relation $C(I) \leq C(J)$ iff $I \subseteq J$. The minimal elements of this set will be called minimal $C$-isolated principal submatrices of $C$.

**Definition 2**

Let $C$ be a $q \times q$ matrix.

$C$ will be called a matrix of first kind if $C$ is singular and $C$ is a (and hence the unique) minimal $C$-isolated principal submatrix of $C$.

$C$ will be called a matrix of second kind if $C$ is singular and there exists a unique minimal $C$-isolated principal submatrix $M$ of $C$ and $M$ is non-singular.

**Definition 3**

Let $C$ be a $q \times q$ matrix and let $Q = \{1, \ldots, q\}$. The set of all principal submatrices of $C$ of first or second kind will be denoted by $K(C)$.
For every $C(I) \in \mathcal{K}(C)$, let $\mu(I)$ be the subset of $Q$ such that $C(\mu(I))$ is the unique minimal $C(I)$-isolated principal submatrix of $C(I)$.

**Definition 4**

Let $b_i, i \in I$, be a family of elements of $\mathbb{R}^n$. The symbol $\langle b_i : i \in I \rangle$ will denote the subspace of $\mathbb{R}^n$ spanned by $\{b_i : i \in I\}$. Coherently with this definition, if $I = \emptyset$, it is $\langle b_i : i \in I \rangle = \{0\}$.

**Theorem 1**

Let $T$ and $H$ be real $n \times n$ and $n \times m$ matrices respectively. The following statements are equivalent.

(a) For every $\pi > 0$ and every $s \in \mathbb{R}^m$, the system

$$\begin{cases}
v_0 - Tv_\infty = Hs \\
(v_01, v_\infty1, \ldots, v_n, v_\infty n)^T \in \Phi_\pi
\end{cases}$$

has a unique solution

(b) $-T$ is a non-singular $P_0$ matrix and, defined $t_1, \ldots, t_n$ the columns of $T$, $h_1, \ldots, h_m$ the columns of $H$ and $N = \{1, \ldots, n\}$, it is:

$$h_1, \ldots, h_m \in \langle t_i : i \in N \setminus \bigcup (-T)^{-1}(I) \in \mathcal{K}((-T)^{-1}) \rangle$$

(The proof is given in Appendix II and is based mainly on a result on non-singular $P_0$ matrices described and proved in Appendix I)

The above results prove the following theorem.

**Theorem 2**

The following statements are equivalent.

(a) For every $\pi > 0$ and every $s \in \mathbb{R}^m$, the network $\mathcal{N}(\pi, s)$ has a unique solution

(b) $\mathcal{R}$ has neither a loop of independent voltage sources nor a cut-set of independent current sources, and $T$ and $H$ verify condition (b) of Theorem 1.

3. NETWORK-THEORETIC MEANING OF THE ALGEBRAIC CONDITION

In this section we give a network-theoretic meaning to the algebraic notions involved in condition (b) of Theorem 1, and give a consequent new version of Theorem 2.

Let $\mathcal{N}$ be the network of Section 2.

**Definition 5**

For every family $g = (g_1, \ldots, g_n)$ of positive reals (in what follows this statement will be referred to by: for every $g > 0$), let $\mathcal{N}(g)$ be the network obtained from $\mathcal{N}$ when each $O_k$, with reference to Figure 3, has constitutive equations:

$$\begin{cases}
i_{0k} = 0 \\
v_{\infty k} = g_kv_{0k}
\end{cases}$$
Remark 3
Let $R$ have neither a loop of independent voltage sources nor a cut-set of independent current sources.

$-T$ is a $P_0$ matrix if and only if for every $g > 0$ the network $\mathcal{N}(g)$ is uniquely solvable.\footnote{For a linear DC network, the existence of a unique solution is a property independent of the values of the independent sources.}

(For the proof, see Appendix III.)

Definition 6
Let $\widetilde{\mathcal{N}}$ be the network obtained from $\mathcal{N}$ by substituting each $O_k$ with the series connection of a nullator and an independent voltage source $V_k$ in place of its input port, and a norator in place of its output port (see Figure 5).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure5.png}
\caption{Structure of the networks $\mathcal{N}(g)$ and $\widetilde{\mathcal{N}}$.}
\end{figure}

Remark 4
Let $R$ have neither a loop of independent voltage sources nor a cut-set of independent current sources.

$-T$ is non-singular if and only if $\widetilde{\mathcal{N}}$ is uniquely solvable.

(For the proof, see Appendix III.)

Definition 7
For every non-empty subset $I$ of $\mathcal{N}$, let $\mathcal{O}(I)$ be the set of op amps of $\mathcal{N}$ defined by $\mathcal{O}(I) = \{O_k : k \in I\}$.

Definition 8
Let $\mathcal{O}(I)$ be a non-empty subset of op amps of $\mathcal{N}$, and let $\mathcal{N}_I$ be the network obtained from $\mathcal{N}$ by substituting each $O_k, k \in I$, with an independent current source of value 0 in place of its input port, and an independent voltage source $W_k$ in place of its output port, and each $O_k, k \notin I$, with a nullator in place of its input port, and a norator in place of its output port (see Figure 6).
The subset $O(I)$ will be called *non-singular* (respectively: *singular*) if $N_I$ is uniquely solvable (respectively: not uniquely solvable).

**Figure 6**: Structure of the network $N_I$.

**Remark 5**

Let $R$ have neither a loop of independent voltage sources nor a cut-set of independent current sources, and $-T$ be non-singular.

For every non-empty subset $I$ of $N$, $(-T)^{-1}(I)$ is singular if and only if $O(I)$ is singular. (For the proof, see Appendix III.)

**Definition 9**

Let $\tilde{N}$ be uniquely solvable, so that each branch voltage and current is a linear function of the values of $S_1, \ldots, S_m, V_1, \ldots, V_n$.

In this case, a non-empty subset $O(J)$ of $O(I)$ will be called an *isolated* subset of $O(I)$ if the voltages $v_{\infty,j}$, with $j \in J$, are independent of the values of $V_i$, with $i \in I \setminus J$.

The isolated subsets $O(J)$ of $O(I)$ are partially ordered by the relation of inclusion. The minimal $O(J)$ will be called *minimal isolated* subsets of $O(I)$.

**Remark 6**

Let $R$ have neither a loop of independent voltage sources nor a cut-set of independent current sources, and $-T$ be non-singular—in this case $\tilde{N}$ is uniquely solvable.

Let $J \subseteq I$ be non-empty subsets of $N$. Then, $(-T)^{-1}(J)$ is a $(-T)^{-1}(I)$-isolated (respectively: minimal $(-T)^{-1}(I)$-isolated) principal submatrix of $(-T)^{-1}(I)$ if and only if $O(J)$ is an isolated (respectively: minimal isolated) subset of $O(I)$.

(For the proof, see Appendix III.)

**Definition 10**

Let $\tilde{N}$ be uniquely solvable.

A subset $O(I)$ of op amps will be called a subset of *first kind* if $O(I)$ is singular and $O(I)$ is a (and hence the unique) minimal isolated subset of $O(I)$.

A subset $O(I)$ will be called a subset of *second kind* if $O(I)$ is singular and there exists a unique minimal isolated subset $O(J)$ of $O(I)$, and $O(J)$ is non-singular.
The set of all $O(I)$ of first or second kind will be denoted by $K(N)$. For every $O(I) \in K(N)$, let $\tilde{\mu}(I)$ be the subset of $N$ such that $O(\tilde{\mu}(I))$ is the unique minimal isolated subset of $O(I)$.

**Remark 7**

Let $R$ have neither a loop of independent voltage sources nor a cut-set of independent current sources, and $-T$ be non-singular—in this case $\tilde{N}$ is uniquely solvable.

Let $I$ be a non-empty subset of $N$. Then, $(\tilde{T})^{-1}(I) \in K((-T)^{-1})$ if and only if $O(I) \in K(N)$. In this case, $\mu(I) = \tilde{\mu}(I)$.

**Remark 8**

Let $R$ have neither a loop of independent voltage sources nor a cut-set of independent current sources, and $-T$ be non-singular.

Let $I$ be a possibly empty subset of $N$; then $h_1, \ldots, h_m \in \{t_i : i \in N \setminus I\}$ if and only if for every $k \in I$ the voltage $v_{\infty k}$ of $\tilde{N}$ is independent of the values of $S_1, \ldots, S_m$.

(For the proof, see Appendix III.)

These definitions and remarks allow us to state Theorem 2 in the following version involving only network-theoretic notions.

**Theorem 3**

The following statements are equivalent.

(a) For every $\pi > 0$ and every $s \in R^m$, the network $N(\pi, s)$ has a unique solution

(b) $R$ has neither a loop of independent voltage sources nor a cut-set of independent current sources, and for every $g > 0$ the network $N(g)$ is uniquely solvable, $\tilde{N}$ is uniquely solvable, and for every $k \in \bigcup_{O(I) \in K(N)} \tilde{\mu}(I)$ the voltage $v_{\infty k}$ of $\tilde{N}$ is independent of the values of $S_1, \ldots, S_m$.

4. CONCLUSIONS AND EXAMPLES

In Section 2 we have considered a network $N$ obtained by connecting independent voltage and current sources, linear resistors of given positive values and nonlinear op amps and the networks $N(\pi, s)$ obtained from $N$ by adopting op amps with ideal characteristic. We have proved that $N(\pi, s)$ has a unique solution for every $\pi > 0$ and every $s \in R^m$ if and only if $R$ has neither a loop of independent voltage sources nor a cut-set of independent current sources, $-T$ is a non-singular $P_0$ matrix and each column of $H$ is linearly dependent on a suitable set of columns of $T$.

**Examples**

As an application of Theorem 2, let $N'$ and $N''$ be the networks shown in Figure 7, in which the six resistors have given positive values $R_1, \ldots, R_6$.

The corresponding networks $R'$ and $R''$ have neither a loop of independent voltage sources nor a cut-set of independent current sources.

Both $N'$ and $N''$ have the same matrix

$$-T = \begin{pmatrix} \beta_{11} & 0 & \beta_{13} \\ 1 & 0 & -1 \\ -1 & \beta_{32} & \beta_{33} \end{pmatrix}$$
Figure 7: Networks $N'$ and $N''$ of the examples.

where: $\beta_{11} = \frac{R_1R_3}{(R_1R_3 + R_2R_3 + R_1R_2)}$, $\beta_{13} = \frac{R_1R_2}{(R_1R_2 + R_1R_3 + R_2R_3)}$, $\beta_{32} = \frac{R_4R_6}{(R_4R_6 + R_5R_6 + R_4R_5)}$, $\beta_{33} = \frac{R_4R_5}{(R_4R_5 + R_4R_6 + R_5R_6)}$. Observe that every $\beta_{ij} \in (0, 1)$.

We have that $-T$ is a non-singular $P_0$ (not $P$) matrix and, since

$$(-T)^{-1} = \frac{1}{\beta_{32}(\beta_{11} + \beta_{13})} \begin{pmatrix}
\beta_{32} & \beta_{32}\beta_{13} & 0 \\
1 - \beta_{33} & \beta_{11}\beta_{33} + \beta_{13} & \beta_{11} + \beta_{13} \\
\beta_{32} & -\beta_{11}\beta_{32} & 0
\end{pmatrix}$$

we have

$$\mathcal{K}((-T)^{-1}) = \{(-T)^{-1}(3), (-T)^{-1}(1, 3)\}$$

and

$$\mu(3) = \{3\}, \mu(1, 3) = \{1\}$$

Hence

$$\langle t_i : i \in N \setminus \bigcup_{(-T)^{-1}(I) \in \mathcal{K}((-T)^{-1})} \mu(I) \rangle = \langle t_i : i \in \{1, 2, 3\} \setminus \{1, 3\} \rangle = \langle t_2 \rangle = \langle (0, 0, \beta_{32})^T \rangle$$

The corresponding matrices $H'$ and $H''$ are

$$H' = \begin{pmatrix}
0 & 0 & -\beta_{13} & -\beta \\
0 & 0 & 1 & 0 \\
-\beta_{33} & -\beta_{32} & -\beta_{33} & 0
\end{pmatrix}, \quad H'' = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}$$

where: $\beta = \frac{R_2R_3}{(R_2R_3 + R_1R_3 + R_1R_2)} \in (0, 1)$.

Since $h'_3, h'_4 \notin \langle t_2 \rangle$, by Theorem 2 there exist $\pi > 0$ and $s \in \mathbb{R}^3$ such that $N'(\pi, s)$ does not have a unique solution.

Since $h''_1, h''_2 \in \langle t_2 \rangle$, by Theorem 2 for every $\pi > 0$ and every $s \in \mathbb{R}^2$ the network $N''(\pi, s)$ has a unique solution.
APPENDIX I. A RESULT ON NON-SINGULAR $P_0$ MATRICES

Apart from Definitions 1–4 and the use of the symbols $\boldsymbol{\pi} > 0$, $\phi_{\pi_k}$ and $\Phi_{\boldsymbol{\pi}}$ introduced in Section 2, this appendix is self-contained.

Let $\mathbf{A} = (\mathbf{a}_1, \ldots, \mathbf{a}_n)$ be a real $n \times n$ non-singular $P_0$ matrix and $\mathcal{N} = \{1, \ldots, n\}$.

We have the following theorem.

**Theorem a.1**

Let $\mathcal{H}$ be the set of the $\mathbf{h} \in \mathbb{R}^n$ such that for every $\boldsymbol{\pi} > 0$ the system
\[
\begin{cases}
\mathbf{x} + \mathbf{A}\mathbf{y} = \mathbf{h} \\
(x_1, y_1, \ldots, x_n, y_n)^T \in \Phi_{\boldsymbol{\pi}}
\end{cases}
\]
has a unique solution.

It is
\[
\mathcal{H} = \{ \mathbf{a}_i : i \in \mathcal{N} \setminus \bigcup_{\mathbf{A}^{-1}(I) \in \mathcal{K}(\mathbf{A}^{-1})} \mu(I) \}
\]

Before giving the proof, we need some preliminaries.

**Definition a.1**

For any positive real $\pi$ let $\gamma_{\pi0}, \gamma_{\pi\infty} : \mathbb{R} \rightarrow \mathbb{R}$ be the continuous piecewise-linear maps defined by

\[
\begin{align*}
\gamma_{\pi0}(\alpha) &= \begin{cases}
\alpha + \pi & \text{for } \alpha < -\pi \\
0 & \text{for } -\pi \leq \alpha \leq \pi \\
\alpha - \pi & \text{for } \alpha > \pi
\end{cases} \\
\gamma_{\pi\infty}(\alpha) &= \begin{cases}
-\pi & \text{for } \alpha < -\pi \\
\alpha & \text{for } -\pi \leq \alpha \leq \pi \\
\pi & \text{for } \alpha > \pi
\end{cases}
\end{align*}
\]

Let
\[
\gamma_{\pi}(\alpha) = (\gamma_{\pi0}(\alpha), \gamma_{\pi\infty}(\alpha))^T
\]

The map $\gamma_{\pi}$ is a homeomorphism of $\mathbb{R}$ onto $\phi_{\pi}$. Moreover, for every $\alpha \in \mathbb{R}$ it is $\gamma_{\pi0}(\alpha) + \gamma_{\pi\infty}(\alpha) = \alpha$.

For any $\pi > 0$ let
\[
\Gamma_{\pi0}(\alpha) = (\gamma_{\pi10}(\alpha_1), \ldots, \gamma_{\pi n0}(\alpha_n))^T, \quad \Gamma_{\pi\infty}(\alpha) = (\gamma_{\pi1\infty}(\alpha_1), \ldots, \gamma_{\pi n\infty}(\alpha_n))^T
\]

and
\[
\Gamma_{\pi}(\alpha) = (\gamma_{\pi10}(\alpha_1), \gamma_{\pi1\infty}(\alpha_1), \ldots, \gamma_{\pi n0}(\alpha_n), \gamma_{\pi n\infty}(\alpha_n))^T
\]

The map $\Gamma_{\pi}$ is a homeomorphism of $\mathbb{R}^n$ onto $\Phi_{\boldsymbol{\pi}}$. Moreover, for every $\alpha \in \mathbb{R}^n$ it is $\Gamma_{\pi0}(\alpha) + \Gamma_{\pi\infty}(\alpha) = \alpha$.

Let $F_{\pi} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be the continuous piecewise-linear map defined by
\[
F_{\pi}(\alpha) = \Gamma_{\pi0}(\alpha) + \mathbf{A}\Gamma_{\pi\infty}(\alpha)
\]

**Definition a.2**

Let $\mathbf{e}_1 = (1, 0, \ldots, 0)^T, \ldots, \mathbf{e}_n = (0, \ldots, 0, 1)^T$. 

For every $\pi > 0$, every partition $L, H, G$ of $N$ and every pair of maps $\sigma : L \to \{-1, 1\}$ and $\zeta : H \to \{-1, 1\}$, define

$$B_\pi(L, \sigma; H, \zeta; G) = \left\{ \sum_{l \in L} \sigma_l (\pi_l + p_l) e_l + \sum_{h \in H} \zeta_h \pi_h e_h + \sum_{g \in G} \alpha_g e_g : p_l > 0, -\pi_g < \alpha_g < \pi_g \text{ for } l \in L, g \in G \right\}$$

For every $\pi > 0$, every $G \subset N$ and every map $\sigma : N \setminus G \to \{-1, 1\}$, define

$$B_\pi^0(N \setminus G, \sigma) = B_\pi(N \setminus G, \sigma; \emptyset, \emptyset; G)$$

**Remark a.1**

Let $\pi > 0$. The set of all $B_\pi(L, \sigma; H, \zeta; G)$ is the partition of $\mathbb{R}^n$ corresponding to the partition

$$(-\infty, -\pi_k), \{-\pi_k\}, (-\pi_k, \pi_k), \{\pi_k\}, (\pi_k, +\infty)$$

of the $k$-th factor $\mathbb{R}$, $k = 1, \ldots, n$.

The sets of the form $B_\pi^0(N \setminus G, \sigma)$ are the elements of the above partition which are open subsets of $\mathbb{R}^n$.

**Remark a.2**

Let us consider the map

$$F_\pi : B_\pi(L, \sigma; H, \zeta; G) \to \mathbb{R}^n$$

For every

$$\alpha = \sum_{l \in L} \sigma_l (\pi_l + p_l) e_l + \sum_{h \in H} \zeta_h \pi_h e_h + \sum_{g \in G} \alpha_g e_g \in B_\pi(L, \sigma; H, \zeta; G)$$

it is

$$F_\pi(\alpha) = \sum_{l \in L} \sigma_l \pi_l a_l + \sum_{h \in H} \zeta_h \pi_h a_h + \sum_{l \in L} \sigma_l p_l e_l + \sum_{g \in G} \alpha_g a_g$$

**Remark a.3**

The following statements are equivalent.

(a) $F_\pi : B_\pi(L, \sigma; H, \zeta; G) \to \mathbb{R}^n$ is injective

(b) $e_l, a_g$ with $l \in L, g \in G$ are linearly independent

In particular

(i) injectivity or less of $F_\pi$ on $B_\pi(L, \sigma; H, \zeta; G)$ does not depend on $\pi, \sigma, \zeta$

(ii) when $F_\pi : B_\pi(L, \sigma; H, \zeta; G) \to \mathbb{R}^n$ is not injective, for every $\alpha \in B_\pi(L, \sigma; H, \zeta; G)$ there are infinitely many elements $\alpha' \in B_\pi(L, \sigma; H, \zeta; G)$ such that $F_\pi(\alpha') = F_\pi(\alpha)$. 
Remark a.4
Since $\Gamma$ is not empty and connected. In particular $F^{-1}\pi(h)$ has either one or infinitely many elements.

Proof
Let $h \in \mathbb{R}^n$. By Definition a.1, for every $\alpha \in \mathbb{R}^n$ it is $\Gamma \eta_0(\alpha) + \Gamma \eta_{\infty}(\alpha) = \alpha$, hence

$$F_{\pi}(\alpha) = \alpha + (A - I) \Gamma \eta_{\infty}(\alpha)$$

Since $\Gamma \eta_{\infty}$ is bounded, by Lemma 2 of [12] the equation $F_{\pi}(\alpha) = h$ has at least one solution. Then $F_{\pi}^{-1}(h) \neq \emptyset$.

For every $\alpha \in F_{\pi}^{-1}(h)$ it is

$$||\alpha|| = ||\Gamma \eta_0(\alpha) + \Gamma \eta_{\infty}(\alpha)|| = ||(h - A \Gamma \eta_{\infty}(\alpha)) + \Gamma \eta_{\infty}(\alpha)||$$

$$= ||h - (A - I) \Gamma \eta_{\infty}(\alpha)|| \leq ||h|| + ||A - I|| ||\Gamma \eta_{\infty}(\alpha)||$$

Since $\Gamma \eta_{\infty}$ is bounded, then $F_{\pi}^{-1}(h)$ is bounded.

Moreover, $F_{\pi}$ is a $P_0$-function (see Section 4 of [13]). Indeed: let $x \neq y$ be elements of $\mathbb{R}^n$. A straightforward argument proves that there exist non negative reals $d_0, \ldots, d_n$ and $d_{\infty}, \ldots, d_{\infty}$, depending on $x$ and $y$, such that, defined $D_0 = \text{diag}(d_0, \ldots, d_n)$ and $D_{\infty} = \text{diag}(d_{\infty}, \ldots, d_{\infty})$, it is $\Gamma \eta_0(x) - \Gamma \eta_0(y) = D_0(x - y)$ and $\Gamma \eta_{\infty}(x) - \Gamma \eta_{\infty}(y) = D_{\infty}(x - y)$. Hence

$$F_{\pi}(x) - F_{\pi}(y) = (D_0 - AD_{\infty})(x - y)$$

Since $D_0 - AD_{\infty}$ is a $P_0$ matrix, there exists $i$ such that $(x - y)_i \neq 0$ and $(F_{\pi}(x) - F_{\pi}(y))_i(x - y)_i \geq 0$. By Section 4 and Remark 1 of Section 3 of [13], $F_{\pi}^{-1}(h)$ is connected.

Remark a.5
Let $h \in \mathbb{R}^n$. The following statements are equivalent:

(a) $F_{\pi}^{-1}(h)$ has infinitely many elements

(b) there exists an improper $B_{\pi}(L, \sigma; H, \zeta; G)$ such that $h \in F_{\pi}B_{\pi}(L, \sigma; H, \zeta; G)$

Remark a.6
For every improper $B_{\pi}(L, \sigma; H, \zeta; G)$ there exists an improper $B_{\pi}^0(N \setminus G', \sigma')$ such that

$$F_{\pi}B_{\pi}(L, \sigma; H, \zeta; G) \subset F_{\pi}B_{\pi}^0(N \setminus G', \sigma')$$

(By Remark a.2, it suffices to set $G' = G \cup H', \sigma' = \sigma$—it is $N \setminus G' = L$—and $\pi_{l} = \pi_{l}$ for $l \in L$, $\pi_{g} = \pi_{g}$ for $g \in G$, $\pi_{h} = \pi_{h} + 1$ for $h \in H$.)
**Definition a.4**

Let \( I, J \) be not empty subsets of \( \{1, \ldots, q\} \). The symbol \( I < J \) will mean that \( i < j \) for every \( i \in I \) and every \( j \in J \).

Let \( v \in \mathbb{R}^q \) and \( I = \{i_1, \ldots, i_r\} \subset \{1, \ldots, q\} \) with \( 1 \leq i_1 < \cdots < i_r \leq m \). The symbol \( v(I) \) will denote the vector \((v_{i_1}, \ldots, v_{i_r})^T\).

Let \( I, J \) be non-empty subsets of \( \{1, \ldots, m\} \). The symbol \( C(I, J) \) will denote the submatrix of \( C \) obtained by omitting the rows and columns with index outside \( I \) and \( J \) respectively.

**Lemma a.1**

Let \( C \) be a real \( q \times q \) matrix such that \( C \) is the unique minimal \( C \)-isolated principal submatrix of \( C \), and let \( w \in \mathbb{R}^q \) be a non-zero vector.

Then there exists \( \lambda \in \mathbb{R}^q \) such that for every \( j = 1, \ldots, q \) it is

\[
\lambda_j(w + C\lambda)_j < 0
\]

**Proof**

The following algorithm gives a \( \lambda \in \mathbb{R}^q \) which verifies the requirements of the lemma.

If every component \( w_j \) of \( w \) is not zero, any vector \( \lambda \) whose components are sufficiently small to ensure that \( \text{sign}(w + C\lambda)_j = \text{sign} w_j \) for every \( j \), and such that every \( w_j\lambda_j < 0 \), verifies the requirements of the lemma.

Otherwise, apart from a reordering of \( 1, \ldots, q \), we may assume \( w_j \neq 0 \) for \( j = 1, \ldots, r_1 < q \) and \( w_j = 0 \) for \( j \geq r_1 + 1 \).

Since \( C(r_1 + 1, \ldots, q) \) is a principal submatrix of \( C \) not \( C \)-isolated, there exists \( \rho_1 \in \{1, \ldots, q - r_1\} \) such that, apart from a reordering of \( r_1 + 1, \ldots, q \), all the rows of \( C(r_1 + 1, \ldots, r_1 + \rho_1; 1, \ldots, r_1) \) are not zero, and, if \( r_1 + \rho_1 < q \), it is \( C(r_1 + \rho_1 + 1, \ldots, q; 1, \ldots, r_1) = 0 \).

Let \( \epsilon_1 > 0 \) be such that for every \( \lambda \in \mathbb{R}^q \) with \( 0 < |\lambda_j| < \epsilon_1 \), \( j = 1, \ldots, q \), it is \( \text{sign}(w + C\lambda)_j = \text{sign} w_j \), \( j = 1, \ldots, r_1 \).

Let \( X = \{(\lambda_1, \ldots, \lambda_{r_1})^T \in \mathbb{R}^{r_1} : 0 < |\lambda_j| < \epsilon_1, w_j\lambda_j < 0, j = 1, \ldots, r_1\} \); since \( X \) is a not empty open subset of \( \mathbb{R}^{r_1} \), we may choose \( (\lambda_1, \ldots, \lambda_{r_1})^T \in X \) such that

\[
(C(r_1 + 1, \ldots, r_1 + \rho_1; 1, \ldots, r_1)(\lambda_1, \ldots, \lambda_{r_1})^T)_j \neq 0, \quad j = 1, \ldots, r_1
\]

Set \( k = 1 \) and apply the following procedure.

**Procedure**

Let \( r_k, \rho_k, \epsilon_k, \lambda_1, \ldots, \lambda_{r_k} \) be such that

(a) \( 1 \leq r_k < q \), \( 1 \leq \rho_k \leq q - r_k \)

(b) \( w_j = 0 \) for \( j \geq r_k + 1 \)

(c) all the rows of \( C(r_k + 1, \ldots, r_k + \rho_k; 1, \ldots, r_k) \) are not zero and, if \( r_k + \rho_k < q \), \( C(r_k + \rho_k + 1, \ldots, q; 1, \ldots, r_k) = 0 \)

(d) every entry of \( C(r_k + 1, \ldots, r_k + \rho_k; 1, \ldots, r_k)(\lambda_1, \ldots, \lambda_{r_k})^T \) is not zero

(e) for every \( \lambda_{r_k+1}, \ldots, \lambda_q \), which verify \( 0 < |\lambda_j| < \epsilon_k \) for \( j = r_k + 1, \ldots, q \), it is \( \lambda_j(w + C\lambda)_j < 0 \) for \( j = 1, \ldots, r_k \)
If \( \rho_k = q-r_k \), a straightforward argument allows us to choose \( \lambda_{r_k+1}, \ldots, \lambda_q \) which verify 
\[
0 < |\lambda_j| < \epsilon_k \quad \text{for } j = r_k + 1, \ldots, q \quad \text{and} \quad \lambda_j(\mathbf{C}(r_k+1, \ldots, q; 1, \ldots, r_k)(\lambda_1, \ldots, \lambda_{r_k})^T + \mathbf{C}(r_k+1, \ldots, q)(\lambda_{r_k+1}, \ldots, \lambda_q)^T)_j < 0 \quad \text{for } j = r_k + 1, \ldots, q.
\]
Hence \( \lambda_1, \ldots, \lambda_q \) verify \( \lambda_j(\mathbf{w} + \mathbf{C}\lambda)_j < 0 \) for \( j = 1, \ldots, q \).

Otherwise, let \( k+1 = r_k + \rho_k \).

Since \( \mathbf{C}(r_k+1, \ldots, q) \) is a principal submatrix of \( \mathbf{C} \) not \( \mathbf{C} \)-isolated, there exists \( \rho_{k+1} \in \{1, \ldots, q-r_k+1 \} \) such that, apart from a reordering of \( r_k+1, \ldots, q \), the all the rows of \( \mathbf{C}(r_k+1, \ldots, q; r_k+1+1, \ldots, r_{k+1}) \) are not zero, and, if \( r_k+1 + \rho_{k+1} < q \), it is \( \mathbf{C}(r_k+1 + \rho_{k+1}, \ldots, q; 1, \ldots, r_{k+1}) = \mathbf{0} \).

A straightforward argument allows us to choose \( \epsilon_{k+1} \) and \( \lambda_{r_k+1}, \ldots, \lambda_{r_k+1} \) which verify 
\[
0 < \epsilon_{k+1} < \epsilon_k, \quad 0 < |\lambda_j| < \epsilon_{k+1} \quad \text{for } j = r_k + 1, \ldots, r_{k+1} \quad \text{and} \quad \text{that for every } \lambda_{r_k+1}, \ldots, \lambda_q \, \text{which verify } 0 < |\lambda_j| < \epsilon_{k+1} \quad \text{for } j = r_k + 1, \ldots, q, \quad \text{it is}
\]
\[
\lambda_j(\mathbf{C}(r_k+1, \ldots, q; r_k+1, \ldots, r_{k+1})(\lambda_1, \ldots, \lambda_{r_k})^T + \mathbf{C}(r_k+1, \ldots, q)(\lambda_{r_k+1}, \ldots, \lambda_q)^T)_j < 0 \quad \text{for } j = r_k + 1, \ldots, r_{k+1}.
\]

Set \( k = k+1 \) and apply the procedure again.

\[ \square \]

Lemma a.2

Let \( \mathbf{C} \) be a real \( q \times q \) \( P_0 \) matrix, and let \( \mathbf{C}(J_1), \ldots, \mathbf{C}(J_h) \) be the minimal \( \mathbf{C} \)-isolated principal submatrices of \( \mathbf{C} \) (by Definition 1 there exists at least one such submatrix).

(a) The subsets \( J_1, \ldots, J_h \) of \( Q = \{1, \ldots, q\} \) are pairwise disjoint

(b) For every \( \mathbf{w} \in \mathbb{R}^q \), the following statements are equivalent.

\[ (i) \quad \text{For every } \mathbf{\lambda} = (\lambda_1, \ldots, \lambda_q)^T \in \mathbb{R}^q \text{ with each } \lambda_k \neq 0, \text{ there exists } j \text{ such that}
\]
\[
\lambda_j(\mathbf{w} + \mathbf{C}\mathbf{\lambda})_j \geq 0
\]

\[ (ii) \quad \mathbf{w} \in \cup_{r=1}^h (\mathbb{R}^q : i \in Q \setminus J_r)
\]

Proof

Let \( \mathbf{C}(I), \mathbf{C}(J) \) be \( \mathbf{C} \)-isolated principal submatrices of \( \mathbf{C} \); if \( I \cap J \neq \emptyset \) then \( \mathbf{C}(I \cap J) \) is a \( \mathbf{C} \)-isolated principal submatrix of \( \mathbf{C} \). This proves (a).

Let \( \mathbf{w} \in \mathbb{R}^q \).

If \( \mathbf{w} \) verifies (ii), then there exists \( J_r \) such that \( w_j = 0 \) for every \( j \in J_r \). Since \( \mathbf{C}(J_r) \) is a \( P_0 \) matrix, for every \( \mathbf{\lambda} \in \mathbb{R}^q \) there exists \( j \in J_r \) such that
\[
\lambda_j(\mathbf{C}(J_r)\mathbf{\lambda}(J_r))_j \geq 0
\]
and then such that
\[
\lambda_j(\mathbf{w} + \mathbf{C}\mathbf{\lambda})_j \geq 0
\]
Hence \( \mathbf{w} \) verifies (i).

Let \( \mathbf{w} \) not verify (ii). Hence \( \mathbf{w}(J_r) \neq 0, r = 1, \ldots, h \). Apart from a reordering of \( \{1, \ldots, q\} \) we may assume \( J_1 < \cdots < J_h \) and \( J_1 \cup \cdots \cup J_h = \{1, \ldots, r\} \). By Lemma a.1 there exist reals \( \lambda_1, \ldots, \lambda_r \) such that
\[
\lambda_j((w_1, \ldots, w_r)^T + \mathbf{C}(1, \ldots, r)(\lambda_1, \ldots, \lambda_r)^T)_j < 0, \quad j = 1, \ldots, r
\]
If \( r = q \) we have proved that \( \mathbf{w} \) doesn’t verify (i).
Otherwise, since \( C(r+1, \ldots, q) \) is a principal submatrix of \( C \) not \( C \)-isolated, there exists \( \rho \in \{1, \ldots, q-r\} \) such that, apart from a reordering of \( r+1, \ldots, q, w_j \neq 0 \) or \( C(j; 1, \ldots, r) \neq 0 \) for \( j = r+1, \ldots, r+\rho \), and that, if \( r+\rho < q \), \( w_j = 0 \) and \( C(j; 1, \ldots, r) = 0 \) for \( j > r + \rho \).

Apart from a slight change of \( \lambda_1, \ldots, \lambda_r \), we may assume also that \( w_j + C(j; 1, \ldots, r)(\lambda_1, \ldots, \lambda_r)^T \neq 0 \) for \( j = r+1, \ldots, r + \rho \).

If \( r + \rho = q \), a straightforward argument allows us to choose \( \lambda_{r+1}, \ldots, \lambda_q \) such that

\[
\lambda_j((w_{r+1}, \ldots, w_q)^T + C(r + 1, \ldots, q)(\lambda_1, \ldots, \lambda_r)^T + C(r + 1, \ldots, q)(\lambda_{r+1}, \ldots, \lambda_q)^T)_j < 0, \quad j = r + 1, \ldots, q
\]

Hence there exists \( \lambda \in \mathbb{R}^q \) such that \( \lambda_j(w + C\lambda)_j < 0 \) for every \( j \) and then \( w \) doesn’t verify (i).

Otherwise, let \( r_1 = r + \rho < q \). Since \( C(r_1 + 1, \ldots, q) \) is a principal submatrix of \( C \) not \( C \)-isolated, there exists \( r_1 \in \{1, \ldots, q-r_1\} \) such that, apart from a reordering of \( r_1 + 1, \ldots, q \), all the rows of \( C(r_1+1, \ldots, r_1+q; 1, \ldots, r_1) \) are not zero, and, if \( r_1 + r_1 < q \), it is \( C(r_1 + r_1 + 1, \ldots, q; 1, \ldots, r_1) = 0 \).

Let \( \epsilon_1 > 0 \) be such that for every \( \lambda_{r_1+1}, \ldots, \lambda_q \) with \( 0 < |\lambda_j| < \epsilon_1 \), \( j = r + 1, \ldots, q \), it is

\[
\lambda_j((w_{r+1}, \ldots, w_{r+\rho})^T + C(r + 1, \ldots, r + \rho; 1, \ldots, r)(\lambda_1, \ldots, \lambda_r)^T + C(r + 1, \ldots, r + \rho; r + 1, \ldots, q)(\lambda_{r+1}, \ldots, \lambda_m)^T)_j < 0, \quad j = r + 1, \ldots, r + \rho
\]

A straightforward argument proves that we may choose \( \lambda_{r_1+1}, \ldots, \lambda_r \) with \( 0 < |\lambda_j| < \epsilon_1 \), \( j = r + 1, \ldots, r_1 \), such that every entry of \( C(r_1 + 1, \ldots, r_1 + r_1; 1, \ldots, r_1)(\lambda_1, \ldots, \lambda_{r_1})^T \) is not zero.

If we set \( k = 1 \) and apply the Procedure described in the Proof of Lemma a.1 we find a \( \lambda \) such that \( \lambda_j(w + C\lambda)_j < 0 \) for \( j = 1, \ldots, q \). Hence \( w \) doesn’t verify (i).

\[\square\]

Lemma a.3

Let \( C \) be a singular \( q \times q \) matrix.

There exists a minimal \( C \)-isolated principal submatrix \( N \) of \( C \) and a principal submatrix \( M \) of \( C \) of first or second kind whose only minimal \( M \)-isolated principal submatrix is \( N \).

Proof

The following algorithm allows us to find submatrices \( N \) and \( M \) of \( C \) which verify the requirements of the lemma.

Let \( C(J_1^{(1)}, \ldots, J_1^{(t)}) \) be the minimal \( C \)-isolated principal submatrices of \( C \).

Apart from a reordering of \( 1, \ldots, q \) we may assume \( J_1^{(1)} < \cdots < J_1^{(t)} \) and \( J_1^{(1)} \cup \cdots \cup J_1^{(t)} = \{1, \ldots, q_1\} \).

If there exists \( k \) such that \( C(J_k^{(1)}) \) is singular, then \( M = N = C(J_k^{(1)}) \) verifies the requirements of the lemma.

Otherwise, set \( r = 1 \) and apply Procedure 1.

Procedure 1

Let \( J_1^{(1)}, \ldots, J_r^{(r)} \) be given subsets of \( \{1, \ldots, q\} \) such that \( J_1^{(1)} < \cdots < J_r^{(r)} \) and \( J_1^{(1)} \cup \cdots \cup J_r^{(r)} = \{1, \ldots, q_r\} \), with \( q_r < q \).

Let \( C(J_1^{(r+1)}, \ldots, J_{t_r}^{(r+1)}) \) be the minimal \( C(q_r + 1, \ldots, q) \)-isolated principal submatrices of \( C(q_r + 1, \ldots, q) \).
Apart from a reordering of \( q_{r+1}, \ldots, q \) we may assume \( J^{(r+1)}_1 < \cdots < J^{(r+1)}_{t_r+1} \) and \( J^{(r+1)}_1 \cup \cdots \cup J^{(r+1)}_{t_r+1} = \{ q_r + 1, \ldots, q_r+1 \} \).

If there exists \( k \) such that \( C(J^{(r+1)}_k) \) is singular, then apply Procedure 2. Otherwise, set \( r = r + 1 \) and apply Procedure 1 again.

Procedure 2

Let \( J^{(1)}_1, \ldots, J^{(1)}_{t_1}, \ldots, J^{(r)}_1, \ldots, J^{(r)}_{t_r}, J^{(r+1)}_1 \) be the subsets of \( \{1, \ldots, q\} \) and \( q_1, \ldots, q_r \) be the integers obtained by the various applications of Procedure 1.

Let \( (\lambda_1, \tau_1) \) be the least pair of integers (with respect to the lexicographic order) such that \( C(J^{(r+1)}_{\tau_1}), J^{(\lambda_1)}_1 \neq 0 \); since \( C(J^{(r+1)}_k) \) is \( C(q_r+1, \ldots, q) \)-isolated in \( C(q_r+1, \ldots, q) \) but not \( C \)-isolated in \( C \), such least element exists and \( \lambda_1 < r + 1 \).

If \( \lambda_1 = 1 \) then \( N = C(J^{(1)}_{\tau_1}) \) and \( M = C(J^{(1)}_{\tau_1} \cup J^{(r+1)}_1) \) verify the requirements of the lemma.

Otherwise, set \( s = 1 \) and apply Procedure 3.

Procedure 3

Let \( (\lambda_1, \tau_1), \ldots, (\lambda_s, \tau_s) \) be given pairs of integers such that (i) \( 1 < \lambda_s < \cdots < \lambda_1 < r + 1 \), and (ii) \( 1 \leq \tau_j \leq \tau_j \) for every \( j = 1, \ldots, s \).

Let \( (\lambda_{s+1}, \tau_{s+1}) \) be the least pair of integers such that \( C(J^{(\lambda_s)}_{\tau_s}), J^{(\lambda_{s+1})}_{\tau_{s+1}} \neq 0 \); since \( C(J^{(\lambda_s)}_{\tau_s}) \) is \( C(q_{\lambda_s}+1, \ldots, q) \)-isolated in \( C(q_{\lambda_s}+1, \ldots, q) \) but not \( C \)-isolated in \( C \), such least element exists and \( \lambda_{s+1} < \lambda_s \).

If \( \lambda_{s+1} = 1 \) then \( N = C(J^{(1)}_{\tau_{s+1}}) \) and \( M = C(J^{(1)}_{\tau_{s+1}} \cup J^{(\lambda_s)}_{\tau_s} \cup \cdots \cup J^{(\lambda_1)}_{\tau_1}) \) verify the requirements of the lemma.

Otherwise, set \( s = s + 1 \) and apply Procedure 3 again.

\[ \square \]

Proof of Theorem a.1

By Definition a.1 it is

\[ \mathcal{H} = \{ h : \text{for every } \pi > 0, F_\pi(\alpha) = h \text{ has a unique solution} \} \]

By Remark a.5 it is

\[ \mathcal{H} = \mathbb{R}^n \setminus \left[ \bigcup_{B_\pi(L, \sigma; H, \zeta; G)} F_\pi B_\pi(L, \sigma; H, \zeta; G) \right] \]

By Remark a.6 it is

\[ \mathcal{H} = \mathbb{R}^n \setminus \left[ \bigcup_{B^0_\pi(N \setminus G, \sigma)} F_\pi B^0_\pi(N \setminus G, \sigma) \right] \]

For every non-empty subset \( G \) of \( N \) such that \( A(G) \) is singular and for every \( \sigma : N \setminus G \to \{-1, 1\} \), define

\[ B_\sigma(G) = \bigcup_{\pi > 0} F_\pi B^0_\pi(N \setminus G, \sigma) \]

by Remark a.2 it is\(^1\)

\[ B_\sigma(G) = \langle \sigma_i a_i, \sigma_i e_l : l \not\in G \rangle_+ + \langle a_g : g \in G \rangle \quad (3) \]

\(^{\ast}\)Let \( X_i, i \in I \) be a family of subsets of \( \mathbb{R}^n \). Remember that when \( I = \emptyset \) then \( \bigcup_{i \in I} X_i = \emptyset \) and \( \cap_{i \in I} X_i = \mathbb{R}^n \) (see [14] Par. 4).

\(^1\)Let \( b_1, \ldots, b_r \in \mathbb{R}^n \). The symbol \( \langle b_1, \ldots, b_r \rangle_+ \) will denote the set of all the \( a_1 b_1 + \cdots + a_r b_r \) with \( a_1, \ldots, a_r \in \mathbb{R}^+ \).
By Definition a.3 it is
\[
\mathcal{H} = \bigcap_{A(G) \text{ singular}} \left[ \mathbb{R}^n \setminus \bigcup_{\sigma} B_\sigma(G) \right]
\]

We have the following claim.

Claim 1
Let \( G \) be a non-empty subset of \( N \) such that \( A(G) \) is singular (since \( A \) is non-singular, it is \( N \setminus G \neq \emptyset \)) and let \( N \setminus G = \{l_1, \ldots, l_\mu\} \) with \( l_1 < \ldots < l_\mu \).

For every \( h \in \mathbb{R}^n \), defined \( w = A^{-1}h \), the following statements are equivalent.

(a) \( h \in \mathbb{R}^n \setminus \bigcup_{\sigma} B_\sigma(G) \)

(b) For every \( \lambda = (\lambda_1, \ldots, \lambda_\mu)^T \in \mathbb{R}^\mu \) with each \( \lambda_k \neq 0 \), there exists \( j \) such that
\[
\lambda_j \left( w(N \setminus G) + A^{-1}(N \setminus G)\lambda \right)_j \geq 0
\]

Indeed: for every \( \sigma : N \setminus G \to \{-1, 1\} \), by (3) it is \( h \in B_\sigma(G) \) if and only if there exist \( \tau_l > 0, \tilde{\tau}_l > 0 \) with \( l \in N \setminus G \) and \( \gamma_g \in \mathbb{R} \) with \( g \in G \) such that
\[
h = \sum_{l \in N \setminus G} \tilde{\tau}_l \sigma_l a_l + \sum_{l \in N \setminus G} \tau_l \sigma_l e_l + \sum_{g \in G} \gamma_g a_g
\]
or, equivalently, such that
\[
h - \sum_{l \in N \setminus G} \tau_l \sigma_l e_l = \sum_{l \in N \setminus G} \tilde{\tau}_l \sigma_l a_l + \sum_{g \in G} \gamma_g a_g
\]

For every \( v \in \mathbb{R}^n \), let \([v]_j, j = 1, \ldots, n\), be defined by \( v = \sum_{j=1}^n [v]_j a_j \); obviously \([v]_j = (A^{-1}v)_j\).

Hence \( h \in B_\sigma(G) \) if and only if there exists \( \tau_l > 0 \) with \( l \in N \setminus G \) such that for every \( k \in N \setminus G \) it is
\[
\sigma_k[h - \sum_{l \in N \setminus G} \tau_l \sigma_l e_l]_k > 0
\]
or, equivalently, such that
\[
\sigma_k(A^{-1}(h - \sum_{l \in N \setminus G} \tau_l \sigma_l e_l))_k > 0
\]

Hence \( h \in \bigcup_{\sigma} B_\sigma(G) \) if and only if there exist \( \tau_l \in \{-1, 1\}, \gamma_l > 0 \) with \( l \in N \setminus G \) such that for every \( k \in N \setminus G \) it is
\[
\sigma_k(A^{-1}(h - \sum_{l \in N \setminus G} \tau_l \sigma_l e_l))_k > 0
\]

As a consequence, \( h \in \mathbb{R}^n \setminus \bigcup_{\sigma} B_\sigma(G) \) if and only if for every \( \sigma_l \in \{-1, 1\}, \gamma_l > 0 \) with \( l \in N \setminus G \) there exists \( k \in N \setminus G \) such that
\[
\sigma_k(A^{-1}(h - \sum_{l \in N \setminus G} \tau_l \sigma_l e_l))_k \leq 0
\]
or, equivalently, such that
\[
-\tau_k \sigma_k (A^{-1}(h - \sum_{l \in N \setminus G} \tau_l e_l))_k \geq 0
\]

Hence, \( h \in \mathbb{R}^n \setminus \bigcup_{\sigma} B_{\sigma}(G) \) if and only if for every \( \tau_l \neq 0 \) with \( l \in N \setminus G \) there exists \( k \in N \setminus G \) such that
\[
\tau_k (A^{-1}(h + \sum_{l \in N \setminus G} \tau_l e_l))_k \geq 0
\]

Since for every \( j = 1, \ldots, \mu \) it is
\[
\tau_j (A^{-1}(h + \sum_{l \in N \setminus G} \tau_l e_l))_l = \tau_j ((w(N \setminus G) + A^{-1}(N \setminus G)(\tau_1, \ldots, \tau_\mu)^T)_l
\]

the claim is proved.

Lemma a.2 proves the following claim.

**Claim 2**

Let \( G \) be as in Claim 1. Let \( A^{-1}(N \setminus (G \cup J_1)), \ldots, A^{-1}(N \setminus (G \cup J_h)) \) be the minimal \( A^{-1}(N \setminus G) \)-isolated principal submatrices of \( A^{-1}(N \setminus G) \) (by Definition 1, there exists at least one such submatrix).

Then
\[
\mathbb{R}^n \setminus \bigcup_{\sigma} B_{\sigma}(G) = \bigcup_{r=1}^h \{ a_i : i \in G \cup J_r \}
\]

For every non-empty subset \( I \) of \( N \), let \( \mathcal{M}(I) \) denote the set of all minimal \( A^{-1}(I) \)-isolated principal submatrices of \( A^{-1}(I) \).

Claim 2 proves that
\[
\mathcal{H} = \bigcap_{A(G) \text{ singular}} \left[ \bigcup_{A^{-1}(J) \in \mathcal{M}(N \setminus G)} \{ a_i : i \in N \setminus J \} \right]
\]

Since \( A(G) \) is singular if and only if \( A^{-1}(N \setminus G) \) is singular, we have
\[
\mathcal{H} = \bigcap_{A^{-1}(I) \text{ singular}} \left[ \bigcup_{A^{-1}(J) \in \mathcal{M}(I)} \{ a_i : i \in N \setminus J \} \right]
\]

Let \( \Psi \) be the set of all maps \( \psi : \{ I \subset N : I \neq \emptyset \text{ and } A^{-1}(I) \text{ singular} \} \rightarrow \mathcal{P}(N) \) such that for every \( I \) it is \( A^{-1}(\psi(I)) \in \mathcal{M}(I) \).

For every \( \psi \in \Psi \) and every \( A^{-1}(I) \in \mathcal{K}(A^{-1}) \) it is \( \psi(I) = \mu(I) \).

Then we have
\[
\mathcal{H} = \bigcup_{\psi \in \Psi} \left[ \bigcap_{A^{-1}(I) \text{ singular}} \{ a_i : i \in N \setminus \psi(I) \} \right]
\]
\[
\bigcup_{\psi \in \Psi} \left[ \left( \bigcap_{A^{-1}(I) \in \mathcal{K}(A^{-1})} \{ a_i : i \in N \setminus \mu(I) \} \right) \cap \left( \bigcap_{A^{-1}(I) \text{ singular and } A^{-1}(I) \not\in \mathcal{K}(A^{-1})} \{ a_i : i \in N \setminus \psi(I) \} \right) \right]
\]

\( ^{\dagger} \mathcal{P}(N) \) is the set of all the subsets of \( N \).
Lemma a.3 proves that there exists a map \( \hat{\psi} \in \Psi \) such that for every singular \( A^{-1}(I) \notin \mathcal{K}(A^{-1}) \) there exists a principal submatrix \( A^{-1}(I') \) of \( A^{-1}(I) \) such that \( A^{-1}(I') \in \mathcal{K}(A^{-1}) \) and \( \hat{\psi}(I) = \mu(I') \). Hence

\[
\mathcal{H} = \bigcap_{A^{-1}(I) \in \mathcal{K}(A^{-1})} \{ a_i : i \in N \setminus \mu(I) \}
\]

Since \( a_1, \ldots, a_n \) are linearly independent, it is

\[
\mathcal{H} = \{ a_i : i \in N \setminus \mu(I) \} = \{ a_i : i \in N \setminus \bigcup_{A^{-1}(I) \in \mathcal{K}(A^{-1})} \mu(I) \}
\]

\[\square\]

**APPENDIX II. PROOFS OF THE STATEMENTS OF SECTION 2**

*Proof of (c) of Remark 2*

Let \( \pi > 0 \) and \( s \in \mathbb{R}^m \). By Definition a.1, the solution of system (1) is equivalent to the solution of

\[
\Gamma_{\pi_0}(\alpha) - T \Gamma_{\pi_\infty}(\alpha) = Hs
\]

and hence to the solution of

\[
\alpha - (T + I) \Gamma_{\pi_\infty}(\alpha) = Hs \tag{4}
\]

Since \( \Gamma_{\pi_\infty} \) is bounded, by Lemma 2 of [12] equation (4) has at least one solution. \[\square\]

*Proof of Theorem 1*

First of all let us observe that (a) implies \(-T\) non-singular. Indeed: if \(-T\) were singular, there would exist \( a \neq 0 \) such that \(-Ta = 0\); hence \((v'_0, v'_\infty) = (0, 0)\) and \((v''_0, v''_\infty) = (0, a)\) would be distinct solutions of

\[
\begin{aligned}
\begin{cases}
v_0 - Tv_\infty = Hs \\
v_01, v_\infty1, \ldots, v_0n, v_\infty n)^T \in \Phi_{\pi}
\end{cases}
\end{aligned}
\]

for \( \pi = (|a_1| + 1, \ldots, |a_n| + 1)^T \) and \( s = 0 \).

Moreover, the proof of Theorem 1 of [11] can now be used to prove that (a) implies also \(-T \in P_0\).

The remaining statements are a consequence of Theorem a.1. \[\square\]

**APPENDIX III. PROOFS OF THE STATEMENTS OF SECTION 3**

*Proof of Remark 3*

By Remark 2, for every \( s \in \mathbb{R}^m \) and every \( g > 0 \) there is a straightforward bijection of the solutions of \( \mathcal{N}(g) \) into the solutions of

\[
\begin{aligned}
\begin{cases}
v_0 = Tv_\infty + Hs \\
v_\infty = Gv_0
\end{cases}
\end{aligned}
\]

19
where \( G = \text{diag}(g_1, \ldots, g_n) \). For every \( s \in \mathbb{R}^m \) and every \( g > 0 \) this system is equivalent to the system

\[
\begin{align*}
\begin{cases}
(G - T)v_\infty &= Hs \\
v_0 &= v
\end{cases}
\end{align*}
\]

where \( T = G^{-1} \). Hence, for every \( g > 0 \) the network \( N(g) \) is uniquely solvable if and only if for every \( g > 0 \) it is \( \det(G - T) \neq 0 \). By [3], this last condition holds if and only if \( T \in P_0 \).

Proof of Remark 4

Let \( v = (v_1, \ldots, v_n) \in \mathbb{R}^n \) where \( v_k \) is the value of \( V_k, k = 1, \ldots, n, \)

By Remark 2, for every \( s \in \mathbb{R}^m \) and every \( v \in \mathbb{R}^n \), there is a straightforward bijection of the solutions of \( \tilde{N} \) into the solutions of

\[
\begin{align*}
\begin{cases}
v_0 &= T v_\infty + Hs \\
v_0 &= v
\end{cases}
\end{align*}
\]

and this system is uniquely solvable if and only if \( -T \) is non-singular. \( \square \)

Proof of Remark 5

By Definition 8, the statement is equivalent to: for every non-empty subset \( I \) of \( N, (-T)^{-1}(I) \) is singular if and only if \( N_I \) is not uniquely solvable.

By Remark 2, and by the non-singularity of \( -T \), for every \( s \in \mathbb{R}^m \) and every value \( w_k \) of \( W_k, k \in I \), there is a straightforward bijection of the solutions of \( N_I \) into the solutions of

\[
\begin{align*}
\begin{cases}
(-T)^{-1}v_0 + v_\infty &= (-T)^{-1}Hs \\
v_\infty &= w_k \text{ for } k \in I \\
v_0 &= 0 \text{ for } k \in N \setminus I
\end{cases}
\end{align*}
\]

and this system is uniquely solvable if and only if \( (-T)^{-1}(I) \) is non-singular. \( \square \)

Proof of Remark 6

By Definition 9, the statement is equivalent to: \( (-T)^{-1}(J) \) is a \( (-T)^{-1}(I) \)-isolated principal submatrix of \( (-T)^{-1}(I) \) if and only if for every \( j \in J, \) the voltage \( v_{\infty j} \) of \( \tilde{N} \) is independent of the values of \( V_i, \) with \( i \in I \setminus J. \)

Let \( v = (v_1, \ldots, v_n) \in \mathbb{R}^n \) where \( v_k \) is the value of \( V_k, k = 1, \ldots, n, \) By Remark 2, and by the non-singularity of \( -T, \) for every \( s \in \mathbb{R}^m \) and every \( v \in \mathbb{R}^n, \) there is a straightforward bijection of the solutions of \( \tilde{N} \) into the solutions of

\[
\begin{align*}
\begin{cases}
(-T)^{-1}v_0 + v_\infty &= (-T)^{-1}Hs \\
v_0 &= v
\end{cases}
\end{align*}
\]

then, for every \( s \in \mathbb{R}^m \) and every \( v \in \mathbb{R}^n, \) it is

\[
v_\infty = -(T)^{-1}v + (T)^{-1}Hs
\]

hence, for every \( i, j \in N, \) \( v_{\infty j} \) is independent of \( v_i \) if and only if \((T)^{-1}_{ji} = 0. \) This proves the statement. \( \square \)

Proof of Remark 8

Let \( L = T^{-1}H \) so that \( H = TL. \) Since \( h_1, \ldots, h_m \in (t_i : i \in N \setminus I) \) if and only if for every \( k \in I \) and \( j \in \{1, \ldots, m\} \) it is \( L_{kj} = 0, \) it is sufficient to prove that for every
k \in I, j \in \{1, \ldots, m\}, it is \( L_{kj} = 0 \) if and only if for every \( k \in I \) the voltage \( v_{\infty} \) of \( \tilde{N} \) is independent of the values of \( S_1, \ldots, S_m \).

Let \( v = (v_1, \ldots, v_n) \in \mathbb{R}^n \) where \( v_k \) is the value of \( V_k, k = 1, \ldots, n \). By Remark 2, and by the non-singularity of \( -T \), for every \( s \in \mathbb{R}^m \) and every \( v \in \mathbb{R}^n \), there is a straightforward bijection of the solutions of \( \tilde{N} \) into the solutions of

\[
\begin{cases}
( -T)^{-1}v_0 + v_{\infty} = ( -T)^{-1}Hs \\
v_0 = v
\end{cases}
\]

then, for every \( s \in \mathbb{R}^m \) and every \( v \in \mathbb{R}^n \), it is

\[ v_{\infty} = -(-T)^{-1}v - Ls \]

hence, for every \( k \in N, j \in \{1, \ldots, m\} \), \( v_{\infty} \) is independent of \( s_j \) if and only if \( L_{kj} = 0 \). This proves the statement.

\[ \square \]

REFERENCES


