Responding to traffic surges: Stochastic networks under
time-space-priority scalings

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January 25, 2011

Abstract

In multi-class communication networks, traffic surges due to one class of users can significantly degrade the performance of other classes. During these transient periods, it is thus of crucial importance to implement priority mechanisms that conserve the quality of service experienced by the affected classes, while ensuring that the temporarily unstable class is not entirely neglected. In this paper, we examine – for a suitably-scaled set of parameters – the complex interaction occurring between several classes of traffic when an unstable class is penalized proportionally to its level of congestion. We characterize the evolution of the performance measures of the network from the moment the initial surge takes place until the system reaches its equilibrium. Using a time-space-transition-scaling, we show that the trajectories of the temporarily unstable class can be described by a differential equation, while those of the stable classes retain their stochastic nature. In particular, we show that the temporarily unstable class evolves at a time-scale which is much slower than that of the stable classes. Although the time-scales decouple, the dynamics of the temporarily unstable and the stable classes continue to influence one another. We further proceed to characterize the obtained differential equations for several simple network examples. In particular, the macroscopic asymptotic behavior of the unstable class allows us to gain important qualitative insights on how the bandwidth allocation affects performance. We illustrate these result on several toy examples and we finally build a penalization rule using these results for a network integrating streaming and elastic traffic.

1 Introduction

The impact of large-scale traffic surges, also known as slash-dot-crowds or flash-crowds, on web servers and content distribution networks has been the subject of several studies \cite{30,17,10}. These mainly focus on designing mechanisms to make the content providers resilient to surges of a given type of traffic. However, in addition to overloading the content providers, a traffic
surge can also negatively impact the performance of other concurrent flows in the network. The temporarily unstable class can potentially starve the other classes from network capacity thereby subjecting them to unreasonable delays and packet losses. In such circumstances, in addition to protection mechanisms in web servers, it is crucial to implement bandwidth-sharing mechanisms inside the network that would protect the stable classes from the adversarial effects of the surge. It seems natural that such mechanisms should penalize the temporarily unstable class more when the level of congestion it creates is larger, without actually throttling it. (Thus, the more significant the surge is, the smaller the bandwidth each flow in this class gets.) The consequences of traffic surges on the performance of the different classes in the presence of such bandwidth sharing mechanisms have not been explored much.

In this paper, we take a more global view at the effects of a traffic surge in a multi-class communication network. Our aim is to present an analytic treatment of the complex interaction that takes place between the temporarily unstable class and the stable class during a traffic surge when the temporarily unstable class is penalized proportionally to its level of congestion. We consider bandwidth allocation networks describing the evolution of the number of flows (or calls) in a communication network where different classes of traffic compete for the bandwidth. Bandwidth-sharing network models \[22, 3, 13\] have become quite a standard modeling tool over the past decade for modeling communication networks. In particular, they have been used extensively to represent the flow level dynamics of data traffic in wireline or wireless networks \[2\], as well as for the integration of voice and data traffic \[3\], hence generalizing more traditional voice traffic models, e.g. \[18\].

Let \(N\) be the number of traffic classes that share a given network. Within each of the \(N\) traffic classes, resources are shared according to a processor-sharing service discipline. The service rates are state-dependent: they may depend on the number of flows within the same class, as well as on the numbers of flows in all other classes. The service rates of the \(N\) traffic classes will be denoted by \(\phi = (\phi_i(\cdot))_{i=1}^{N}\). Several examples are considered in the next section. Note that the service rate function \(\phi\) captures the allocation of bandwidth which is determined by the specific network topology and congestion control mechanisms. Special allocation functions that have received much attention in literature include the celebrated max-min fair allocation and the proportional fair allocation.

We assume that class-\(i\) customers arrive subject to a Poisson process of intensity \(\lambda_i\) and require exponentially distributed\(1\) service times of mean \(\mu_i^{-1}\) for class-\(i\). The arrival processes of all classes are mutually independent. Our main results allow for time-varying arrival rates for the class exhibiting a traffic surge. When applicable, we reflect this dependence in the notation by adding the time parameter to the arrival rates and then \(\lambda_i(t)\) is the arrival rate of class-\(1\) at time \(t\). For ease of exposition, however, we restrict ourselves to constant arrival rates for all classes in this section and will formulate our results with time-varying arrival rates in Section 4.

Let \(X\) be the stochastic process describing the number of flows (or calls) in progress. In the absence of priority mechanisms, and under the assumptions of Poisson arrivals and exponential flow sizes, \(X\) is a multi-dimensional birth and death process with transition rates:

\[
q(x, x - e_i) = \mu_i \phi_i(x),
\]

\[
q(x, x + e_i) = \lambda_i,
\]

with \(x \in \mathbb{N}^N\). Assume now that priority mechanisms are employed in the network such that the actual bandwidth allocation depends on the variables \(r_i x_i, i = 1 \ldots N\) rather than simply on \(x_i, i = 1 \ldots N\). Hence, if \(x_i\) is thought of as a measure of the level of congestion of class-\(i\), a differentiation between classes can be enforced by giving different weights to the different classes. (Such a differentiation can be enforced at lower time-scales by packet schedulers like weighted deficit round robin.)

It can also be the case that each class of traffic has a limited peak rate (because of access constraints for instance). It could then be advantageous for providers, in order to meet the demand, to share capacity as a function of the demanded rates \(r_i x_i\) rather than as a function of the number of flows of each class in the network. In both configurations, \(X\) can now be

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1Such assumptions are certainly not necessary to obtain the results we are aiming at; however, a rigorous generalization would be technically very involved and is beyond the scope of the present paper.
described as multi-dimensional birth and death process with transition rates:

\[ q(x, x - e_i) = \mu_i \phi_i(r.x), \]
\[ q(x, x + e_i) = \lambda_i, \]

where \( r.x = (r_i x_i)_{i=1..N} \) for some \( r \in \mathbb{R}_+^N \). To avoid confusion, we emphasize once more that reflecting the dependence on the control parameters \( r_i \) in our notation will be more convenient for the purposes in this paper, rather than making this dependence implicit, i.e., through the allocation \( \tilde{\phi}(x) = \phi(r.x) \).

We model a traffic surge by a large number of initial flows and we suppose that the temporarily unstable class, which we shall denote by class-1 is penalized proportionally to its level of congestion. To be more specific, in the sequel we study the case where:

1. the number of initial class-1 flows is of order \( K \) (to obtain limit theorems, we shall naturally make \( K \to \infty \)),
2. we scale (accelerate) time by a factor \( K \),
3. we scale class-1 states by a factor \( 1/K \),
4. the prioritization weight \( r_1 \) of class-1 is of order \( 1/K \).

Accelerating time together with re-scaling the first class (conditions 2 and 3) allow to “zoom out” the process, just as for usual fluid limits and obtain a bird’s-eye view of the large scale class-1 dynamics. Condition 4 expresses that the priority weight is very small compared to the offered traffic. This may reflect several situations. For example, the surge of traffic may have been caused by an inappropriately small level of priority. Alternatively, if the surge is externally caused, perhaps due to a network attack, the network may be reacting to it by penalizing class-1 according to its level of congestion, so that other classes do not starve, see for instance [25] for practical considerations on the matter.

The convergence of dynamics to a differential equation has some similarities with the classical fluid limits of the G/G/1 queue where time and space are jointly scaled. More generally in order to obtain a classical fluid limit for Jackson networks [21] or for more complex bandwidth-sharing networks [13], all the classes are jointly scaled in time and in space. This yields a set of differential equations that govern the dynamics of all the classes. Under additional assumptions on the drift \( \delta \) of the considered Markov process, the differential equation is simply of the form \( \dot{x}(t) = \delta(x(t)) \).

In our case, the situation is different as the transitions of class-1 are also scaled to model that the priority weight of class-1 is inversely proportional to the level congestion. This has far-reaching consequences for the structure of the limiting process. Under this scaling, we will show that the dynamics of the temporarily unstable class can be described by a deterministic differential equation, while the stable classes retain their stochastic nature. Hence, a time-scale separation occurs: the temporarily unstable class evolves on the much slower time-scale compared to the stable classes. However even with this separation of time-scales, a strong coupling in the dynamics of the temporarily unstable and the stable classes remains. The dynamics of the temporarily unstable class is influenced by the stable classes through their conditional stationary distribution (conditional on the level of congestion of class-1 flows being fixed to its present macroscopic value), which in turn depends on the temporarily unstable class. Hence, for class-1 the differential equation obtained is of the form \( \dot{x}_1(t) = \bar{\delta}_1(x_1(t)) \), where \( \bar{\delta}_1 \) is an average of the first coordinate drift according to the conditional distribution of the other classes, given \( x_1(t) \). This phenomenon is usually known in the probability literature as averaging principle.

**Contribution:**

Our contribution first consists in establishing the convergence in \( L^1 \) uniformly on compact sets for stochastic processes commonly adopted in the modeling and analysis of communications networks under the scaling considered. Since the slow part of the processes (class-1) remains coupled (at a macroscopic scale) to the fast part (the remaining classes), such a proof is not standard and has to be decomposed in several steps. While preliminary results for monotone
networks were presented in [16], a proof for general bandwidth sharing networks was still missing.

Second, we characterize the responses (evolutions of queue length) of different networks to the surge of traffic. In particular, the limiting macroscopic state of class-1 gives precious indications on the (macroscopic) stationary regime of the system. We first show that for work conserving allocations, the unstable class, at its macroscopic time scale, sees the other classes as having full priority, while the effect of the first class on the other classes gradually vanishes (again at a macroscopic time scale). Hence class-1 tends macroscopically to 0 under the stability condition of the system $p_1 < 1 - \sum_{j=2}^{N} \rho_j$. The situation is much more complex for non work-conserving networks, where the behavior of the unstable class depends in an intricate manner upon that of the other classes. In particular, under the usual stability conditions of the network, the macroscopic state of class-1 might converge to 0 or to a strictly positive number, depending on the conditional distribution of the other classes. We observe on simple examples that only when the allocation giving full priority to classes 2 to $N$ that is the allocation in a network where class-1 has 0 arrival rate, (which corresponds to the worst case for class-1) is stable, then class-1 will converge to 0 on the macroscopic time scale. We illustrate these concepts on several simple network topologies.

Finally, we use our analytical results to build an implementable penalization rule allowing to adapt the level of priority of streaming traffic in a network integrating streaming and elastic traffic, such as to target a given loss probability threshold/ quality of service.

The rest of the paper is organized as follows. Related work is presented in the next Section. In Section 3 we give several examples of network topologies. In Section 4 we present the convergence theorem for the considered scaling. In Section 5, we analyze the qualitative behavior of networks after a traffic surge in different cases. In Section 6, we give numerical examples of applications of the main result to bandwidth sharing on some simple network topologies and construct a practical penalization rule for streaming traffic. Finally, we conclude in Section 7.

2 Related work

The transient behavior of communications networks – though of crucial importance as underlined previously – has received relatively less focus for performance evaluations purposes compared to the extensive studies of their stationary regimes (see, for instance, [18, 15, 29, 2]). There has been however a considerable amount of work on fluid limits and ODE methods both for Markov processes and for communications networks [21, 26, 8, 7, 13, 23].

On the other hand, the time-scale separation that takes place between the temporarily unstable and the stable class is closer in spirit to the one observed in singularly perturbed or nearly decomposable Markov chains [31] though most of the studies in that context concern only stationary aspects of the chain. In the analysis of the fluid limit of bandwidth sharing networks a time-scale separation between classes usually occurs when one class of traffic reaches equilibrium faster than the others, and hence when the fluid limit hits an hyperplane of the state state space. Simple examples of this phenomenon can be found in [26]. A more complex example can be found in [12]. The interesting feature in our scaling is the appearance of the averaging principle in the whole state space. Similar averaging phenomena have also been studied in statistical physics [6] as well as chemistry and biochemistry [28] where the kinetics of chemical reactions can be described by systems of ordinary differential equations. Usually these works assume that one of the dependent variable is in steady state with respect to the instantaneous values of the other dependent variables. Taking this time-scale separation as an assumption, an efficient approximation method called the quasi-steady-state is commonly used in that context. This is however in contrast with our situation where we show that the time-scale decoupling occurs as a consequence of the scaling of the parameters of the transitions of the stochastic processes considered.
3 Examples of topologies and traffics

Notation
In the sequel, for $x \in \mathbb{Z}^N$, $|\cdot|$ denotes the $l_1$-norm:

$$ |x| = \sum_{i=1}^{N} |x_i|. $$

For $x, y \in \mathbb{Z}^N$, we also use the notation $x \leq y$ to denote the partial order $x_i \leq y_i$ for all $i = 1 \ldots N$.

3.1 Several elastic classes on one link

The simplest instance of a network consists of one link shared by several competing classes of traffic. If the initial policy is supposed to be the classical processor sharing policy: $\phi_i(x) = \frac{x_i}{|x|}$ then the prioritized version of the model becomes the so-called discriminatory processor sharing (DPS): $\phi_i(r.x) = \frac{r_i x_i}{\sum_j r_j x_j}$.

3.1.1 Bandwidth sharing networks

Bandwidth sharing networks constitute a natural extension of a multi-class processor sharing queue, and have become a standard stochastic model for the flow level dynamics of Internet congestion control (they were introduced by [22]).

Consider for example the tree network represented on the left of Figure 1 with two traffic routes, each passing through a dedicated link, followed by a common link. If each dedicated link has a capacity $c_i \leq 1, i = 1, 2$, and the common link has capacity 1, the flow on each route gets a capacity $\phi_i(x)$ that lies in the polyhedron $C$:

$$ \sum_{i=1}^{2} \phi_i(x) \leq 1, \quad \phi_i(x) \leq c_i, \quad i = 1, 2. $$

Another example of interest is the linear network represented on the right of Figure 2 with 3 routes sharing two links. While the first route passes through both links, routes 2 and 3 only use one of the links (one each). This gives the following capacity constraints:

$$ \phi_1(x) + \phi_2(x) \leq c_1, \quad \phi_1(x) + \phi_3(x) \leq c_2. $$

In general, like for the specific foregoing examples, the capacity constraints determine the space over which a network controller can choose a desired allocation function. It has been argued by [20] that a good approximation of current congestion control algorithms such as TCP (the Internet’s predominant protocol for controlling congestion) can be obtained by using the weighted proportional fair allocation, which solves an optimization problem for each vector $x$ of instantaneous numbers of flows. Specifically, the weighted proportional fair allocation $\eta(x)$ for state vector $x$ maximizes

$$ \sum_{i=1}^{N} w_i x_i \log(\eta_i), \eta \in C, $$

where the weights $w_i$ are class-dependent control parameters.
Remark 1. By definition of this optimization program, if $\phi(\cdot) = \eta(\cdot)$ is the standard (unweighted) proportional fair allocation with $w_i \equiv 1$, then the allocation $\phi^r(\cdot) = \phi(r \cdot)$ corresponds to the weighted proportional fair allocation with weights $w_i \equiv r_i$.

This framework has been generalized to so-called weighted $\alpha$-fair allocations, which provide flexibility to model different levels of fairness in the network. Another important alternative is the balanced fair allocation [2], which allows a closed form expression for the stationary distribution of the numbers of flows in progress. In addition, the balanced fair allocation gives a good approximation of the proportional fair allocation while being easily evaluated, which is attractive for performance evaluation.

Figure 2: Tree network and linear network

3.1.2 Integration of streaming and elastic traffic

Consider now a system where two intrinsically different types of traffic – “streaming” and “elastic” traffic – coexist and share a given link. Such models have been considered by [24, 9, 4]. It is natural to equip streaming traffic with a fixed required rate, say, $c$ per flow. Giving priority to streaming traffic (class 2) the allocation of service may be chosen as:

$$
\phi_1(x) = \max \left( \frac{r_1 x_1}{r_1 x_1 + c x_2}, 1 - c x_2 \right),
$$

$$
\phi_2(x) = c x_2,
$$

where the parameter $r_1$ quantifies the level of priority. The allocated capacity cannot exceed the total capacity. If the latter is normalized to 1, the state space must be restricted to states $x_2$ such that

$$
\phi_1(x) + \phi_2(x) \leq 1.
$$

Then, if the number of current streaming flows $x_2$ is such that $\phi_1(x + c_2) + \phi_2(x + c_2) > 1$, arriving streaming flows must be blocked from the network.

4 Main result

We now consider a network with several classes of traffic and with class-1 going through a temporary surge of traffic. Recall that we focus on a regime where $r_1 \equiv \frac{1}{K}$ and $K \to \infty$. We further let $Y^K$ denote the (scaled) process:

$$
Y^K(t) = \left( \frac{X^K(t)}{K}, (X_i^K(t))_{i=2...N} \right).
$$

In the following we show that, as $K \to \infty$, $Y^K$ converges to a stochastic process with a deterministic first coordinate, which is a solution of a differential equation which we describe in terms of an averaged rate $\bar{\phi}_1$. In the limit, the result implies a time-scale separation between the first class and all other classes.

Define $U^{z_1}$ to be an $N-1$ dimensional Markov birth-and-death process with arrival rates $\lambda_i$ and death rates $\phi_i(z_1, \cdot)$, $i = 2...N$ ($z_1 \in \mathbb{R}^+$ being a fixed number here) and denote by $\pi^{z_1}(\cdot)$ its stationary probability. When we do not use a time index, we implicitly suppose that we consider stationary versions of the processes.

Let $u_1(t)$ be the solution (assuming it is unique) of the differential equation:

$$
u_1(t) = \begin{cases} 
\dot{u}_1(t) - \bar{\phi}_1(u_1(t)), & \text{if } u_1(t) > 0, \\
0, & \text{if } u_1(t) = 0
\end{cases}
$$

with $\bar{\phi}_1(z_1) = \sum_y \phi_1(z_1, y) \pi^{z_1}(y)$. To establish our main result, we shall make the following assumptions:
(A1): $\phi_i(\cdot, x_2, \ldots, x_N)$ can be extended to Lipschitz-continuous functions from $\mathbb{R}^+ \setminus \{0\}$ to $\mathbb{R}^+$.

(A2): for all fixed $z_1$, the process $U^{z_1}$ is ergodic. We can thus define $E^{U^{z_1}}$ the mean under the stationary distribution of the process $U^{z_1}$.

(A3): $\frac{1}{T} \int_0^T \lambda^K_i(s) \, ds \to a_1(t)$.

We can now proceed to state our main result:

**Theorem 1.** Under the assumptions (A1), (A2) and (A3), the process $Y^K_1(t)$ converges in $L^1$, uniformly on compact intervals, to the deterministic trajectory $u_1(t)$, i.e.,

$$
\mathbb{E} \left[ \sup_{0 \leq s \leq t} \left| Y^K_1(s) - u_1(s) \right| \right] \to 0, \quad K \to \infty.
$$

Moreover, for all times $t$, and for all bounded continuous functions $f$:

$$
\lim_{K \to \infty} \mathbb{E} \left[ \sup_{0 \leq s \leq t} \left| \int_0^t f \left( Y^K_s(u) \right) - \mathbb{E}^{U^{Z_i^{(0)}}} \left( f \left( Z_1(s), U^{Z_i(s)}(s) \right) \right) \right| ds \right] = 0.
$$

**Remark 2.** The fact that there is a unique class entering a traffic surge plays no role in the proof of the Theorem: we could as well have supposed that classes $1 \geq k < N$ play jointly the role of class $1$.

The details of the proof are in the Appendix. We underline here the main steps:

- We first prove tightness of the laws of the scaled process, and show that the limit-points of $Y^K_1$ are continuous processes.
- Supposing the convergence in distribution of the first class we characterize the limit of the functional $\int_s^t \int_{\Gamma} \mathbb{1}_{\{(X^K_i(s), \ldots, X^K_N(s)) \in \Gamma\}} \, ds$, $\forall \Gamma \subset \mathbb{R}^{N-1}$ and prove the limits are unique (and deterministic given the value of the first class). A key step is the useful characterization of bimeasures.
- Finally, we show that $Y^K_1$ converges in distribution towards a deterministic process which allows to prove, using the previous step, the convergence in $L^1$, uniformly on compact sets.

## 5 Qualitative behaviors of the limiting processes

Assume that class $1$ has entered a traffic surge (see remark 2), and let $a(t)$ be its macroscopic arrival intensity. Under the scaling considered in Theorem 1, we observe three qualitative types of behaviors for the network responses, which are completely characterized using the stationary distributions of the family of processes $U^K_i$, $i = 2, \ldots, N$. Defining $\tilde{h}_1(x_i) = a_1(t) - \mu_1 \hat{\phi}_1(x_i)$, let $\mathcal{S}$ the set of positive solutions of the equation

$$
\tilde{h}(x) = 0.
$$

Given the classical results on asymptotic stability of non-linear autonomous systems, we can partially classify the possible situations using in particular the Hartman-Grobman theorem (see for instance [14]). For a $C^1$ flow $\delta$, we write $D\delta(x) < 0$ if the linearization of $\delta$ has only eigenvalues with strictly negative real parts and no eigenvalue on the unit complex circle.

Assume in the following $\delta$ is $C^1$. We have the possible behaviors:

1. The traffic surge will be absorbed after a finite (macroscopic) time, i.e., the differential equation is asymptotically stable with stable point $0$. A sufficient condition is that:
   - (a) $0 \in \mathcal{S}$
   - (b) $D\delta(0) < 0$.
   - (c) the initial condition is close enough to $0$, or $\mathcal{S} = \{0\}$.

In this case, note that the stationary measure of $(Y^K_i(t))_{i=2}^{\ldots,N}$ converges when $K \to \infty$ to the stationary measure of the original system with full priority given to classes $2, \ldots, N$ (i.e., the $N-1$ system with service rates $\phi_i(0, x_2, \ldots, x_N)$), which boils down to the fact that the limit in time and in $K$ commute for classes $2$ to $N$. 

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2. The network continues to see class-1 saturated (at a macroscopic state), even after any large amount of (macroscopic) time. A sufficient condition is that there exists $x \in \mathcal{S}$ such that $D\delta(x) < 0$ and $x > 0$, with initial conditions sufficiently close to $x$. Then the differential equation is asymptotically stable with stable point $x > 0$. In this case, limits in time and $K$ cannot commute since there is always a part of the bandwidth of the network used for class-1, while taking first the limit in time and then the limit in $K$ always converge to the system priority with full priority given to classes $2, \ldots, N$.

3. The differential equation (6) governing the macroscopic dynamics of class-1 is unstable, which means that the traffic surge cannot be absorbed and keeps building up. It might lead to the instability of other classes in the network.

We now look at more specific situations encountered in usual bandwidth allocations. We first show that for work-conserving allocations, the limit point of the differential equation is either zero or infinity depending on whether $\sum \rho_i < 1$ or not. On the other hand, the situation is much more complex for non-work-conserving allocations. Contrary to the previous situation, the presence of class-1 might not be asymptotically transparent to the other classes while class-1 might get asymptotically strictly more bandwidth than in the case of the allocation giving full priority to classes $2$ to $N$ (see for instance Figure 6 in the examples below). For monotonic networks, we however prove that if the allocation giving full priority to classes $2$ to $N$, i.e., the allocation defined by

$$
\phi_{PR}^1(x) = \phi_1(x_1, 0, \ldots, 0) \mathbb{1}_{x_2 = 0, \ldots, x_N = 0},
\phi_{PR}^i(x) = \phi_i(0, x_2, \ldots, x_N), \forall i \geq 2
$$

is stable, then the differential equation (6) has the stable point 0. This is consistent with the fact that the priority allocation constitutes a worst case scenario for class-1. Necessary and sufficient conditions of stability stay a challenging direction future research. We now give more details of these findings. For the ease of exposition, in this section, we suppose that the arrival rate of class-1 is fixed.

5.1 Work-conserving allocations

Consider a work conserving allocation such that

$$\forall x \neq 0, \sum_{i=1}^{N} \phi_i(x) = 1.$$

Every work-conserving allocation has the same stability region, namely $\sum_{i=1}^{N} \rho_i < 1$. If the stability condition is satisfied, then the priority mechanism considered is asymptotically equivalent to giving full priority to class-$i$, $i \geq 2$. In other words, the fluid limit obtained for class-1 is in that case the same as the fluid limit of an allocation that gives a full priority to class-$i$, $i \geq 2$, which we prove in the following Proposition.

**Proposition 1.** For a work conserving network,

$$Y_1^K(t) \xrightarrow{L^1} u_1(t) = \left(u_1(0) + a_1(t) - \mu_1 \left(1 - \sum_{i=2}^{N} \rho_i\right) t\right)^+.$$

**Proof.** Assume $\sum_{i=1}^{N} \rho_i < 1$ in which case the network is stable. Fix $z_1 \in \mathbb{R}$. Using the conservation of the rates at equilibrium for the process $U^{z_1}$ (which boils down in the Markovian context to saying that at equilibrium the drift of $y \to y_i$ should be 0), we can write that:

$$\sum_{i=2}^{N} \sum_{y} \phi_i(z_1, y) \pi^{z_1}(y) = \sum_{i=2}^{N} \rho_i.$$
We now calculate $\bar{\phi}_1$ for $z_1 > 0$:

$$\bar{\phi}_1(x) = \sum_y \phi_1(z_1, y)\pi^{z_1}(y),$$
$$\bar{\phi}_1(x) = \sum_y (1 - \sum_{j \geq 2} \phi_j(z_1, y))\pi^{z_1}(y),$$
$$\bar{\phi}_1(x_1) = 1 - \sum_{j \geq 2} \rho_j.$$

Hence, the capacity seen asymptotically by class-1 is $(1 - \sum_{j \geq 2} \rho_j)$, which concludes the proof.

As a first example, we consider a single link of capacity 1 shared by three classes. The bandwidth is allocated according to DPS with weight $r_i$ for class $i$, $i = 1, 2, 3$. Proposition 1 says that $u_1(t)$ is a straight line with slope $\lambda_1 - \mu_1(1 - (\rho_2 + \rho_3))$. This behavior is illustrated in Figure 3 for which $\lambda_1 = 0.5$, $\mu_1 = 1$, $\rho_2 = 0.3$, $\rho_3 = 0.1$. The slope calculated using the proposition is thus 0.1, which is verified in the figure.

In Figure 3, we plot the empirical mean of class-2 at a macroscopic scale, (i.e. $\frac{1}{s}\int_{t-s}^{t} f(Y^K(h)) \, dh$) for a temporal window of $s = 0.1$.

5.2 Non work-conserving allocations

For non work-conserving allocations the dynamics of $u_1(t)$ can be considerably more complex to describe as they depend on the allocation itself and the conditional stationary measure of the other classes. We can only partially characterize conditions for the convergence to 0 of $u_1(t)$.

Proposition 2. Consider a monotonic allocation (i.e. such that $\phi_i$ is decreasing in $x_j$, $j \neq i$). If the network is stable under an allocation giving full priority to class 2 to $N$, then the differential equation governing class 1 is asymptotically stable with stable point 0.

Proof. Denote $\bar{\phi}_1^{PR}$ the macroscopic service rate of class 1 under the priority allocation, $V = (X_2, \ldots, X_N)$ the Markov process describing the state of class 2 to $N$ under the priority allocation, and $v_1(t)$ the corresponding solution of the differential equation [9]. Now note that under the full priority policy, the dynamics of classes 2 to $N$ are completely independent of class 1. Also, the transitions of the Markov process do not depend on the parameter $r_1$ and hence the scaling considered for class-1 in Theorem 4 boils down to the usual fluid limit of a modulated process. Moreover, (since the transitions of $X_1$ do not depend on $r_1 = \frac{1}{K}$) the stochastic stability of the network implies that:

$$E \left[ \frac{X^K(Kt)}{K} \right] \rightarrow 0, \quad \text{when } K \rightarrow \infty,$$

which implies, using Theorem 4 for the priority allocation that there exists a finite time $T$ such that $v_1(T) = 0, \forall t \geq T$.

On the other hand, using stochastic comparisons (see [9] for more details on stochastic comparisons of multidimensional birth-and-death processes with monotonic allocations), we obtain that

$$U_i^{z_1} \geq_{st} V_i, \forall z_1,$$
which implies in turn that $\forall z_1$, $\tilde{\phi}_1(z_1) \geq \tilde{\phi}_1^{PR}(z_1)$, and $u_1(t) \leq v_1(t)$, $\forall t$, which concludes the proof.

For some special cases, the conditions of convergence of the differential equation (9) are exactly those described by the previous Proposition, the limit point being zero, strictly positive and finite, or infinite, depending on whether the parameters lie inside the stability region of the priority allocation, outside the stability region of the priority allocation but inside the stability region of the network, or outside the stability region of the network. We now illustrate this trichotomy on a few examples.

6 Specific networks

6.1 Tree network

Let us consider the tree network shown in Figure 2 with $c_1 = 0.4$ and $c_2 = 0.8$. We shall assume the following bandwidth allocation: Define $S_1 = \{(x_1, x_2) : (r_1 x_1 + r_2 x_2) c_1 < r_1 x_1\}$. For $x_1 > 0$ and $x_2 > 0$,

$$
\phi_1(x_1, x_2) = \begin{cases} 
  c_1, & \text{if } (x_1, x_2) \in S_1, \\
  \max\left(\frac{r_1 x_1}{r_1 x_1 + r_2 x_2}, 1 - c_2\right), & \text{if } (x_1, x_2) \in S_1^c.
\end{cases}
$$

(9)

and $\phi_2 = 1 - \phi_1$.

For this network, the allocation becomes a strict priority allocation for class-2 when $r_1 = 0$, in which case class-1 gets capacity $c_1$ if there are no class-2 flows, and $1 - c_2$ otherwise. Thus, for a fixed value of $\rho_2$, class-1 is stable if $\rho_1 < \left(1 - \frac{c_2}{c_1}\right) c_1 + \frac{c_2}{c_1} (1 - c_2)$. The stability regions for $r_1 = 0$ and $r_1 > 0$ are shown in Figure 4.

![Figure 4: Partitioning of the stability region for the tree network](image)

The dynamics of $u_1(t)$ for two different values of $\rho_1$ – one in each region – is plotted in Figure 5 for which $\rho_2 = 0.5$.

![Figure 5: Tree network: scaling of class-1 (left) and of class-2 (right) (image)](image)
For class 2, when the priority allocation is stable the dynamics of the average number of customers converges to the one of the priority allocation, that is $\rho_2/(c_2 - \rho_2)$, as is illustrated in Figure [1].

In Figure [6] we show how class-1 is actually favored by asymptotically using the bandwidth of class-2, compared to the case where class-2 is given a strict priority.

Figure 6: Tree network: comparisons of trajectories of class-1 for a proportional fair allocation and a priority (to class-2) allocation

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure6}
\caption{Tree network: comparisons of trajectories of class-1 for a proportional fair allocation and a priority (to class-2) allocation}
\end{figure}

6.2 Linear network

Next, we consider a linear network with two links of capacity $c_1 < c_2$, and three classes of flows as shown in Figure [2].

Let $\alpha_i$ be the capacity allocated to a flow of class-$i$. The capacity allocated to class-$i$, $\phi_i$, is then $x_i\alpha_i$. The bandwidth is allocated according to the weighted proportional fair allocation, that is, $(\alpha_1, \alpha_2, \alpha_3)$ is the solution of the following maximization problem:

\begin{align*}
\text{maximize} & \quad r_1 x_1 \log(\alpha_1) + r_2 x_2 \log(\alpha_2) + r_3 x_3 \log(\alpha_3) \\
\text{subject to} & \quad x_1\alpha_1 + x_2\alpha_2 \leq c_1, \\
& \quad x_1\alpha_1 + x_3\alpha_3 \leq c_2,
\end{align*}

where $r_i$ is the weight of class $i$.

As before, when $r_1 = 0$, the allocation becomes a strict priority for classes 2 and 3 over class-1, for which the condition of stability is given by $\rho_1/c_1 < (1 - \rho_2/c_2)(1 - \rho_3/c_2)$. If $\rho_1$ satisfies this condition, then we observe the limit point to be zero, while the limit point is positive otherwise. For $\rho_2 = 0.2$, $\rho_3 = 0.7$, $c_1 = 1$ and $c_2 = 2$, this dichotomy is illustrated in Figure [7]. For $\rho_1 = 0.5$, the priority allocation is stable, and as expected class-1 queue empties out eventually. On the other hand, for $\rho_1 = 0.7$, the priority allocation is unstable but the original network is stable. In this case, class-1 cannot rely only on the capacity available when
all the other classes are zero. It needs to get additional capacity for which it needs to remain of the order of \( r_1 \).

The dynamics of the average number class-2 flows is illustrated in Figure 7. When \( \rho_1 = 0.5 \), it follows that class-2 will get strict priority over class-1 after the number of class-1 flows are no longer of the order of \( r_1 \). As can be verified from the figure, the average number of class-2 flows tends to \( \rho_2/(1 - \rho_2) \).

### 6.3 Streaming and elastic traffic

The third example we consider is that of the integration of streaming and elastic traffic as described in Section 3.2. In this example, if accepted in the network, the capacity allocation is given by

\[
\phi_2(x_2) = \frac{z_1}{z_1 + cx_2} + cx_2 \leq 1,
\]

the state space of class-2 conditioned on \( u_1(t) = z_1 \). Define \( \rho_2 = \lambda_2/\mu_2 \). The process \( \bar{U}_1^{z_1} \) is a birth-death process with birth rate \( \lambda_2 \) and death rate \( \mu_2 \), and whose stationary distribution is given by

\[
\bar{\pi}_2(x_2) = \frac{1}{\sum_{j \in S_{z_1}} \rho_2^j / j!} x_2^j.
\]

For the priority allocation, class-1 is stable if and only if \( \rho_1 < \pi_2(0) \). Thus, if \( \rho_1 < \pi_2(0) \), then the limit point of \( u_1(t) \) is 0, and if \( \pi_2(0) < \rho_1 < 1 \), then the limit point is positive.

Performing the scaling previously defined, remark that the state space depends for a fixed macroscopic state \( z_1 \) on both \( z_1 \) and \( c \). We can apply Theorem [1] with \( \phi_1 \) being defined by:

\[
\phi_1(z) = \sum_{x_2 \in S_{z_1}} \frac{z_1}{z_1 + cx_2} \frac{\rho_2^x}{x_2!} C(z_1),
\]

where \( C(z_1) = (\sum_{x_2 \in S_{z_1}} \frac{\rho_2^x}{x_2!})^{-1} \). In the case that \( c \) is very small (\( c << 1 \)), we might consider as a reasonable approximation a Poisson distribution for class-2, whatever the state of class-1. In that case, \( \phi_1 \) takes a slightly simpler form. After simple calculations:

\[
\phi_1(z_1) = H(z_1) = \frac{z_1 \int_0^{\rho_2} u^{z_1-1} \exp(u) \, du}{\rho_2^z \exp(\rho_2)}.
\]

This allows a recursive evaluation for integer-valued \( z_1 \). Using simple calculus, for \( n \in \mathbb{N} \):

\[
H(n + 1) = \frac{n + 1}{\rho_2} (1 - H(n))
\]

We can also evaluate \( H \) in terms of special functions:

\[
H(n) = \frac{n! - n \Gamma(n, -1)}{(-\rho_2)^n \exp(\rho_2)},
\]

where \( \Gamma(n, -1) \) is the incomplete \( \Gamma \) function.

In Figure 8 we plot the \( u_1(t) \) for \( \rho_1 = 0.6, \rho_2 = 0.2, \) and \( c = 0.01 \).

### 6.3.1 Quality-of-Service guarantee

The Quality-of-Service (QoS) for streaming flows is mainly characterized by the probability that an incoming flow does not find sufficient capacity in the network for the flow to be accepted. In networks in which streaming and elastic traffic do interact, this probability can be computed using an Erlang Fixed-Point approximation. However, in the context of the present example, the interaction of these two types of traffic makes it more difficult to apply these fixed-point approximations, mainly due to the fact that the state space of the elastic flows is unbounded. However, in the limiting regime under consideration, we can come
up with a rule-of-thumb that can be used to guarantee a blocking probability smaller than a desired value.

First, we consider a single link whose capacity is shared by the two types of flows as described in 3.1.2 Let $p_n$ denote the desired maximal blocking probability of class-2 flows. We shall set the priority level of class-1 (by varying $r_1$) such that only the unconditional blocking probability satisfies a given constraint.

The blocking probability for a given value of $\bar{z}_2$ is always less than $p_n$. The unconditional blocking probability of class-2 flows is appropriately decreased so that the blocking probability constraint of class-2 flows is not violated.

First, we consider a single link whose capacity is shared by the two types of flows as described in 3.1.2 Let $\bar{z}_2$ denote the maximum number of simultaneous flows of class-2 in the system, and an arrival of class-2 is blocked if and only if $\bar{z}_2 = \lceil \frac{1-x_1}{c} \rceil - 1 = x_2 \leq \frac{1-x_1}{c}$. The term $\frac{1-x_1}{c}$ is the number of circuits of size $c$ available when the total capacity is $1 - z_1$. Thus,

$$\bar{N}_2 = \left[ \frac{1 - z_1}{c} \right],$$

is the maximum number of simultaneous flows of class-2 in the system, and an arrival of class-2 is blocked if and only if the number of circuits of class-2 is $\bar{N}_2$.

Let $g(n)$ denote the blocking probability when the number of circuits is the network is $n$. From the Erlang-B formula,

$$g(n) = \frac{\rho_2^n}{\sum_{j=0}^{\infty} \rho_2^j / j!}.$$

The inverse function $g^{-1}(p_n)$ gives the minimum number of circuits required to ensure a blocking probability smaller than $p_n$. In order to guarantee a maximal blocking of $p_n$, the number of circuits, $\bar{N}_2$ has to be larger than $g^{-1}(p_n)$ at all instant of time, which leads us to the following necessary and sufficient condition for guaranteeing the QoS of class-2 flows:

$$\bar{u}_1 := \sup_{0 \leq t < \infty} u_1(t) < 1 - c[g^{-1}(p_n)].$$

We can ensure the above inequality by scaling the process $u_1(t)$ by a factor $\frac{1-c[g^{-1}(p_n)]}{\bar{u}_1}$. This, in turn, can be achieved by scaling the priority level (or, equivalently, $r_1$) by this very same factor. This additional scaling results in a larger share of the bandwidth for class-1 flows in case $1 - c[g^{-1}(p_n)] > \bar{u}_1$. Conversely, if $1 - c[g^{-1}(p_n)] < \bar{u}_1$, the priority level of class-1 flows is appropriately decreased so that the blocking probability constraint of class-2 flows is not violated.

Using the monotonicity of $\phi_1$ in its first variable, we get that if $\lambda_1 > \hat{\phi}_1(u_1(0))$, then $u_1(t)$ converges monotonically to its limit point. Hence,

$$\bar{u}_1 = \begin{cases} u_1(0), & \text{if } \lambda_1 < \hat{\phi}_1(u_1(0)); \\ \phi_1^{-1}(\bar{u}_1), & \text{otherwise}. \end{cases}$$

Remark 3. The blocking probability for a given value of $z_1$ is in fact a conditional blocking probability in the sense that it is the fraction of calls dropped when the class-1 flows take away a capacity of $z_1$. The unconditional blocking probability of class-2 flows can be computed by integrating over $z_1$, which is rather conservative. An alternative scaling could be constructed such that only the unconditional blocking probability satisfies a given constraint.
Remark 4. In a network of links shared by several classes of streaming flows and one class of elastic flow, we could use fixed-point approximations to compute the blocking probability for the different classes of streaming flows as a function of $z_1$. Assuming that this probability is increasing in $z_1$, we could then compute the maximum value that $z_1$ can attain without the streaming classes violating their individual blocking probability.

7 Conclusions

We analyzed the flow-level performance of multi-class communication networks when one of the classes undergoes a traffic surge. We showed that, under an appropriate scaling of space and time, the dynamics of the temporarily unstable class can be described by a deterministic differential equation in which the time derivative at a given point depends on the conditional stationary distribution of the other classes calculated at that point. For work-conserving allocations, the differential equation is the same as the one of the network in which other classes have strict priority over the temporarily unstable class, that is, the scaled process evolves linearly and is either absorbed at zero or grows indefinitely depending on whether the network is stable or not.

For non-work conserving allocations, the trajectory is much more complex to describe as it depends on the mean residual bandwidth left over by the other classes which in turn depends on the current state of the first class. The limit point of the fluid trajectory can hence be non-zero and finite. We showed that the limit point is zero if the priority allocation is stable; finite if the original network is stable; and is infinite otherwise. We illustrated this behavior through several examples of network topologies and bandwidth allocations that are commonly used to model communication networks.

The time-space-transitions scaling that we considered raises several open questions which would give a better understanding of the network dynamics. wonder how the stochastic stability relates to the properties of the limiting processes of this type of scaling. In particular, finding necessary and sufficient conditions for the limit point of non work-conserving allocations to be zero would constitute a very interesting result. Also, error bounds estimates would be necessary to obtain a reliable performance evaluation tool.

References


A Proof of Theorem 1

Step 1:

We consider the process \( Y^K(t) = \left( \frac{X^K(Kt)}{K}, (X^K(Kt))_{i=2...N} \right) \) as defined by \([8]\). We define \( \hat{N} = N \cup \{ +\infty \} \) and for each \( K \), we define the following random measure on \([0, \infty) \times \hat{N}^{N-1} \):

\[
\nu^K((0, t) \times \Gamma) = \int_0^t \mathbf{1}_{\{(X^K(Ks),...,X^K(Ks)) \in \Gamma\}} \, ds, \forall \Gamma \subset \hat{N}^{N-1}, \text{ and } \forall t \geq 0.
\]

We denote \( \mathcal{L}_0(\hat{N}^{N-1}) \) the set of measures on \([0, \infty) \times \hat{N}^{N-1} \) such that, for all measure \( \nu \) in \( \mathcal{L}_0(\hat{N}^{N-1}) \) and all \( t \geq 0 \), we have \( \nu((0, t) \times \hat{N}^{N-1}) = t \). Since \( \hat{N} \) is compact, we have that \( \mathcal{L}_0(\hat{N}^{N-1}) \) is compact and we deduce that \( \{\nu^K, K \in \mathbb{N}\} \) is relatively compact.

In order to prove the relative compactness of \( \{Y^K_1, \nu^K \}, K \in \mathbb{N}\} \), we then just have to prove the relative compactness of \( \{Y^K_1, K \in \mathbb{N}\} \). We define the following process

\[
M^K(t) = Y_1(t) - \frac{1}{K} \int_0^{Kt} \lambda_1(s) \, ds + \frac{1}{K} \int_0^{Kt} \phi_1 \left( \frac{X^K(s)}{K}, X^K(s) \right) \, ds
\]

The martingale characterization of jump processes (see \([22]\)) shows that \( M^K_1 \) is a local martingale and its increasing process is given by

\[
\langle M^K_1 \rangle = \frac{1}{K^2} \int_0^{Kt} \lambda_1(s) \, ds + \frac{1}{K} \int_0^{Kt} \phi_1 \left( \frac{X^K(s)}{K}, X^K(s) \right) \, ds
\]

Using Doob’s inequality \([3]\), it follows that \( M^K \) converges in probability to 0 on any compact set when \( K \to \infty \), i.e., for any \( T \geq 0 \) and any \( \varepsilon > 0 \),

\[
\lim_{K \to \infty} \mathbb{P} \left( \sup_{0 \leq s \leq T} |M^K(s)| > \varepsilon \right) = 0.
\]

We then define \( w_\eta \) the modulus of continuity for any function \( h \) defined on \([0, t]\):

\[
w_\eta(\delta) = \sup_{s, u \leq t, |s-u| < \delta} |h(s) - h(u)|.
\]

Using Equations \((10)\) and \((11)\), we are able to prove that for any \( \varepsilon > 0 \) and \( \eta > 0 \), there exists \( \delta > 0 \) and \( A \) such that for \( K > A \), we have

\[
\mathbb{P} \left( w_{Y^K_1}(\delta) > \eta \right) \leq \varepsilon.
\]

The conditions of \([1] \), 7.2 p81] are then fulfilled and the set \( \{Y^K_1, K \in \mathbb{N}\} \) is relatively compact. Moreover, any limiting point is a continuous process.

Step 2:

We now suppose that \( \{Y^K_1\} \) converges in distribution to a limit \( Z_1 \). We have to characterize any limiting point of the sequence \( \{\nu^K\} \) and then deduce the existence and uniqueness of the limit of \( \{\nu^K\} \). In the following, we consider a convergent subsequence \( \{Y^{K_i}, \nu^{K_i}\} \) and its limit process \( \{Z_1, \nu\} \).

\[For\ any\ martingale\ \( M \),\ using\ Cauchy\ Schwartz\ and\ Doob’s\ inequality,\ \( \mathbb{E}\left( \sup_{0 \leq s \leq t} M_s \right)^2 \leq \mathbb{E}\left( \sup_{0 \leq s \leq t} |M_s| \right)^2 \leq 4 \mathbb{E} (M^2_t) \).\]
Let \((\Omega, \mathcal{F}, \mathbb{P})\) be the probability space on which they are defined. We call \(\{\mathcal{F}_t\}\) the natural filtration of \((Z(t), \nu)\). We then define \(\gamma\) such that

\[
\forall A \in \mathcal{F}, \forall B \in \mathcal{B}([0,\infty)), \forall C \in \mathcal{B}(\mathbb{R}^{N-1}) \quad \gamma(A \times B \times C) = \mathbb{E}(\mathds{1}_A \nu(B \times C))
\]

According to [11, appendix 8], \(\gamma\) can be extended to a measure on \(\mathcal{F} \otimes \mathcal{B}([0, \infty)) \otimes \mathcal{B}(\mathbb{R}^{N-1})\) and there exists \(\vartheta\) such that for all \(t, \vartheta(t, )\) is a random probability measure on \(\mathbb{R}^{N-1}\) and for any \(B \in \mathcal{B}(\mathbb{R}^{N-1})\), \((\vartheta(t, B), t \geq 0)\) is \(\{\mathcal{F}_t\}\)-adapted and for any \(A \in \mathcal{F} \otimes \mathcal{B}([0, \infty))\),

\[
\gamma(A \times B) = \mathbb{E} \left( \int_{0}^{\infty} \mathds{1}_A(s) \vartheta(s, B) \, ds \right)
\]

(12)

\[
M_B(t) = \nu([0, t] \times B) - \int_{0}^{t} \vartheta(s, B) \, ds
\]

\(M_B\) is \(\{\mathcal{F}_t\}\)-adapted and continuous. We consider \(t \geq s\), and \(D \in \mathcal{F}_s\). We define \(\mathds{1}_C(\omega, u) = \mathds{1}_D(\omega) \mathds{1}_{(s,t)}(u)\) and we have

\[
\mathbb{E}(\mathds{1}_D \nu([s,t] \times B)) = \gamma(D \times [s,t] \times B),
\]

\[
= \gamma(C \times B),
\]

\[
= \mathbb{E} \left( \int_{0}^{\infty} \mathds{1}_C(u) \vartheta(u, B) \, du \right), \text{ (according to (12))}
\]

\[
= \mathbb{E} \left( \mathds{1}_D \int_{s}^{t} \vartheta(u, B) \, du \right).
\]

Since the previous equality is true for all \(D \in \mathcal{F}_t\), it follows that

\[
\mathbb{E}(\nu([s,t] \times B) | \mathcal{F}_s) = \mathbb{E} \left( \int_{s}^{t} \vartheta(u, B) \, du \bigg| \mathcal{F}_s \right).
\]

and immediately, we have

\[
\mathbb{E}(M_B(t) | \mathcal{F}_s) = M_B(s).
\]

Then, \(M_B\) is a continuous \(\{\mathcal{F}_t\}\)-martingale. It has finite sample paths and then is almost surely identically null. Almost surely, the following equation holds for all \(t\),

\[
\forall B \subset \mathbb{R}^{N-1}, \quad \nu([0, t] \times B) = \int_{0}^{t} \vartheta(s, B) \, ds.
\]

(13)

We have to characterize the random measures \(\vartheta(t, .)\) associated to \(\nu\). For any uniformly continuous bounded function \(g\) on \(\mathbb{R}^{N-1}\) and any \(K \in \mathbb{N}\), we define,

\[
M_g^K(t) = \frac{1}{K} \left( g(X^K_1(Kt), \ldots, X^K_N(Kt)) - g(0) \right)
\]

\[
- \sum_{i=2}^{N} \lambda_i \int_{0}^{t} \left( g \left( X^K_1(Kt), \ldots, X^K_i(Kt) + e_i, \ldots, X^K_N(Kt) \right) - g \left( X^K_1(Kt), \ldots, X^K_N(Kt) \right) \right) \, ds
\]

\[
- \sum_{i=2}^{N} \mu_i \int_{0}^{t} \left( g \left( X^K_1(Kt), \ldots, X^K_i(Kt) - e_i, \ldots, X^K_N(Kt) \right) - g \left( X^K_1(Kt), \ldots, X^K_N(Kt) \right) \right) \phi_i \left( Y^K_1(s), X^K_2(Kt), \ldots, X^K_N(Kt) \right) \, ds.
\]

As \(M^K\) is a martingale, \(M^K_g\) is a martingale. We have that \(M^K_g\) converges in distribution to 0. \(|K|^{-1}(g(X^K_i(Kt), \ldots, X^K_N(Kt)) - g(0))\) also converges to 0 because \(g\) is bounded. As a consequence, the following term

\[
\sum_{i=2}^{N} \lambda_i \int_{0}^{t} \left( g \left( X^K_1(Kt), \ldots, X^K_i(Kt) + e_i, \ldots, X^K_N(Kt) \right) - g \left( X^K_1(Kt), \ldots, X^K_N(Kt) \right) \right) \, ds
\]

\[
- \sum_{i=2}^{N} \mu_i \int_{0}^{t} \left( g \left( X^K_1(Kt), \ldots, X^K_i(Kt) - e_i, \ldots, X^K_N(Kt) \right) - g \left( X^K_1(Kt), \ldots, X^K_N(Kt) \right) \right) \phi_i \left( Y^K_1(s), X^K_2(Kt), \ldots, X^K_N(Kt) \right) \, ds
\]
also converges in distribution to 0. But, by the continuous mapping theorem and \( \mathbb{E} \), it converges in distribution to

\[
\int_0^t \sum_{i=2}^N \left( \lambda_i \sum_{y \in \mathbb{N}^{N-1}} g(y + e_i) - g(y) + \mu_i \sum_{y \in \mathbb{N}^{N-1}} (g(y - e_i) - g(y)) \phi_i(Z_1(s), y) \right) \vartheta(s, y) \, ds.
\]

Consequently, this is null almost surely for all \( t \) and we have then, for Lebesgue-almost every \( t \),

\[
\sum_{i=2}^N \left( \lambda_i \sum_{y \in \mathbb{N}^{N-1}} g(y + e_i) - g(y) + \mu_i \sum_{y \in \mathbb{N}^{N-1}} (g(y - e_i) - g(y)) \phi_i(Z_1(t), y) \right) \vartheta(t, y) = 0.
\]

We deduce immediately that

\[
\int_{\mathbb{R}^{N-1}} \Omega^{Z_1(t)}(g)(y) \vartheta(t, dy) = 0
\]

where \( \Omega^{Z_1(t)} \) is the infinitesimal generator of \( (U^{Z_1(t)}(s)) \). This proves exactly that \( \vartheta(t, \cdot) \) is invariant for \( U^{Z_1(t)} \). By uniqueness of the invariant distribution of \( (U^{Z_1(t)}(s)) \), this implies that, given \( Z_1 \), \( \vartheta(t, \cdot) \) is a deterministic measure for all \( t \). We can deduce that, if \( (Y^{K_i}) \) is a converging subsequence, then \( (\nu^{K_i}) \) is also converging and its limit is a random measure in \( \mathcal{L}_0(\mathbb{R}^{N-1}) \). This implies in particular that \( (\nu^{K_i}) \) is tight in \( \mathcal{L}_0(\mathbb{N}^{N-1}) \). We can now proceed of the last part of this step.

We consider \( \varepsilon > 0, \eta > 0 \) and \( t \geq 0 \). Because the sequence \( (\nu^{K_i}) \) is tight in \( \mathcal{L}_0(\mathbb{N}^{N-1}) \), there exists \( \kappa > 0 \) and a compact \( \Gamma \subset \mathbb{N}^{N-1} \) such that:

\[
\mathbb{P} \left( \sup_{t \geq \kappa} \nu^{K_i}([0, t] \times \Gamma^c) \geq \varepsilon \right) \leq \eta/2.
\]

Because \( Z_1 \) is almost surely continuous and \( f \) is Lipschitz-continuous, we have

\[
\mathbb{P} \left( \sup_{t \geq \kappa, y \in \Gamma, s \geq t} \left| f(Y^{K_i}_1(t), y) - f(Z_1(t), y) \right| \geq \varepsilon \right) \leq \eta/2.
\]

Since \( f \) is bounded, we can deduce:

\[
\mathbb{P} \left( \sup_{t \geq \kappa} \left| \int_{[0, t] \times \mathbb{R}^{N-1}} f(Y^{K_i}_1(s), y) \nu^{K_i}(ds \times dy) - \int_{[0, t] \times \mathbb{R}^{N-1}} f(Z_1(s), y) \nu(ds \times dy) \right| \geq 2\varepsilon \|f\| \right) \leq \eta.
\]

According to \( \underline{15} \), there exists a family \( (\vartheta(t, \cdot)) \) of random measures on \( \mathbb{R}^{N-1} \) such that

\[
\sup_{0 \leq s \leq t} \left| \int_0^s f(Y^{K_i}_1(u), X^{K_i}_i(K_i u)) - \sum_{y \in \mathbb{N}^{N-1}} f(Z_1(u), y) \vartheta(u, y) \, du \right|
\]

converges in probability to 0 when \( K_i \) tends to infinity.

Since \( f \) is bounded, we can apply the dominated convergence theorem and we have that

\[
\lim_{K_i \to \infty} \mathbb{E} \left( \sup_{0 \leq s \leq t} \left| \int_0^s f(Y^{K_i}_1(u), X^{K_i}_i(K_i u)) - \sum_{y \in \mathbb{N}^{N-1}} f(Z_1(u), y) \vartheta(u, y) \, du \right| \right) = 0.
\]

We further have that

\[
\lim_{K \to \infty} \mathbb{E} \left( \sup_{0 \leq s \leq t} \left| \int_0^s f(Y^{K}_1(u), X^{K}_i(K u)) - \mathbb{E} \left( f \left( Z_1(u), U^{Z_1}_i(u) \right) \mid Z_1(u) \right) \, du \right| \right) = 0.
\]

Step 2 is complete.
Step 3:
Using the martingale decomposition of $X^K_1$,
\[
\frac{X^K_1(Kt)}{K} = x_1 + M_K(t) + \frac{1}{K} \int_0^{Kt} \lambda^K_i(s) \, ds - \frac{1}{K} \int_0^{Kt} \phi_1 \left( \frac{X^K_i(s)}{K}, \ldots, X^K_N(s) \right) \, ds.
\]
As already remarked in step 1, since $\phi$ is bounded it follows that
\[
\mathbb{E} \left( M_K(t)^2 \right) \leq \frac{A t}{K}
\]
which implies using Doob’s inequality that there exists a constant $A'$ such that for $K$ big enough:
\[
\mathbb{E} \left( \sup_{0 \leq s \leq t} M_K(s) \right) \leq A' \sqrt{\frac{t}{K}} \leq \epsilon.
\]

Using the convergence of the arrival process together with the convergence of the martingale $M_K$, we obtain the uniform integrability of $Y^K_1$. (The tightness of $Y^K_i$ has already been obtained in step 1). Now consider a converging subsequence $Y^K_i$ towards $Z_1$. Using the results of step 2, the convergence of the arrival process together and the convergence of the martingale $M_K$, we obtain that $Z_1$ must satisfy:
\[
Z_1(t) = x_1 + 0 + a_1(t) - \int_0^t \phi_1(Z_1(s)) \, ds.
\]
Hence the limit is unique and deterministic.

This in turn shows the convergence of $Y^K_i$ in distribution and completely characterize the measure $\theta$ introduced in Step 2 as a deterministic measure. We can now prove the convergence in $L^1$. Let $\epsilon$ be given. Define the error estimate:
\[
n_K(t) = \sup_{0 \leq s \leq t} |Y^K_1(s) - u_1(s)|.
\]
Define now the noise amplitude as:
\[
\bar{M}_K(t) = \sup_{0 \leq s \leq t} |M_K(s)|.
\]
Using the convergence of the intensity of the arrival process,
\[
n_K(t) \leq \bar{M}_K(t) + \epsilon + \sup_{s \leq t} \left| \frac{1}{K} \int_0^{Ks} \phi_1 \left( \frac{X^K_i(z)}{K}, X^K_i(z) \right) \, dz - \int_0^s \phi_1(u(z)) \, dz \right|.
\]
Using step 2, $\phi_1$ being Lipschitz and bounded, for $K$ large enough:
\[
\mathbb{E} \left( \sup_{s \leq t} \left| \frac{1}{K} \int_0^{Ks} \phi_1 \left( \frac{X^K_i(z)}{K}, X^K_i(z) \right) \, dz - \int_0^s \phi_1(u(z)) \, dz \right| \right) \leq \epsilon,
\]
which concludes the proof for the $L^1$ convergence of $Y^K_1$.

B The proportional fair allocation for the 3-class linear network

The weighted proportional fair allocation is the solution of the following maximisation problem:

maximise \[ r_1 x_1 \log(\alpha_1) + r_2 x_2 \log(\alpha_2) + r_3 x_3 \log(\alpha_3) \]
subject to \[ \begin{align*}
x_1 \alpha_1 + x_2 \alpha_2 & \leq c_1, \\
x_1 \alpha_1 + x_3 \alpha_3 & \leq c_2, \\
\alpha_i & > 0, \; i = 1, 2, 3.
\end{align*} \]

We assume that $x_i > 0, \forall i$ so that $\alpha_i > 0, \forall i$. In this case the link capacities will be satisfied with equality. Further, from the Karush-Kuhn-Tucker optimality conditions, if $(\alpha_1, \alpha_2, \alpha_3)$ is
optimal then there exist Lagrange multipliers $z_1 > 0$ and $z_2 > 0$ corresponding to the link capacity constraints such that

$$\frac{r_1 x_1}{\alpha_1} - z_1 x_1 - z_2 x_2 = 0, \quad (14)$$
$$\frac{r_2 x_2}{\alpha_2} - z_1 x_2 = 0, \quad (15)$$
$$\frac{r_3 x_3}{\alpha_3} - z_2 x_3 = 0. \quad (16)$$

We can compute $z_1$ and $z_2$ using the equations

$$x_1 \alpha_1 + x_2 \alpha_2 = c_1, \quad (17)$$
$$x_1 \alpha_1 + x_3 \alpha_3 = c_2. \quad (18)$$

Using first (14), (15), and (17), and then (14), (16), and (18), we get the following two equations for $z_1$ and $z_2$:

$$\frac{r_1 x_1}{z_1 + z_2} + \frac{r_2 x_2}{z_1} = c_1,$$
$$\frac{r_1 x_1}{z_1 + z_2} + \frac{r_3 x_3}{z_3} = c_2.$$

These equations can be rewritten as

$$r_1 x_1 z_1 + r_2 x_2 (z_1 + z_2) = c_1 (z_1 + z_2) z_1,$$
$$r_1 x_1 z_2 + r_3 x_3 (z_1 + z_2) = c_2 (z_1 + z_3) z_2,$$

which give two quadratic equations for $z_1$ and $z_2$. We can simplify them further by summing the two equations to get one quadratic equation and one linear equation:

$$r_1 x_1 z_1 + r_2 x_2 (z_1 + z_2) = c_1 (z_1 + z_2) z_1,$$
$$r_1 x_1 + r_2 x_2 + r_3 x_3 = c_1 z_1 + c_2 z_2.$$

Assuming $c_1 < c_2$ we get

$$z_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a},$$
$$z_2 = \frac{(r_1 x_1 + r_2 x_2 + r_3 x_3 - c_1 z_1)}{c_2},$$

where

$$a = c_2 c_1 - c_1^2,$$
$$b = c_1 (r_1 x_1 + 2r_2 x_2 + r_3 x_3) - c_2 (r_1 x_1 + r_2 x_2),$$
$$c = -(r_1 x_1 + r_2 x_2 + r_3 x_3) r_2 x_2.$$

The rate allocations in this case are

$$\phi_1 = \frac{r_1 x_1}{z_1 + z_2}, \quad \phi_2 = \frac{r_2 x_2}{z_1}, \quad \phi_3 = \frac{r_3 x_3}{z_2}.$$

When either $x_2$ or $x_3$ is zero, the allocation reduces to the single link case for which it is DPS with appropriate rate limitation, and is given as follows. For $x_2 = 0$, and $x_3 > 0$,

$$\phi_1 = \min \left( c_1, \frac{r_1 x_1}{r_1 x_1 + r_3 x_3} c_2 \right), \quad \phi_3 = c_2 - \phi_1.$$

For $x_2 > 0$, and $x_3 = 0$,

$$\phi_1 = \frac{r_1 x_1}{r_1 x_1 + r_2 x_2} c_1, \quad \phi_2 = c_1 - \phi_1.$$

For $x_2 = 0$, and $x_3 = 0$,

$$\phi_1 = c_1.$$

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