Technical Memorandum Number 763R

Control of Dividends, Capital Subscriptions, and Physical Inventories

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Revised January 2009

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January 26, 2009

Abstract

Manufacturers are faced with the financial and physical task of managing interrelated flows of material and cash. Material needs capital, and the sale of finished goods contributes to cash reserves. We present and study a dynamic model of managerial decisions in a manufacturing firm in which inventory and financial decisions interact and are coordinated in the presence of demand uncertainty, financial constraints, and a risk of default. The criterion is to maximize the expected present value of dividends net of capital subscriptions. We contrast coordination with the usual decentralized separation of operational and financial decisions, and give an example in which the relative financial value of coordination can be made unboundedly large. We establish conditions which imply that the optimal base-stock inventory level and financial decision variables are nondecreasing functions of the levels of inventory and retained earnings. We show that some important attributes of an optimal policy remain the same regardless of whether default precipitates Chapter 7 or Chapter 11 bankruptcy. The optimal policy is myopic, and if the inventory-related cost and default penalty (in the case of Chapter 11 bankruptcy) are piece-wise linear, the optimal coordinated policy is characterized with simple formulas. The effect of inventory on the risk and cost of default causes the target levels of inventory and residual retained earnings to be different in a coordinated dividend-maximizing firm than in one which decentralizes financial and operational decisions and makes the latter decisions to maximize profit.
1 Operations and Finance

Business organizations are faced with the financial and physical task of managing flows of material and cash. In large firms, the task is decentralized into separate functional responsibilities, although the flows are interrelated because material needs capital and the sale of finished goods contributes to cash reserves. The same dichotomy into operations and finance occurs in management education and research. Although decentralization in professional practice was obligatory in the absence of enterprise-wide information systems, technology has advanced and in many firms it is technically feasible to coordinate financial and operational decisions. This paper contributes to understanding what coordination would entail and to determining its financial value.

Ease of analysis and concentration on special function are two reasons in practice to separate operational and financial decisions. In practice, the production manager has much to learn about sales, production/procurement, and inventories without being concerned directly with finance. But it is a practical question to decide when the interaction between finance and operations can be completely ignored, as a good enough approximation to reality, or when it is sufficiently important that the functions should be coordinated. Unlike some large firms, many small growing firms are severely cash constrained and could not survive without coordinating financial and operational decisions.

The large literature on asset pricing in financial economics argues that the primary measure of the financial value of an investor-owned firm is the expected present value of the time stream of its dividends (cf. Lucas (1973), Brodie, Kane and Marcus (1995, page 326), and Cochrane (2001)), namely its \( EPV \). One may argue that operations models should not use this criterion due to the Modigliani-Miller Theorem (Modigliani and Miller, 1958) which is predicated on perfect markets. However, many firms do not operate in perfect markets and are influenced by the specter of bankruptcy. This is particularly true of those which are small and entrepreneurial. Our modeling is motivated by firms such as these where prescriptions for operating behavior can be ineffective if they overlook the interaction of financial and operational decisions. Therefore, we use the EPV criterion net of capital subscriptions. In contrast, the research methodologies and qualitative results in inventory theory respond to cost and/or profit criteria.

There is a potential for liquidity crises in the firm facing production/procurement lags, uncertain
demand and financial constraints. In young firms, as much as the stockholders might like to see their dividends flow and grow, the payouts may sometimes be negative in the sense that the stockholders may be called for capital subscriptions, namely more cash to keep the enterprise solvent. At first, we consider this possibility by allowing the dividends to be negative. The model which we define and analyze in Sections 2, 3, and 4 does not constrain dividends to be nonnegative. One may interpret a negative dividend for a small or medium-sized business as an action by the owners to subscribe additional capital. So, the model corresponds to a small or medium-sized business whose owners may be obliged to subscribe additional capital. Later, in §6, we impose a nonnegativity constraint on dividends.

We make two kinds of contributions in highly stylized settings. First, we characterize optimal policies for coordinating operational and financial decisions. Second, we characterize the market value of a firm that adopts such policies. The financial value of coordination is the difference between the market value under coordination and the market value of a corresponding firm which continues to decentralize its decisions.

1.1 Background Literature

Although there is a well developed literature on models of production and inventory systems (cf. Graves, Rinnoy Kan, and Zipkin (1993)), until recently, only a few models of operational decisions included financial considerations. An early analysis of financial considerations is based on a model of the dynamics of the growth of a dividend-paying non-financial firm as a controlled random walk between a reflecting and an absorbing barrier (Shubik and Thompson, 1959). The reflecting barrier is created by dividend payments and the absorbing barrier by bankruptcy conditions which put the firm out of business. The separation of ownership and management in a game of economic survival provides the mathematical structure to reflect the potential differences in goals of management and ownership; in particular concerning the roles of dividends and bankruptcy. The model below reflects the strategic aspects of the tradeoffs between bankruptcy and the paying of dividends. Alternatively, if the firm is penalized by an insolvency, but is in a position to continue to operate, this too can be modeled as a reflecting barrier. See Radner and Shepp (1996) for recent work of this kind. Hadley and Whitin (1963) and Sherbrooke (1968) are early papers on inventory management with budgetary
constraints. The treatment of the maintenance of cash safety levels as an inventory control problem (cf. Porteus (1972) and Shubik and Sobel (1992)) is another research connection between operations and finance.

Most of the fusion of operational and financial considerations is recent in origin. A series of papers model operational decisions in the presence of foreign exchange exposure (e.g., Kogut and Kulatilaka (1994), Huchzermeier and Cohen (1996), Dasu and Li (1997), Aytekin and Birge (2004), and Dong et al. (2006)). Papers in another series analyze capacity-expansion problems with financial constraints (e.g., Birge (2000), Van Mieghem (2003), and Babich and Sobel (2004)) and use approaches similar to ours.

Some recent work on the coordination of operational and financial decisions can be partitioned according to whether or not the models and criteria are influenced by the capital structure of the firm. Perhaps earliest among the papers that are orthogonal to capital structure considerations are Archibald et al. (2002) who optimize the probability of survival of a growing firm which manages an inventory, and Buzacott and Zhang (2003) who model a growing manufacturer which finances production both with loans secured by inventory and with unsecured loans. The latter paper demonstrates the importance of jointly considering production and financing decisions in a dynamic but deterministic setting and proposes a single-period newsvendor model to examine the incentives for a lender and a borrower to engage in asset-based financing.

Capital structure plays a role in the static model that is essentially common to Xu and Birge (2004), Xu and Birge (2005), and Dada and Hu (2008). In these papers, a financially constrained firm coordinates its operating decisions and short-term borrowing. Xu and Birge (2004), assuming that the creditor is non-strategic, comment that bankruptcy costs remove the firm from the Modigliani-Miller world. As a result, the firm’s borrowing capacity is limited, and it may have to produce less than the unconstrained optimum. Comparative statics implies that the firm’s optimal production quantity decreases as its debt increases. The authors conclude that “a low-margin company should [select] a conservative output level and an aggressive financial decision, while [a] high-margin company” should do the opposite.

Xu and Birge (2005) numerically examine the dependence of the value of the firm on production unit cost and debt-equity ratio, and then compare their results with market data. They find that
the model predicts lower debt-equity ratios than the data exhibit, and they attribute this difference to the absence of long-term debt in the model. They show that a low-margin producer faces higher agency costs than a high-margin one. Dada and Hu (2008) assume that the creditor is strategic and, using game theory, show (a) that the firm produces less than the unconstrained optimum even without bankruptcy costs, and (b) the optimal production quantity increases as a function of the firm’s equity.

The model that underlies these three papers is static and yields conclusions that differ from some of those in Hu and Sobel (2008) which is based on a dynamic model that distinguishes between long-term debt and short-term loans. In the latter paper, as the long-term debt increases, (i) the optimal inventory level increases and then decreases, and (ii) the optimal short-term loan decreases. As is typically true of dynamic newsvendor-like models without capacity constraints, the optimal production quantity (in any period after the first) is the previous period’s demand. So inventory levels are driven by the amount of long-term debt, but production amounts do not depend on it at all.

In the valuation model in Xu and Birge (2006), the firm maximizes a combination of the EPV of the net cash flow to shareholders and multiples of other firm attributes, and it decides whether to default and liquidate the firm or to continue to produce. The contingent default opportunity raises the expected present value of the net cash flow to shareholders, so it yields higher equity valuations than traditional valuation and planning models. The authors conclude that valuations are understated if they stem from models which exclude the possibility of contingent defaults.

In §2 we model the decisions of a simply-structured firm which operates in perpetuity (or, in §5, until bankruptcy occurs), makes a sequence of operational and financial decisions, and strives to maximize its market value. One of our purposes is to establish a basis for comparison with models which relax the Draconian assumptions in §2. To some extent, that has already occurred; see §5 and §6 for the effects of precluding capital subscriptions and of resorting to Chapter 7 bankruptcy instead of Chapter 11. In reality, people (“managers”) – not “firms” – make decisions in response to incentives that may not be consistent with optimizing the firm’s value. We do not examine the consequent agency issues of non-owner-managers, but see Xu and Birge (2005) for their inclusion. Real owner-managers have shorter time-horizons than our “firm;” Babich and Sobel (2004) optimize
a cash-out goal of an owner-manager.

Other papers leverage the results in this paper but change the assumptions in §2. Brunet and Babich (2007) compute the signaling value of trade credit financing for the acquisition of goods. Hu and Sobel (2005) design the firm’s capital structure before operations commence; the resulting mixture of equity and long-term debt affects the optimal base-stock levels for inventory and working capital. Sobel and Zhang (2003) have nonlinear costs of production and, as a result, optimal inventory decisions become more complicated than the base-stock levels in this paper. Sobel and Turcic (2007) consider how to adapt to evolving market conditions that are modeled with a more flexible and realistic model of demand than in §2. That raises issues that cannot be addressed here, such as the change in the levels of inventory and working capital as the firm grows.

1.2 The Value of Coordination

The initial version of this paper (Li et al., 1997) neglected to quantify the comparison of coordination versus continued decentralization. This subsection presents some of the emerging knowledge about such comparisons due to more recent numerical examples and research which leverages earlier versions of the paper. Let $\Delta$ be the difference between the market value of a firm which coordinates its financial and operational decisions, and the market value of a corresponding firm which decentralizes those decisions. We refer to $\Delta$ as the financial value of coordination. The relative value of coordination divides $\Delta$ by the market value of a firm which decentralizes its operational and financial decisions.

We present a numerical example in §4.4 in which the relative value can be made unboundedly large. That is, there are combinations of model parameters where decentralization yields an EPV that is nearly zero (the denominator) while $\Delta$ (the numerator) is strongly positive. The numerical examples in §4.4 also demonstrate that the dividend loss in decentralization can stem primarily from the failure to coordinate short-term financial decisions with operational decisions rather than incorrect operational decisions. This contrasts with (but is not contrary to) the finding in Xu and Birge (2004) that the value of the firm is more sensitive to erroneous production decisions than to incorrect financing decisions.

There is an explicit formula for $\Delta$ in Sobel and Turcic (2007) which has essentially the same model as ours except that the demand process is autoregressive (demands here are independent from
one period to the next), excess demand is lost (here it is backordered), and revenue for excess demand is received when demand arises (here it is received when demand is satisfied). The explicit formula permits the following conclusions which should remain valid here but would be more difficult to confirm analytically due to the model differences. The relative value increases if there is a reduction in \( r \), the exogeneous price at which goods are sold, or if there is an increase in \( c \), the unit purchasing/production cost, or in the unit holding cost. If demand is normally distributed, the relative value increases sharply if the standard deviation rises, and it is insensitive to the mean demand.

The contrary effects on relative value of \( r \) and \( c \) reflect differences in their effects on bankruptcy risk. Expenditures to produce or buy goods may be financed with short-term loans that raise the \textit{a priori} default risk with attendant higher default costs and lower future dividends. This contributes to the discrepancy between these assertions and the aforementioned conclusions in Xu and Birge (2004) which are based on a static model. Here, an increase in \( c \) can be offset by an increase in \( r \) only to the extent that all of the newly produced goods are sold. Thus, the effects on relative value due to changes in \( r \) and \( c \) cannot be compressed into a composite effect due to changes in the difference \( r - c \). In summary, the relative value would be particularly enhanced in firms with high unit costs and volatile demands.

1.3 Contributions of this Paper

§2 specifies the model to which most of the paper is devoted. In particular, dividends can be negative, connoting capital subscriptions, and default results in a penalty and subsequent resumption of operations. The U.S. Bankruptcy Code has two routes for a corporation to enter bankruptcy (http://www.uscourts.gov/bankruptcycourts/bankruptcybasics/process.html). “Chapter 7,” which we term \textit{wipeout} bankruptcy, entails a permanent halt of operations and an immediate liquidation of assets. “Chapter 11,” which we term \textit{reorganization} bankruptcy, consists of a restructuring of debt and a continuation of operations. Thus, the model in §2 corresponds to Chapter 11. Along with other assumptions, this results in a dynamic stochastic optimization problem in which the randomness is due to market demand.

§3 establishes key properties of a dynamic programming representation of the optimization problem. §3.1 reduces the dimensionality of the dynamic program and §3.2 shows that the reduced
dynamic program has a concave structure. Also, the optimal values of dividends, production, and short term borrowing are nondecreasing functions of the levels of inventory and retained earnings, and the optimal level of inventory (residual retained earnings) is a nondecreasing (nonincreasing) function of the distribution of demand (in the sense of first-order stochastic dominance). §3.3 significantly simplifies the optimization by showing that the dynamic problem has a myopic optimum, and §3.4 observes that an optimal policy is consistent with the “pecking-order” principle of corporate finance.

§4 considers the important case in which the default penalty and the gross profit from sales (net of inventory costs) are piece-wise linear functions. Since some venture capitalists impose a “no-borrowing” constraint in return for providing funds, §4.1 constrains the problem with the requirement that short-term loans be zero. This leads to a nearly explicit solution in which the optimal level of goods inventory rises if there are increases in $\beta$, the discount factor, or $\gamma$, the unit backorder cost, or there are decreases in $h$, the unit holding cost, $c$, or $p_1$, the unit default penalty. The optimal level of residual retained earnings rises if there are increases in $\gamma, h, \beta,$ or $p_1$.

§4.2 is the counterpart to §4.1 when short-term loans are feasible. This results in a more complicated solution, but the optimal goods inventory level has the same monotonicity relationships as when the no-borrowing constraint is in force. The optimal level of residual retained earnings loses monotonicity with respect to $\gamma$ but gains it with respect to $\rho$, the interest rate on short-term loans. As a result, the optimal short-term loan rises if $r$ or $\gamma$ increase, or if there is a decrease in $h, \beta, p_1$, or $\rho$.

§4.3 compares the operational decisions stemming from financial considerations with their corresponding values which optimize the expected present value of the profits (not dividends), without cash constraints. It turns out that the profit-maximizing base-stock level may be greater than, less than, or the same as the dividend-maximizing base-stock level.

Besides the examples in §4.4 that we have already mentioned, we numerically investigate the effects of the discretized time scale and the source of the financial advantage of coordination. We find that the opportunity cost of decentralization instead of coordination can sometimes be due primarily to the uncoordinated financial decisions rather than erroneous inventory base-stock levels.

The next two sections investigate the extent to which the conclusions would remain valid if
major features of the model were changed. In §5, default precipitates dissolution of the firm; so this section corresponds to Chapter 7 instead of Chapter 11 in the U.S. Bankruptcy Code. The primary conclusions are that major features of the optimal coordination of operations and finance transcend the particular form of bankruptcy, i.e., some important results in §3 and §4 remain valid. Also under Chapter 7, the life-time of the optimally coordinated firm has a geometric distribution. In §5.2 we alter the model by endowing the firm with the option to declare Chapter 7 bankruptcy even if it is not forced into it. As consequences, the set of circumstances in which the firm should opt for bankruptcy is determined by a barrier limit, and the production quantities and dividends should continue to be selected according to order-up-to levels. However, the order-up-to levels become functions of retained earnings instead of being scalars. In §6, we preclude capital subscriptions, so dividends are constrained to be nonnegative. We find that some of the results in §3 remain valid and that inventories are lower, short-term loans are larger, and the default risk is higher than if capital subscriptions were permitted. We conclude with §7.

2 The Model

Consider a discrete-time multi-period model in which a firm which lives in perpetuity decides each period how much money to borrow, how many units to produce, and how much dividend to issue before the realization of the period’s demand. Although there may be several means of financing, we assume that only short-term loans are available. The specifics of the model are as follows.

In each period, the firm makes three decisions:

\[ b_n \equiv \text{amount of money borrowed at the start of period and repaid at the end of period } n; \]
\[ z_n \equiv \text{quantity of goods procured/produced during period } n; \text{ assumed available to satisfy demand in period } n; \]
\[ v_n \equiv \text{dividend issued in period } n \text{ if } v_n > 0; \text{ capital subscription if } v_n < 0. \]

Notation for other variables:

\[ w_n \equiv \text{amount of retained earnings at the beginning of period } n; \]
\[ x_n \equiv \text{number of units in inventory (net of cumulative unsatisfied demand, if any) at the beginning of period } n; \]
\[ D_n \equiv \text{new demand in period } n; \text{ we assume that } D, D_1, D_2, \ldots \text{ are independently, identically} \]
distributed and nonnegative, and let \( F \) denote the distribution function of \( D \);

\( c \equiv \text{unit procurement cost; we assume } c > 0; \)

\( r \equiv \text{unit selling price; we assume } r > c; \text{ revenue is received in the period when demand is met;} \)

\( g(y_n, D_n) \equiv \text{sales revenue made on new demand in period } n \text{ minus inventory-related costs where } y_n = x_n + z_n \text{ is the total amount of product available to satisfy the current demand; example:} \)

\[
g(y, d) = r \min\{y, d\} - h(y - d)^{+} - \gamma(y - d)^{-} = (r + \gamma)y - (r + h + \gamma)(y - d)^{+} - \gamma d \quad \text{where } d, h \text{ and } \gamma \text{ denote realization of demand, unit inventory cost and unit backorder cost respectively;}
\]

\( \rho \equiv \text{interest rate on short-term loans; } \rho b_n \text{ is interest paid at the beginning of period } n; \text{ the results would not significantly change if we assumed that interest were paid at the end of the period;} \)

\( \beta \equiv \text{single period discount factor } (0 < \beta < 1); \text{ borrowing would not occur unless the interest rate is less than the opportunity cost of capital } (\rho < 1/\beta - 1); \)

\( p(w_n) \equiv \text{default penalty exercised if } w_n < 0 \text{ but it is convenient to define } p(\cdot) \text{ as a function on } \mathbb{R}. \)

Although many of our results depend on \( p(\cdot) \) being a convex decreasing function, the most important results do not depend on this assumption and we make no assumptions regarding \( p(\cdot) \) now.

Although we have called \( p(\cdot) \) a “default” penalty, it encompasses three related phenomena. First, if the retained earnings go negative and the firm cannot refinance, this could be the terminal bankruptcy costs. Second, if the firm can refinance, it can do so by borrowing externally and paying legal fees and other financing costs. Third, it can raise money by subscriptions from current stockholders. Our current model illustrates the latter two choices (the terminal bankruptcy case will be dealt with in §5). Thus, the default penalty can be thought of as the amount of total corporate losses, including the utility losses of stockholders, due to an insolvency.

The sequence of decisions and events in period \( n \) is as follows:

1. Each period starts with computing the amount of retained earnings \( w_n \) and the current inventory level \( x_n \).

2. The default penalty \( p(w_n) \) is paid if the amount of retained earnings is negative.

3. Choose the levels of borrowing, production, and dividend, \((b_n, z_n, v_n)\), subject to a liquidity constraint ((7)) below.
4. The dividend and loan interest are paid, \( v_n \) and \( \rho b_n \), the production decision is implemented at a cost of \( cz_n \), and the backordered demand is satisfied with a revenue of \( rx_n^- \).

5. The demand is realized, \( D_n \), and sales revenue on current demand net of inventory costs \( g(y_n, D_n) \) is realized.

6. The principal of the loan \( b_n \) is repaid.

We define two new decision variables which are convenient for analysis.

\[
y_n \equiv x_n + z_n \\
s_n \equiv w_n + rx_n^- - p(w_n) - v_n - cz_n - \rho b_n
\]  

We assume that ordered goods are delivered immediately. Therefore, \( y_n \) is the amount of goods that is available to satisfy demand in period \( n \). We assume that \( y_n \geq 0 \), or \( z_n \geq -x_n \). (Under the backordering assumption that will be made shortly, standard arguments imply that the model corresponds to one in which there is a constant lag between the time at which goods are produced and the time at which they are available to satisfy demand.) That means, the production quantity, \( z_n \), is no less than the backordered demand, \( -x_n \) if \( x_n < 0 \), namely, the backordered demand will always be satisfied first. This is a reasonable assumption because the margin that could made on the backordered demand is positive and risk-free. We interpret \( s_n \) as the amount of internally generated working capital that is available. That is, \( s_n \) is working capital after dividend, loan interest, and production cost are paid and the revenue on backordered demand is received and before the loan is made and revenue and inventory costs are realized. So \( b_n + s_n \) is the total working capital available in period \( n \), i.e., after event 4 and before event 5. By the end of the period, revenues and inventory costs have been realized, and the loan is straightforwardly repaid if \( b_n + s_n + g(y, D_n) \geq b_n \). Otherwise, there is a delay in repayment and the default penalty \( p(w_{n+1}) \) is levied.

We assume that demand is backordered if it exceeds the supply of goods; so \( x_{n+1} = x_n + z_n - D_n \) and

\[
w_{n+1} = w_n + rx_n^- - p(w_n) - v_n - cz_n + g(y_n, D_n) - \rho b_n
\]
Equation (3) balances the cash flow. Using the convenient notation (1) and (2), the two equations become:

\[ x_{n+1} = y_n - D_n \]  
\[ w_{n+1} = s_n + g(y_n, D_n) \]

If excess demand were lost instead of backordered, (5) would be replaced by \( x_{n+1} = (y_n - D_n)^+ \). This would complicate the formulas in §4.1 and §4.2 but all of the paper’s qualitative results would be preserved.

We assume that the loan and production quantities are nonnegative:

\[ b_n \geq 0 \quad \text{and} \quad z_n \geq 0 \]

The liquidity constraint is the inequality \( w_n + rx_n^+ + (1 - \rho)b_n \geq p(w_n) + v_n + cz_n \) which corresponds to

\[ b_n + s_n \geq 0. \]

Inequality (7) prevents the expenditures in period \( n \) from exceeding the sum of retained earnings plus the loan proceeds. If \( w_n + rx_n^+ + (1 - \rho)b_n - p(w_n) - cz_n < 0 \) then (7) forces \( v_n \) to be negative. In §6 we analyze the consequences of imposing the constraint that dividends are nonnegative, i.e. \( v_n \geq 0 \).

Given \( x_n \) and \( w_n \), from (1) and (2) the decision variables in period \( n \) can be specified as \( y_n, s_n \) and \( b_n \) instead of \( z_n, v_n \) and \( b_n \). Let \( \beta \) be the single period discount factor \( (0 < \beta < 1) \), for \( n = 1, 2, \ldots \) let \( H_n \equiv (x_1, w_1, b_1, s_0, y_0, D_1, \ldots, x_{n-1}, w_{n-1}, b_{n-1}, s_{n-1}, y_{n-1}, D_{n-1}, x_n, w_n) \), and let

\[ B = \sum_{n=1}^{\infty} \beta^{n-1}v_n \]

denote the present value of the dividends net of capital subscriptions. We treat the stockholders, who may be diverse, as a representative agent. A policy is a nonanticipative rule for choosing \( y_1, s_1, b_1, y_2, s_2, b_2, \ldots \). That is, a policy is a rule that, for each \( n \), chooses \( y_n, s_n \) and \( b_n \) as a function of \( H_n \). An optimal policy maximizes \( E[B|x_1 = x, w_1 = w] \) for each \( (x, w) \in \mathbb{R}^2 \). The goal is to characterize an optimal policy.
Our model, like all models, suppresses many details, and the inclusion of some of those details, such as shareholder income taxes, would complicate the analysis and affect the results. However, four kinds of additional details could be inserted in the model at the expense of expository clarity. These details would complicate the exposition but would not cause any significant change in the results. First, the dividend decision would be made less frequently than the borrowing decision, and the borrowing decision would be made less often than the production quantity decision. That is, borrowing decisions would be made only in periods 1, 1+\(r_b\), 1+2\(r_b\), ..., and dividend decisions would be made only in periods 1, 1+\(r_d\), 1+2\(r_d\), ... where \(r_b\) and \(r_d\) are positive integers. Second, the right side of (3) would include a credit for interest earned on retained earnings. Third, instead of the borrowing interest rate, \(\rho\), being constant, the rate in period \(n\) would be a random variable whose distribution depends on state and decision variables. Similarly, the discount factor in period \(n\) would be the \(n\)-fold product of random variables whose distributions depend on state and decision variables. See Babich and Sobel (2003) for an analysis of a model with such dependencies. Similarly, our assumption that loan interest is prepaid at the beginning of a period could be replaced by the assumption that it is repaid at the end of the period. All of our results would remain valid except in Section 4.2 where straightforward changes would be necessary. Fourth, the representative shareholder’s attitude towards risk could be included in the model with an exponential utility function. This would slightly complicate the formulas in §4 but preserve the paper’s qualitative results.

3 Problem Simplification and Analysis

In this section we show that the problem has a more compact structure and a simpler optimal policy than might first appear. Although there seem to be two state variables (\(x_n\) and \(w_n\)) and three decision variables (\(b_n, s_n, y_n\)), we reduce the problem to one with only one state variable (\(x_n\)) and two decision variables (\(s_n\) and \(y_n\)) and its solution can be specified almost explicitly.

From (1), (2) and (5), \(v_n = w_n + r x_n^- - p(w_n) - s_n - c y_n + c x_n - \rho b_n\) and \(x_n = y_{n-1} - D_{n-1}\) \((n > 1)\). Substitution in (8) and rearranging terms yields

\[
B = r x_1^- + c x_1 + \sum_{n=1}^{\infty} \beta^n c D_n + \sum_{n=1}^{\infty} \beta^{n-1} [w_n + r \beta(y_n - D_n)^- - p(w_n) - s_n - (1 - \beta)c y_n - \rho b_n]
\]
3 PROBLEM SIMPLIFICATION AND ANALYSIS

Inserting (5) and \((y_n - D_n)^- = (y_n - D_n)^+ - y_n + D_n\) and rearranging terms produces
\[
B = rx_1^- + cx_1 + w_1 - p(w_1) + (\beta r - c) \sum_{n=1}^{\infty} \beta^n D_n
+ \sum_{n=1}^{\infty} \beta^{n-1}[-(1 - \beta)(s_n + cy_n) - \beta ry_n + \beta r(y_n - D_n)^+ + \beta g(y_n, D_n)
- \beta p(s_n + g(y_n, D_n)) - \rho b_n] \tag{9}
\]

For \((b, y, s) \in \mathbb{R}^3\) let
\[
K(b, s, y) = -(1 - \beta)s - [\beta r + (1 - \beta)c]y + \beta E[r(y - D)^+ + g(y, D) - p(s + g(y, D))] - \rho b \tag{10}
\]

3.1 Reduction of Dimensionality

Since a policy maximizes \(E[B|H_1]\) if and only if it maximizes \(E[B|H_1] - (rx_1^- + x_1 + w_1 - p(w_1) - (\beta(\beta r - c)/(1 - \beta))E[D])\), we utilize (9) and (10) and optimize the following criterion:
\[
E[\sum_{n=1}^{\infty} \beta^{n-1}K(b_n, s_n, y_n)] \tag{11}
\]

The constraints on the decision variables are
\[
y_n \geq 0, \quad y_n \geq x_n, \quad b_n + s_n \geq 0, \quad \text{and} \quad b_n \geq 0 \tag{12}
\]

The optimization of the expected value of (8) subject to (6) and (7) is equivalent to maximizing (11) subject to (12). However, a straightforward dynamic program for the former problem has a state consisting of a pair of scalars, whereas the latter has a single scalar state variable, namely the inventory level. This reduces the computational effort to obtain numerical solutions, simplifies the characterization of an optimal policy, and justifies the following statement.

**Proposition 3.1** The optimization of the expected value of (8) subject to (6) and (7) corresponds to the following dynamic program:

\[
\psi(x) = \sup_{b, s, y} \{J(b, s, y) : y \geq 0, y \geq x, b \geq 0, b + s \geq 0\} \text{ where } \tag{13}
\]

\[
J(b, s, y) = K(b, s, y) + \beta E[\psi(y - D)] \tag{14}
\]

A finite-horizon recursion that corresponds to (13) and (14) is \(\psi_{N+1}(\cdot) \equiv 0\) and

\[
\psi_n(x) = \max_{b, s, y} \{J_n(b, s, y) : y \geq 0, y \geq x, b \geq 0, b + s \geq 0\} \text{ where } \tag{15}
\]

\[
J_n(b, s, y) = K(b, s, y) + \beta E[\psi_{n+1}(y - D)] \tag{16}
\]
for each \( n = 1, 2, \ldots, N \) and \( x \in \mathbb{R} \). Let \( b_n(x) \), \( s_n(x) \), and \( y_n(x) \) be optimal values of \( b, s, \) and \( y \), respectively, in (15). We deduce properties of (13) and (14) via the finite-horizon approximation \( \psi_n \) because \( \psi_1 \) converges pointwise to \( \psi \) as \( N \to \infty \), and \( \psi \) inherits the essential properties of \( \psi_1 \).

### 3.2 Concavity

The following result gives conditions which imply that the marginal value of inventory increases as the planning horizon lengthens. This is an intuitive property of many dynamic concave resource allocation models (Mendelssohn and Sobel, 1980). If the horizon is longer, then there is greater opportunity to make productive use of an additional unit of resource. Here and throughout the paper, “decrease” and “increase” are used in the weak sense.

**Proposition 3.2** Suppose that \( p(\cdot) \) is a decreasing convex function on \( \mathbb{R} \) and \( g(\cdot, d) \) is a concave function on \( \mathbb{R} \) for each \( d \in \mathbb{R}_+ \).

1. The value function in (15), \( \psi_n(\cdot) \), is a concave function on \( \mathbb{R} \) and \( J_n(\cdot, \cdot, \cdot) \) is a concave function on \( \mathbb{R}^3 \) for each \( n \).

2. Let \( \psi'_n(x) \) be the right-hand derivative of \( \psi_n(x) \); then \( \psi'_n(x) \leq \psi'_{n+1}(x) \) for each \( n \) and \( x \).

**Proof.** For 1, start with \( \psi_{N+1}(\cdot) \equiv 0 \) and perform an induction on \( n \) in (15) and (16). Part 2 can be established with lattice programming, e.g., Heyman and Sobel (2003, p.398).

Concavity of the dynamic program value function leads to sensitivity analysis results for the effects of changing the demand distribution. Denote \( s_n(x) \) and \( y_n(x) \) by \( s_n(x, F) \) and \( y_n(x, F) \) to make explicit the dependence of \( s_n(x) \) and \( y_n(x) \) on \( F \), the distribution function of demand, \( D \).

**Corollary 3.1** Under the assumptions of Proposition 3.2, if \( F_1(a) \geq F_2(a) \) for all \( a \in \mathbb{R} \), then \( y_n(x, F_1) \leq y_n(x, F_2) \) and \( s_n(x, F_1) \geq s_n(x, F_2) \) for all \( x \in \mathbb{R} \).

**Proof.** See Veinott (1965).

The partial ordering of demand in Corollary 3.1 is first-order stochastic dominance. So stochastically greater demand leads to higher product stock levels and lower cash stock levels. The former effect is intuitive. The rationale for lower cash stock levels is that higher demand yields greater
end-of-period revenue, so the firm can accept a lower cash level at the beginning of the period. Also, Corollary 3.1 can be exploited in an algorithm. One uses the bounds to limit the search space at each iteration for a sequence of distribution functions that begins with a unit step function that jumps at zero, and ends with the actual distribution function of demand.

We assume for each \( n \) and \( x \) that the maximum on the right side of (15) is achieved, say at \((b, s, y) = (b_n(x), s_n(x), y_n(x))\). From (2),

\[
v_n = w_n + rx_n - p(w_n) + cx_n - cy_n(x_n) - s_n(x_n) - \rho b_n(x_n).
\]

If the inventory level is \( x \) and the amount of retained earnings is \( w \), the optimal dividend in period \( n \) is

\[
v_n(x, w) = w + rx - p(w) + cx - cy_n(x) - s_n(x) - \rho b_n(x) \tag{17}
\]

**Proposition 3.3** Under the assumptions of Proposition 3.2, for each \( n \), \( y_n(x) \) is increasing and \( z_n(x) (y_n(x) - x) \) is decreasing with respect to \( x \in \mathbb{R} \). If \( p(\cdot) \) is decreasing on \( \mathbb{R} \) then \( v_n(x, \cdot) \) is increasing on \( \mathbb{R} \) for each \( x \in \mathbb{R} \).


The monotonicity of the target inventory level and the production/purchase quantity is an attribute of optimal policies in many production/inventory models. It reflects the substitutability of current inventory and additional goods that are procured or produced. The monotonicity of the dividend originates from the fact that an increase in the “cash on hand,” \( w \), can only ease the liquidity constraint. If there is an increment in \( w \), then there is at least a small increase in the dividend that would not preclude the selection of the order-up-to inventory level \( (y) \) and residual retained earnings \( (s) \) that were optimal at the original level of \( w \).

It is convenient to define

\[
G(y) = \sup_{b, s} \{ K(b, s, y) : b \geq 0, b + s \geq 0 \} \tag{18}
\]

and rewrite (13) and (14) as

\[
\psi(x) = \sup_{y} \{ G(y) + \beta E[\psi(y - D)] : y \geq 0, y \geq x \}. \tag{19}
\]

This dynamic program corresponds to the dynamic newsvendor model and, as a result, there is an optimal base-stock level policy for inventory replenishment. More generally, there is a myopic optimum.
We do not need the convexity and concavity assumptions in Proposition 3.2 for any of the remaining results in this section.

### 3.3 A Myopic Optimum

Let \((b^*, s^*, y^*)\) maximize \(K(b, s, y)\) subject to \(b \geq 0, b + s \geq 0,\) and \(y \geq 0:\)

\[
K(b^*, s^*, y^*) = \sup_{b, s, y} \{K(b, s, y) : b \geq 0, b + s \geq 0, y \geq 0\}.
\]  

Under reasonable assumptions (see Proposition 3.2), this numerically easy nonlinear programming problem is a concave maximization problem with three variables and three polyhedral constraints. Let \(L\) denote the maximal expected present value of (11). Then

\[
L \leq \frac{K(b^*, s^*, y^*)}{1 - \beta}
\]

with equality if \((b_n, s_n, y_n) = (b^*, s^*, y^*)\) is feasible in (11) (i.e., satisfies (12)) for all \(n\). A simple condition is sufficient for feasibility. If \(x_n \leq y_n\) for some \(n\), then \((b_n, s_n, y_n) = (b^*, s^*, y^*)\) is feasible because \(b^* \geq 0, b^* + s^* \geq 0,\) and \(y^* \geq 0\) from (20). If \(y^*_n = y^*_n\), then \(x_n + 1 = y_n - D_n = y^* - D_n \leq y^*;\) so \(y^*_n = y^*_n\) is feasible. Therefore, \(x_1 \leq y^*\) permits \((b_n, s_n, y_n) = (b^*, s^*, y^*)\) for all \(n\). This argument yields the next result which is most significant if \(x_1 \leq y^*\).

**Proposition 3.4** If \(x_k \leq y^*\) for some \(k\) then \((b_n, s_n, y_n) = (b^*, s^*, y^*)\) for all \(n \geq k\) is optimal.

So \(x_1 \leq y^*\) permits \(b_n = b^*, s_n = s^*,\) and \(y_n = y^*\) for all \(n\). That is, if the initial inventory of goods is not excessive, then an optimal decision rule is determined by three scalars. Another result is that the sequence of production quantities and dividends is a sequence of independent and identically distributed random vectors.

**Proposition 3.5** For all \(n\) if \(b_n = b^*, s_n = s^*,\) and \(y_n = y^*\), then \((v_2, z_2), (v_3, z_3), \ldots\) are independent and identically distributed random vectors with the same joint distribution as the random vector

\[
(r(y^* - D) - g(y^*, D) - cD - p(s^* + g(y^*, D)) - \rho b^*, D).
\]

**Proof.** Utilizing (1) and (2) with \(b_n = b^*, s_n = s^*,\) and \(y_n = y^*\) for \(n = 1, 2, \ldots,\) if \(n > 1\) then \(x_n = y^* - D_{n-1}\) which causes \(z_n = y_n - x_n = y^* - (y^* - D_{n-1}) = D_{n-1}\) and

\[
v_n = w_n + rx_n - p(w_n) - s^* - cz_n - \rho b^*.
\]
From (5), if \( n > 1 \) then
\[
w_n = s^* + g(y^*, D_{n-1}).
\]
Substitution in (21) yields
\[
v_n = r(y^* - D_{n-1})^{-} + g(y^*, D_{n-1}) - cD_{n-1} - p(s^* + g(y^*, D_{n-1})) - \rho b^*.
\]

Here, the myopic policy (which is base-stock) stipulates that the firm produces just enough to raise the product stock to a target level \( y^* \) and issues dividends to bring the cash stock to a target level \( s^* \). Therefore, production just offsets the consumption of goods or demand, and the realized dividend equals the revenue net of production/inventory costs, default penalty and loan interest payment. It is well known that production exactly offsets the most recent demand in a dynamic newsvendor model without a capacity constraint. However, the joint distribution of production and dividend, and the probability distribution of the dividend are new results. A consequence of Proposition 3.5 is that each period’s dividend would be a reflection of the previous period’s actual financial performance (not its expected value) which depends on operational decisions \( y^* \), operations structure \( g(\cdot, \cdot) \), market characteristics \( r \), market events \( D \), and financial characteristics \( p(\cdot) \) and \( \rho \).

We assume for each \( x \) that the maximum on the right side of (13) is achieved, say at \((b^*, s^*, y^*) = (b(x), s(x), y(x))\). Of course, if \( x \leq y^* \), then Proposition 3.4 implies that \((b^*, s^*, y^*)\) achieves the maximum. If follows from definition (2) of \( s_n \) that in a period with inventory level \( x \) and retained earnings \( w \), the optimal dividend is
\[
v(x, w) = w + rx^* - p(w) + cx - cy(x) - s(x) - \rho b(x).
\]

**Proposition 3.6** Suppose \( x \leq y^* \) and \( w \geq 0 \). (a) \( v(\cdot, w) \) is differentiable respectively on \((-\infty, 0)\) where its derivative is \(-(r - c)\) and on \((0, y^*)\) where its derivative is \( c \). (b) If \( p(w) = 0 \) when \( w \geq 0 \), then \( v(x, \cdot) \) is differentiable on \([0, \infty)\) where its derivative is 1.

**Proof.** If \( x \leq y^* \), then (22) becomes
\[
v(x, w) = w - p(w) + rx^* + cx - cy^* - s^* - \rho b^*.
\] If \( w \geq 0 \) then \( p(w) = 0 \).}

Therefore, if the inventory level changes from \( x \) to \( x + \Delta I \leq y^* \) and retained earnings is \( w + \Delta E \) instead of \( w \), the dividend change is \( \Delta E + c \Delta I \).
3.4 Pecking Order Optimality

We now show that \( b(x) = (-s(x))^+ \) is optimal for all \( x \). That is, it is optimal to borrow the smallest amount that satisfies the liquidity constraint. This is consistent with the well known “pecking order” in finance which advises a firm to resort to internal equity before it borrows externally.

Proposition 3.7 In (13), \( b(x) = (-s(x))^+ \) is optimal for all \( x \in \mathbb{R} \).

**Proof.** From definition (10) and (15),

\[
\sup_{b,s} \left\{ -(1-\beta)s - [\beta r + (1-\beta)c]y + \beta E[r(y-D)^+ + g(y,D) - p(s + g[y,D])] - \rho b : \right. \\
\left. b \geq 0, b+s \geq 0 \right\} \\
= -[\beta r + (1-\beta)c]y + \beta E[r(y-D)^+ + g(y,D)] + \sup_s \{ \sup_b (-\rho b : b \geq 0, b+s \geq 0) \} \\
- \beta E(p[s + g(y,D)]) \\
= \sup_s \{ -\rho(-s)^+ - (1-\beta)(s + cy) - \beta ry + \beta E[r(y-D)^+ + g(y,D) - p(s + g[y,D])] \}.
\]

4 Characterization of the Optimal Policy

This section characterizes the myopic optimal policy and provides an explicit solution for the following case in which the default penalty and the gross profit from sales (net of inventory costs) are piece-wise linear functions:

\[
p(x) = (p_1 x)^- 
\]

\[
g(y,d) = r\min\{y,d\} - \gamma(d - y)^+ - h(y - d)^+
\]

It is convenient to rewrite the second equation as

\[
g(y,d) = ry + \gamma(y - d) - (r + h + \gamma)(y - d)^+
\]

Here, \( p_1 > 0 \) is the rate of default penalty. Notice that \( p(\cdot) \) is decreasing and convex, and \( g(\cdot, d) \) is concave. These properties are essential in this section which has the following organization. We exploit piece-wise linearity to specialize (10), and then examine the respective cases in which
To compute the expected default penalty cost (26), note that the density function of demand is given by \( s \), when demand matches supply. Shortly we discuss the importance of this term depending on the sign of \( s \). The challenging component of (25) is the expected default penalty. The following expansion shows how to determine it:

\[
K(b, s, y) = -(1 - \beta)s - [\beta r + (1 - \beta)c]y + \beta E[r(y - D)^+ + ry + \gamma(y - D) - (r + h + \gamma)(y - D)^+] - p_1(s + ry + \gamma(y - D) - (r + h + \gamma)(y - D)^+)^- - \rho b
\]

\[
= -(1 - \beta)s + [\beta \gamma - (1 - \beta)c]y - \rho b + \beta \gamma y - \beta\gamma E[D]
\]

\[
- \beta E[(h + \gamma)(y - D)^+ + p_1(s + ry + \gamma(y - D) - (r + h + \gamma)(y - D)^+)^-]
\]

\[
= K(s, y) \equiv -(1 - \beta)s + [\beta \gamma - (1 - \beta)c]y - \rho(-s)^+ - \beta\gamma E(D)
\]

\[
- \beta E[(h + \gamma)(y - D)^+ + p_1(s + ry + \gamma(y - D) - (r + h + \gamma)(y - D)^+)^-]
\]

(25)

The challenging component of (25) is the expected default penalty. The following expansion shows how to determine it:

\[
E[p_1(s + ry + \gamma(y - D) - (r + h + \gamma)(y - D)^+)^-]
\]

\[
= \begin{cases} 
  p_1(\int_{s+\gamma y}^{\infty} (\gamma x - s - (r + \gamma)y) f(x) \, dx & \text{if } s + ry > 0; \\
  p_1(\int_{s+\gamma y}^{\infty} (x - y) f(x) \, dx + (r + h) \int_{0}^{\gamma y} (y - x) f(x) \, dx) & \text{if } s + ry = 0; \\
  p_1(-s - ry + \gamma \int_{0}^{\infty}(x - y) f(x) \, dx + (r + h) \int_{0}^{\gamma y} (y - x) f(x) \, dx) & \text{if } s + ry < 0.
\end{cases}
\]

(26)

To compute the expected default penalty cost (26), note that

\[
s + ry \geq 0 \quad \text{if and only if} \quad \frac{hy - s}{r + h} = y - \frac{s + ry}{r + h} \leq y \leq \frac{s + (r + \gamma)y}{\gamma} = y + \frac{s + ry}{\gamma}.
\]

(27)

Thus, if \( s + ry \geq 0 \),

\[
(s + ry + \gamma(y - D) - (r + h + \gamma)(y - D)^+)^-
\]

\[
= \begin{cases} 
  \gamma D - s - (r + \gamma)y & \text{if } D > (s + (r + \gamma)y)/\gamma; \\
  0 & \text{if } (hy - s)/(r + h) < D \leq (s + (r + \gamma)y)/\gamma; \\
  hy - s - (r + h)D & \text{if } D \leq (hy - s)/(r + h),
\end{cases}
\]

and if \( s + ry \leq 0 \),

\[
[s + ry + \gamma(y - D) - (r + h + \gamma)(y - D)^+]^-
\]
4 CHARACTERIZATION OF THE OPTIMAL POLICY

\[
= \begin{cases} 
-\left[ s + ry + \gamma(y - D) \right] & \text{if } D > y; \\
-\left[ s + ry - (r + h)(y - D) \right] & \text{if } D \leq y.
\end{cases}
\]

Let \( \Gamma(s, y) \) be the set of values of demand \( D_n \) which do not precipitate default in period \( n + 1 \) if \( s_n = s \) and \( y_n = y \), namely \( \Gamma(s, y) = \{ d : s + g(y, d) > 0 \} \). Let \( q(s, y) \) be the probability that default does not occur in period \( n + 1 \) if \( s_n = s \) and \( y_n = y \), i.e., \( q(s, y) = P\{ D \in \Gamma(s, y) \} \).

Since \( s + ry \) is the level of the retained earnings when demand matches supply, it is the maximum amount of earnings the firm can achieve, and each unit of inventory will reduce that amount by \( r + h \) while each unit of backorder will reduce the amount by \( \gamma \). It follows that if \( s + ry < 0 \), then the firm will default with certainty regardless of demand realizations; that is, \( \Gamma(s, y) = \emptyset \) and \( q(s, y) = 0 \). If \( s + ry = 0 \), then any amount of inventory or supply shortfall will result in default; so default does not occur if and only if supply matches demand perfectly. In this case, \( \Gamma(s, y) = \{ y \} \). Finally, if \( s + ry > 0 \), then the firm will remain solvent as long as neither the inventory nor the backorder is too high; that is, if the demand is between \( (hy - s)/(r + h) \) and \( (s + (r + \gamma)y)/\gamma \). In this case, \( \Gamma(s, y) = \{ d : (hy - s)/(r + h) < d \leq (s + (r + \gamma)y)/\gamma \} \) and \( q(s, y) = F[(s + (r + \gamma)y)/\gamma] - F[(hy - s)/(r + h)] \). If demand is lower than the lower limit in \( \Gamma(s, y) \), then default is precipitated by insufficient revenue and high holding costs. If demand is above the upper limit, then default is caused by the penalty cost for the demand in excess of supply and the delay in receiving revenue for that excess.

In the next two subsections, first we examine the case in which the firm does not have the borrowing option by constraining \( b = 0 \) in optimization (20) (“No Borrowing”), and then we look at the whole problem with the borrowing opportunity, namely, \( b \geq 0 \) (“With Borrowing”).

4.1 No Borrowing

This subsection analyzes the coordination problem when the firm’s owners impose a no-borrowing constraint. Some owners put a higher value on privacy of information, completeness of control, and avoidance of default risk than on profitability. Also, some venture capitalists insist on a “no-borrowing” constraint in return for providing funds to an entrepreneurial venture.

Proposition 3.7, namely \( b = (-s)^+ \), yields the cases \( b = 0 \) if \( s \geq 0 \) and \( b > 0 \) if \( s < 0 \). In this
subsection, we assume \( b = 0 \) so optimization (20) becomes

\[
K(0, s^*, y^*) = \sup \{ K(0, s, y) : s \geq 0, y \geq 0 \}. \tag{28}
\]

Since \( K(0, s, y) \) is concave in \( s \) and \( y \), there exist \( \lambda_1^* \) and \( \lambda_2^* \) so that the optimal solution satisfies the following conditions:

\[
\frac{\partial K(0, s^*, y^*)}{\partial s} + \lambda_1^* = 0 \tag{29}
\]

\[
\frac{\partial K(0, s^*, y^*)}{\partial y} + \lambda_2^* = 0 \tag{30}
\]

\[
\lambda_1^* s^* \geq 0, \quad \lambda_2^* y^* \geq 0, \quad y^* \geq 0, \quad \text{and} \quad s^* \geq 0.
\]

In order to specify the impacts on dividends of adjusting the working capital or the goods supply level, we differentiate \( K(0, s, y) \) with respect to \( s \) and \( y \). If \( p(\cdot) \) and \( g(\cdot, \cdot) \) are bilinear as in (23) and (24), then \( s \geq 0 \) and \( y \geq 0 \) imply \( s + ry \geq 0 \). Therefore, from (26),

\[
E[p(s + g[y, D])] = E[p_1(s + ry + \gamma(y - D) - (r + h + \gamma)(y - D)^+) - ]
\]

\[
= p_1 \left( \int_{(s+(r+\gamma)y)/\gamma}^{\infty} (\gamma x - s - (r + \gamma)y) f(x) dx + \int_0^{(hy-s)/(r+h)} (hy - s - (r + h)x) f(x) dx \right). 
\]

Thus, in (25),

\[
\frac{\partial K(0, s, y)}{\partial s} = -(1 - \beta) + \beta p_1 \left( 1 - F \left( \frac{s + (r + \gamma)y}{\gamma} \right) + F \left( \frac{hy - s}{r + h} \right) \right) \tag{29}
\]

\[
\frac{\partial K(0, s, y)}{\partial y} = -(1 - \beta) c + \beta \gamma - \beta(\gamma + h)F(y) + \beta p_1 (r + \gamma) \left( 1 - F \left( \frac{s + (r + \gamma)y}{\gamma} \right) \right) - \beta p_1 h F \left( \frac{hy - s}{r + h} \right) \tag{30}
\]

These impacts on dividends of adjusting the working capital level, \( s \), or the goods supply level, \( y \), are intuitive. The essential tradeoff is between the current dividend and the future retained earnings level which in turn will affect the probability of default. When we increase \( s \) by one dollar, the current dividend will decrease by one dollar while next period’s retained earnings will increase by one dollar and, hence, the cost of default will go down by \( p_1 \) dollars in the event of a default (with probability \( 1 - F[(s + (r + \gamma)y)/\gamma] + F[(hy - s)/(r + h)] \)). Therefore, equation (29) follows. The effects of increasing \( y \) on dividends and retained earnings are indirect and through revenues and costs. When we increase \( y \) by one unit, the current production cost increases by \( c \) dollars and, hence,
the current dividend goes down by $c$ dollars. On the other hand, next period’s retained earnings will increase by $\gamma$ dollars in the event that the unit would have been backordered otherwise (i.e., demand exceeds supply, with probability $1 - F(y)$) while it will decrease by $h$ dollars if the extra unit produced cannot be sold and must be held in inventory for one period (with probability $F(y)$).

Furthermore, in the event of default, extra unit revenue plus one unit of savings in backorder cost will decrease the default cost by $p_1(r + \gamma)$ if default is due to the upside risk of demand (with probability $1 - F[(s + (r + \gamma)y)/\gamma]$) while the extra holding cost will increase the default cost by $p_1h$ if default is due to the downside risk (with probability $F[(hy - s)/(r + h)]$). Equation (30) summarize the above tradeoffs with respect to a marginal change of $y$.

The following results are obtained by examining the aforementioned optimality conditions.

**Proposition 4.1** Suppose that the default penalty and gross profit functions are piece-wise linear as in (23) and (24).

1. With "no borrowing," i.e. (28), the solution can be determined as follows. Let $y_1^*$ be the solution to equation

\[-(1 - \beta)c + \beta \gamma - \beta(\gamma + h)F(y_1^*) + \beta p_1(r + \gamma) \left[1 - F\left(\frac{(r + \gamma)y_1^*}{\gamma}\right)\right] - \beta p_1 h F\left(\frac{hy_1^*}{r + h}\right) = 0 \tag{31}\]

Note that $y_1^*$ exists and is nonnegative if $\beta \gamma + \beta p_1(r + \gamma) \geq (1 - \beta)c$.

(a) If

\[1 - F\left(\frac{(r + \gamma)y_1^*}{\gamma}\right) + F\left(\frac{hy_1^*}{r + h}\right) > \frac{1 - \beta}{\beta p_1} \tag{32}\]

then $y^* = y_0^*$ and $s^* = s_0^*$ where $y_0^*$ and $s_0^*$ solves

\[(1 - \beta)(r - c) + \gamma - \beta(h + \gamma)F(y_0) - \beta p_1(r + \gamma + h)F\left(\frac{hy_0 - s_0}{r + h}\right) = 0 \tag{33}\]

and

\[1 - F\left(\frac{s_0 + (r + \gamma)y_0}{\gamma}\right) + F\left(\frac{hy_0 - s_0}{r + h}\right) = \frac{1 - \beta}{\beta p_1} \tag{34}\]

(b) If

\[1 - F\left(\frac{(r + \gamma)y_1^*}{\gamma}\right) + F\left(\frac{hy_1^*}{r + h}\right) \leq \frac{1 - \beta}{\beta p_1} \tag{35}\]

then the optimal solution is $y^* = y_1^*$ and $s^* = 0$.  


2. (a) $y^*$ increases if $\gamma$ or $\beta$ increases, or if $h$, $c$ or $p_1$ decreases.

(b) $s^*$ increases if $\gamma$, $h$, $\beta$ or $p_1$ increases.

**Proof.** Part 2 follows by applying the theorem of implicit functions to the optimality conditions.

From this proposition, the optimal production/inventory policy has the same structure as in a profit-maximizing firm without cash constraints, namely, a base-stock policy. That is, the production quantity in each period is determined so that the total available goods is kept at the level $y^*$. However, the optimal base-stock level for a dividend-maximizing firm with cash constraints might differ from the one prescribed by a standard inventory model because over-production not only implies higher holding costs or lower backorder costs but also affects the possibility and the cost of default. In addition, an optimal base-stock policy for working capital, $s^*$, regulates the balance between current dividends and future retained earnings.

Suppose that the optimal base-stock policy $(s^*, y^*)$ is determined in the two newsvendor formulas (33) and (34). Equation (34) sets the optimal cash stock level $s^*$ (given the goods stock level $y_0^*$), or equivalently, it sets the optimal no-default limits for demand, namely the upper limit $s^* + (r + \gamma)y^*/\gamma$ and the lower limit $(hy^* - s^*)/(r + h)$. If the demand is outside the two limits, then default will occur, so this might be thought of as the case of “overage” in a newsvendor model. One dollar increase in $s$ will reduce the dividend by one dollar in the current period while increasing the retained earnings by one dollar and decreasing the default cost by $p_1$ in the next period. Therefore, the “overage cost” is $C_o = \beta p_1 - (1 - \beta)$. On the other hand, if the demand is within the two limits, then default will not occur. So one dollar increase in $s$ simply decreases the current dividend by one dollar and increases the next period’s retained earnings by one dollar. Thus, the “underage cost” is $C_u = 1 - \beta$, and the optimal no-default limits for demand should be set so that the default probability equals the “critical ratio”:

$$\frac{C_u}{C_u + C_o} = \frac{1 - \beta}{\beta p_1} = \frac{\alpha}{p_1}$$

where $\alpha \equiv (1 - \beta)/\beta$ is the discount rate.

Rewriting the critical ratio as the ratio of the discount rate to the default cost rate gives a very intuitive interpretation for formula (34). Think of a one dollar increase in working capital as deferring
a one dollar dividend payment to the next period. Therefore, the tradeoff is between saving $p_1$ dollars in the event of default versus losing $\alpha$ dollars in dividend regardless of whether default occurs or not, i.e., $C_o = p_1 - \alpha$ and $C_u = \alpha$. Equation (33) sets the optimal goods stock level $y^*$ in a similar fashion. Notice that a unit increase in $y^*$ would reduce the default cost by $p_1(r + \gamma)$ in the event that demand is higher than $s^* + (r + \gamma)y^*/\gamma$ (the upside risk) and increase the default cost by $p_1h$ if the demand is lower than $(hy^* - s^*)/(r + h)$ (the downside risk). These are in addition to the usual tradeoffs in a standard inventory model because the probability of default is already set by (34) (see §4.2 for more detailed comparisons with the standard newsvendor inventory model).

Three considerations play an implicit role in sensitivity analysis although they are not model parameters: current dividend, future retained earnings, and default cost. An increase in production reduces the current dividend due to the associated production cost while, if demand is high but not too high, it may increase retained earnings at the beginning of next period through successful sales (reduced backorder cost) and reduced default cost due to lower backorder cost. On the other hand, an increase in production may decrease the retained earnings at the beginning of next period and increase default costs if unsold goods must be held in inventory. So (i) an increase in current dividend or retained earnings induces an increase in the optimal product base-stock level, $y^*$, (ii) an increase in default cost due to inventory induces a decrease in $y^*$, and (iii) an increase in default cost due to backorder induces an increase in $y^*$.

The sensitivity analysis with respect to $\gamma$, $h$, $c$ or $p_1$ is obvious because an increase in $\gamma$ increases the next period’s retained earnings in case of supply shortage while it reduces the default cost due to backorder. An increase in $h$ not only reduces retained earnings but also increases the default cost due to inventory, an increase in $c$ reduces the current dividend, and an increase in $p_1$ is just an increase in unit default cost. Thus, one would expect the optimal product stock, $y^*$, to increase when $\gamma$ increases, or when $h$, $c$, or $p_1$ decreases. Finally, a higher discount factor, $\beta$, implies that the future is more important, and hence, the optimal product base-stock level is larger because it is a cash generator for the future.

On the other hand, an increase in current dividend or retained earnings induces a decrease in the optimal cash base-stock, $s^*$; an increase in default costs induces an increase in $s^*$. However, the sensitivity analysis of $s^*$ is complicated by the fact that one must also consider the effects of possible
4 CHARACTERIZATION OF THE OPTIMAL POLICY

changes in $y^*$ on the default probability. Thus, we are able to show the monotonicity of $s^*$ only with respect to $\beta$, $\gamma$, $h$ and $p_1$. Intuitively, an increase in the discount factor $\beta$ implies that the future is more important, and hence, the optimal cash stock (as a cash reserve for the future) should be larger.

4.2 With Borrowing

In this subsection, the firm is allowed to borrow, so $b \geq 0$, and we solve

$$K(b^*, s^*, y^*) = \sup \{ K(b, s, y) : b \geq 0, b + s \geq 0, y \geq 0 \}.$$ (36)

The Kuhn-Tucker condition is as follows:

$$\frac{\partial K(b^*, s^*, y^*)}{\partial s} + \lambda_1^* = 0,$$

$$\frac{\partial K(b^*, s^*, y^*)}{\partial y} + \lambda_2^* = 0,$$

$$\frac{\partial K(b^*, s^*, y^*)}{\partial b} + \lambda_1^* + \lambda_3^* = -\rho + \lambda_2^* + \lambda_3^* = 0,$$

$$\lambda_2^* y^* = \lambda_1^* (b^* + s^*) = 0, \quad y^* \geq 0, \quad \text{and} \quad \lambda_3^* b^* = 0.$$

Note that $\lambda_1^* + \lambda_3^* = \rho$ implies that at least one of the $\lambda_i^*$’s is positive. Therefore, either $b^* = 0$ or $b^* + s^* = 0$. So, the firm’s borrowing policy is either not to borrow or to borrow just enough to bring the cash level to zero before the realization of revenue and inventory costs. In effect, this is an alternative proof of pecking-order optimality (Proposition 3.7) when there are piece-wise linear functions for the default penalty and the gross profit from sales (net of inventory costs).

We again consider the transformed single-period payoff as given in (25). It follows from (26) that

$$\frac{\partial K}{\partial s} = -(1 - \beta) + \left\{ \begin{array}{ll} \frac{\beta p_1}{\beta p_1} (1 - F[(s + (r + \gamma)y)/\gamma] + F[(hy - s)/(r + h)]) & \text{if } s + ry > 0; \\ \frac{\beta}{\beta p_1} & \text{if } s + ry < 0, \end{array} \right.$$ (37)

and

$$\frac{\partial K}{\partial y} = -(1 - \beta)c + \beta \gamma - \beta (\gamma + h)F(y)$$

$$+ \left\{ \begin{array}{ll} \frac{\beta p_1}{\beta p_1} ((r + \gamma)(1 - F[(s + (r + \gamma)y)/\gamma]) - hF[(hy - s)/(r + h)]) & \text{if } s + ry > 0; \\ \frac{\beta}{\beta p_1} (r + \gamma - (r + \gamma + h)F[y]) & \text{if } s + ry < 0. \end{array} \right.$$ (38)

Note that $K$ is not differentiable at $s + ry = 0$, and (37) and (38) provide left- and right-derivatives at that point. Recall that if $s + ry > 0$, then the firm will not default if demand is within the limits
(hy - s)/(r + h) < D < s + (r + γ)y)/γ. On the other hand, s + ry < 0 represents a situation where the internal working capital, s, and/or the total goods supply, y, are set so low that the firm will default with certainty. The effects of changing the base-stock level y on the objective are different in the two cases. In both cases, a unit increase in the base-stock level y will reduce the default cost by \( p_1(r + γ) \) if the default is due to the upside risk and increase the default cost by \( p_1h \) if the default is due to the downside risk, i.e., when demand is outside the band of no default or \( D \not\in Γ(s, y) \). However, an increase in y has an additional effect on the default cost in the latter case (s + ry < 0). Since the firm will default regardless of demand’s realization, one more unit of supply will generate r more dollars of sales and save γ more dollars of backorder cost and, hence, reduce the default penalty by \( p_1(r + γ) \) dollars as long as the unit can be sold (with a probability of 1 − F(\( y^* \))). On the other hand, one more unit of supply will increase the default cost by \( p_1h \) dollars when the unit cannot be sold (with probability of F(\( y^* \))).

Proposition 4.2 Suppose that there are piece-wise linear default costs and the firm may borrow if it wishes, so the objective function is (25), and that \( βγ ≥ (1 − β)c \).

1. The solution to (36) can be determined as follows. Let \( y_1^* \) solve equation (31).

(a) If (32) holds, then \( y^* = y_0^* \) and \( s^* = s_0^* \) where \( y_0^* \) solves (33) and \( s_0^* \) solves (34).

(b) If

\[
\frac{1 − β − ρ}{βp_1} ≤ 1 − F \left( \frac{(r + γ)y_1^*}{γ} \right) + F \left( \frac{hy_1^*}{r + h} \right) \leq \frac{1 − β}{βp_1},
\]

then the optimal solution is \( y^* = y_1^* \) and \( s^* = b^* = 0 \).

(c) If

\[
1 − F \left( \frac{(r + γ)y_1^*}{γ} \right) + F \left( \frac{hy_1^*}{r + h} \right) < \frac{1 − β − ρ}{βp_1},
\]

then:

1. If

\[
\frac{1 − β − ρ}{βp_1} < 1,
\]

\( y^* = y_2^* \), \( s^* = s_2^* \) and \( b^* = −s^* \) where \( y_2^* \) and \( s_2^* \) solve

\[
(1 − β)(r − c) + γ − (r + γ)ρ − β(γ + h)F(y_2^*) − βp_1(r + γ + h)F \left( \frac{hy_2^* − s_2^*}{r + h} \right) = 0
\]
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\[ 1 - F \left( \frac{s^*_2 + (r + \gamma)y^*_2}{\gamma} \right) + F \left( \frac{hy^*_2 - s^*_2}{r + h} \right) = \frac{1 - \beta - \rho}{\beta p_1}. \]  

(43)

ii. If

\[ \frac{1 - \beta - \rho}{\beta p_1} \geq 1, \]  

(44)

then \( y^* = y^*_3 \) where \( y^*_3 \) solves

\[ F(y^*_3) = \frac{\beta \gamma - (1 - \beta)c + \beta p_1 (r + \gamma)}{\beta (\gamma + h) + \beta p_1 (r + \gamma + h)}, \]  

(45)

\( s^* = -ry^* \) and \( b^* = ry^* \) if (44) holds in equality, otherwise \( s^* = -\infty \) and \( b^* = \infty \).

2. Suppose that (41) holds, i.e., \((y^*, s^*) \) is \((y^*_0, s^*_0)\), \((y^*_1, 0)\), or \((y^*_2, s^*_2)\). Then,

(a) \( y^* \) increases when \( \gamma \) or \( \beta \) increases, or when \( h, c \) or \( p_1 \) decreases.

(b) \( s^* \) increases when \( h, \beta, p_1 \) or \( \rho \) increases, or when \( r \) decreases. \( s^* = s^*_0 \) increases in \( \gamma \) while \( s^* = s^*_2 \) decreases in \( \gamma \).

(c) \( b^* \) increases when \( r \) or \( \gamma \) increases or when \( h, \beta, p_1 \) or \( \rho \) decreases.

When the firm may borrow, the previous proposition asserts that the optimal policy again consists of a cash base-stock level \( s^* \) and a product base-stock level \( y^* \). Also, if short-term borrowing is optimal, the firm should borrow just enough to bring the total working capital level \( (b^* + s^*) \) to zero. Note that if \( y^* \) is \( y^*_0, y^*_1, \) or \( y^*_2 \), then \( s^* + ry^* > 0 \) implies that the optimal loan is less than the value of the product base-stock \( (b^* = (-s^*)^+ < ry^*) \), and default does not occur even when inventory or backorder is at some positive level. In this case, we show that all the sensitivity analysis results in Proposition 4.1 carry over and the additional sensitivity analysis results are quite intuitive. Obviously, when the loan rate \( \rho \) is high, the firm will set the cash stock level high to reduce the need to borrow. A higher \( h, \beta \) or \( p_1 \), or a lower \( r \) or \( \gamma \) implies a higher optimal cash stock level, \( s^* \), and, hence, a smaller loan is needed to bring the total working capital to zero. It is interesting to observe that the firm would finance the cash needs for demand risk by internal capital (raising \( s^* \)) if there is an increase in inventory cost while by external funding or borrowing (raising \( b^* \)) if the backorder cost increases.

However, when \( y^* = y^*_3 \), some sensitivity analysis results do not hold. When \( y^* = y^*_3 \) and (44) holds in strict inequality, \( s^* + ry^* < 0 \) implies that the optimal loan is more than the value of the
product base stock \((b^* = -s^* > ry^*)\) and default will occur with probability one. An increase in \(y\) will have no effect on the default probability. Consequently, all results in part 2 of Proposition 4.2 remain true except one, which will change to: \(y^*\) is increasing in \(p_1\) for \(y^* = y_3^*\). However, lenders in reality are unlikely to grant a larger loan than the value of the product base stock.

Also, Corollary 3.1 has the following implication for the sensitivity analysis of demand when the assumptions of Proposition 3.2 are valid. If demand increases in the sense of first-order stochastic dominance, then \(y^*\) increases, \(s^*\) decreases and \(b^*\) increases.

### 4.3 Comparison with a Standard Newsvendor Inventory Model without Cash Constraints

Now we compare the operating decisions that are determined with financial considerations with those that are determined without them. For simplicity, we base the comparison on the case in §4.2 where the firm cannot make short-term loans. Consider a standard dynamic newsvendor production/inventory model in which we maximize the expected present value of the profits (without cash constraints). For a fair comparison, we assume that the unit revenue \(r\), production cost \(c\) and inventory cost \(h\) remain the same. We also assume that the revenue net of inventory-related costs is realized in the next period because in our model the dividend paid in a period is based on realization of the revenue and inventory cost in the previous period. That is, the firm maximizes the expected value of

\[
\Pi = \sum_{n=1}^{\infty} \beta^{n-1} \left[ rx_n - cz_n + \beta g(y_n, D_n) \right]
\]

subject to the nonnegativity constraint for the production quantity \(z_n\). From \(z_n = y_n - x_n\) and \(x_n = y_{n-1} - D_{n-1}\) \((n > 1)\), the objective function can be rewritten as

\[
\Pi = r x_1^- + c x_1 + (\beta r - c) \sum_{n=1}^{\infty} \beta^n D_n + \sum_{n=1}^{\infty} \beta^{n-1}[-(1 - \beta) cy_n - \beta r y_n + \beta r (y_n - D_n)^{+} + \beta g(y_n, D_n)].
\]

Therefore, the optimal base-stock level in the standard model must satisfy the following newsvendor formula:

\[
F(\tilde{y}^*) = \frac{\beta \gamma - (1 - \beta)c}{\beta(\gamma + h)}.
\] (46)

Incidentally, if the default penalty \((p(\cdot))\) and financial constraints \((b_n \geq 0; b_n + s_n \geq 0)\) are ignored, \(\tilde{y}^*\) can also be obtained by choosing \(y\) to maximize \(K\) in (10). Thus, \(\tilde{y}^*\) can be viewed as the production
decision without the considerations of financial costs and constraints regardless of what criterion is used. It is straightforward to obtain the following results by comparing the optimality conditions in (31), (33), (42), (45) and (46).

**Proposition 4.3** Suppose that \( \bar{y}^* \) is the base-stock level in the traditional inventory model determined in (46).

1. If \( y^* = y^*_0 \) or \( y^*_1 \) or \( y^*_2 \), then \( \bar{y}^* \) may be greater than, less than or equal to \( y^* \), and, furthermore, \( \bar{y}^* \leq y^* \) if and only if
   \[
   hF \left( \frac{hy^* - s^*}{r + h} \right) \leq (r + \gamma) \left[ 1 - F \left( \frac{s^* + (r + \gamma)y^*}{\gamma} \right) \right].
   \]

2. If \( y^* = y^*_3 \), then \( \bar{y}^* < y^* \).

Therefore, the traditional inventory model may set the base-stock level too high, too low or just right. The reason is that the traditional model fails to consider the effects of the base-stock level on the expected default cost due to mismatch between supply and demand (see equations (30) and/or (38)). One unit increase in the base-stock level \( y \) has two effects on the expected default cost: the positive effect of reducing the cash shortage due to supply shortage and the negative effect of increasing the cash shortage due to supply surplus. In (47), the left-hand side, \((r + \gamma)(1 - F[(s^* + (r + \gamma)y^*)/\gamma])\), measures the expected marginal increase in cash position for an additional unit of base stock in the event of a supply shortage, and the right-hand side, \(hF[(hy^* - s^*)/(r + h)]\), measures the expected marginal cash loss in the event of a supply surplus. If the positive effect exceeds the negative effect, the optimal base stock should be set higher than that the traditional inventory model suggests, and if the opposite is true, the optimal base stock should be lower. The traditional inventory model sets the correct stock level if and only if the two effects on the default costs cancel out, i.e.,

\[
(r + \gamma) \left[ 1 - F \left( \frac{s^* + (r + \gamma)y^*}{\gamma} \right) \right] = hF \left( \frac{hy^* - s^*}{r + h} \right).
\]

### 4.4 Numerical Examples

We illustrate the solution via the following numerical examples. We first consider an example in which the time period is a year and then we show how results differ if the length of a period is
shortened, i.e., if there are several time periods per year. Suppose that \( r = \$8 \), \( c = \$5 \), \( \gamma = \$6 \) and \( h = \$4 \) per unit per year, \( \beta = 0.8333 \) (the discount rate, or opportunity cost of capital, is \( 1/\beta = \alpha = 0.2 \)), \( \rho = 0.12 \), and each year’s demand is normally distributed with mean 100 units and standard deviation 30 units. Notice that there is an incentive to borrow because the opportunity cost of capital is higher than the interest rate on loans, i.e., \((1/\beta - 1) - \rho = 0.2 - 0.12 > 0\). We let the unit default cost \( p_1 \) vary from \$0.06 per dollar default to \$20 per dollar default. This range of \( p_1 \) will cover all five solution scenarios mentioned in Proposition 4.2.

Figures 1 and 2 show how the optimal product base-stock level and cash base-stock level vary as the unit default penalty \( p_1 \) increases from 0.06 to 20. Figure 3 shows the dependence of the global maximum of the transformed single-period payoff (the function \( K \)) on the unit default penalty. First, we consider the case with the borrowing opportunity. For \( p_1 \in [0, 0.056) \), the optimal product base-stock level is \( y^*_3 \) which increases from 100 to 101.914 (the firm borrows as much as possible). For \( p_1 \in [0.056, 4.472) \), the optimal product stock level is \( y^*_2 \) which decreases from 101.914 to 98.319 and the optimal cash base-stock level is \( s^*_2 \) which increases from \(-815.889\) to \(0\) (the borrowing amount decreases from \(815.889\) to \(0\)). When \( p_1 \) further increases, borrowing is no longer optimal. For \( p_1 \in [4.472, 18.00) \), the optimal product base-stock level is \( y^*_1 \) which decreases from 98.319 to 94.012 and the optimal cash base-stock level remains at \(0\). For \( p_1 \in [18.00, 20) \), the optimal product base-stock level is \( y^*_0 = 94.012 \) and the optimal cash base-stock level is \( s^*_0 \) which increases from \(0\) to \(13.87\).

If there is no borrowing opportunity, it is intuitive that the optimal policy should be the same if it is optimal not to borrow in the presence of the borrowing opportunity. This is the case when \( p_1 \geq 4.472 \). For \( p_1 \in [0.06, 4.472) \), the optimal product base-stock level is \( y^*_1 \) which decreases from 99.976 to 98.319 and the optimal cash base-stock level remains at \(0\). Also, the payoff gain due to the borrowing opportunity could be as high as \$27.58 per period when \( p_1 = 0.056 \) (for any smaller \( p_1 \), the problem becomes unbounded). This is about 11.03% of the potential contribution margin per period assuming that supply matches demand perfectly because \( \beta(r - c)\mu = \$250 \).

Note that the optimal base-stock level in the standard inventory model without financial considerations is \( \bar{y}^* = 100.00 \). When there is no borrowing, \( \bar{y}^* \) is always higher than the optimal product base-stock level and it is the limit case when \( p_1 \) approaches zero (\( y^*_1 = \bar{y}^* = 100.00 \) when \( p_1 = 0 \)).
With borrowing, $\bar{y}^*$ coincides with our optimal product base-stock level in two cases. The first is the trivial case when there is no default penalty ($y_3^* = \bar{y}^* = 100.00$ when $p_1 = 0$). The second case occurs when the unit default penalty cost $p_1 = 0.195$. In this case, $y^* = y_3^* = \bar{y}^* = 100.00$. Otherwise, $\bar{y}^*$ is either too high or too low. In particular, it could be 6.3% higher than the optimal level, $y^* = y_0^* = 94.012$ when $p_1 \geq 18.00$.

We also demonstrate in the numerical examples that the largest profit loss does not necessarily arise from the incorrect product base stock level; rather it comes from the lack of coordination between the product stock and cash stock. Suppose we use the standard product base-stock level of $\bar{y}^* = 100.00$ in the above numerical example and compute the payoff loss in two cases. First, suppose we employ the optimal cash base-stock level $s^*$; that is, although the production decision is made in isolation, the firm makes the correct financial decisions. The payoff loss, measured as a percentage of the potential contribution margin, ranges from 0.09% (when $p_1 = 0.06$) to 0.96% (when $p_1 = 30$). In the second case, we use a fixed cash base-stock level, for example, $s = 0$. This represents the case when the firm makes financial decisions without considering production and/or default costs. Then the payoff loss ranges from 11% at one extreme (when $p_1 = 0.06$) to 3.5% at the other extreme (when $p_1 = 30$).

We generate a model in which controls are exercised on a quarterly basis by changing the following parameter values: $\gamma = $1.5 and $h = $1 per unit per period, $\beta = 0.9524$ (discount rate $\alpha = 0.05$), $\rho = 0.03$, and each quarter’s demand is normally distributed with mean 25 units and standard deviation 15 units. The unit revenue and unit production cost remain unchanged, i.e., $r = $8 and $c = $5. We let the unit default cost, $p_1$, vary from $0.0185 per dollar default to $10 per dollar default. In this range of unit default cost, the optimal product base-stock level, $y^*$, decreases from 26.098 to 24.248 and the optimal cash base-stock level, $s^*$, increases from $-208.83 to 146.98. Now, the optimal base-stock level in the standard inventory model is $\bar{y}^* = 25.00$, about 3% higher than the optimal level $y^* = y_0^* = 24.498$ when $p_1 = 10$. If the firm uses a base-stock level of $\bar{y}^* = 25.00$ instead of the optimal $y^*$ while making the correct financial decisions with $s^*$, the highest payoff loss is only 0.05% (when $p_1 = 0.0185$). On the other hand, if the firm uses a fixed cash base-stock level, for example, $s = 0$, then the payoff loss could go as high as 3.5% at one extreme (when $p_1 = 0.0185$) and 41.7% at the other extreme (when $p_1 = 10$).
In §1.2 we define the financial value of coordination, $\Delta$, as the difference between the EPV (expected present value) of dividends (net of capital subscriptions) of a firm which coordinates its financial and operational decisions, and of a corresponding firm which decentralizes those decisions and makes the latter ones to maximize the EPV of profits. The relative value of coordination divides $\Delta$ by the EPV of dividends of the decentralizing firm.

We use the definition $K_C$ for the value of the function $K$ in (25) plus $\beta(\beta r - c)E(D)$, when it is evaluated at $(s,y) = (s^*_2,y^*_2)$. Note that $K_C/(1 - \beta)$ is the present value of the expected dividends $B$ defined in (8) evaluated at the optimal coordination solution when $x_1 = w_1 = 0$. For simplicity of exposition, the term $\beta(\beta r - c)D_n$ was separated in (9) from the terms whose conditional expectations become the function $K$ in (10). Similarly, we define $K_D$ for the value of the function $K$ in (25) plus $\beta(\beta r - c)E(D)$, when it is evaluated at $(s,y) = (0,\bar{y}^*)$. The subscripts $C$ and $D$ are mnemonics for “coordinated” and “decentralized,” respectively. The following parameters initiate a series of examples (in which short-term loans may be made) in which the relative values become arbitrarily large: $x_1 = w_1 = 0, \beta = 0.6, r = 10, c = 1, h = 0.5, \gamma = 0.8, \rho = 0.03, p_1 = 1.1$, and demand is uniformly distributed on the interval $[0,1]$.

With these parameters, the optimal coordinated decision is $(s,y) = (s^*_2,y^*_2) = (-4.9,0.52)$, the decentralized base-stock level is $\bar{y}^* = 0.1026$, $K_C = 1.54$, and $K_D = 0.54$, so the relative value is $1.85$. If the other parameters remain invariant while $\beta$ increases toward $0.6185$, $K_D$ diminishes to 0 and $K_C$ slowly increases toward 1.62. So the relative value “explodes.” That is, there are parameter settings where the decentralization EPV (the denominator) can be driven to zero while $\Delta$ (the numerator) and the coordination EPV remain strongly positive.

5 Bankruptcy Variations

Thus far, bankruptcy in the model is consistent with Chapter 11 and generates substantial costs of reorganizing the firm. In the first subsection we consider an extreme alternative version of bankruptcy that is consistent with Chapter 7, namely dissolution of the firm. In the second subsection we insert the option for the firm to declare bankruptcy even if it is not forced into it.
5 BANKRUPTCY VARIATIONS

5.1 Wipeout Bankruptcy

Here we maximize the expected present value of dividends prior to dissolution. There are three important insights in this section. First, key features of an optimal policy with “reorganization” bankruptcy remain valid with “wipeout” bankruptcy. These are the myopic optimum property and Propositions 3.1, 3.4, 3.5, and 3.7 (with minor changes). Second, the firm should be more shortsighted because its discount factor is reduced from \( \beta \) to \( \beta q(s, y) \) (recall that \( q(s, y) \) is the probability that bankruptcy does not occur in period \( n + 1 \) if \( s_n = s \) and \( y_n = y \)). In effect, the firm’s choices influence its time preference as well as conversely. Third, the life-time of the optimally operated firm has a geometric probability distribution.

Let \( T \) denote the lifetime of the firm, so we let \( T = \sup \{ n : w_n > 0 \} \) and maximize \( E[B] \) where \( B = \sum_{n=1}^{T} \beta^{n-1} v_n \). The same substitutions that lead from (8) to (9) yield

\[
B = \sum_{n=1}^{T} \beta^{n-1} v_n = r x_1^- + c x_1 + w_1 - \beta^{T-1} (s_T + c y_T + \rho b_T) + \sum_{n=1}^{T-1} \beta^{n-1} \left( \beta r (y_n - D_n) + \beta g(y_n, D_n) + \beta (r - c) D_n - (1 - \beta) s_n - [\beta r + (1 - \beta)c] y_n - \rho b_n \right)
\]

Therefore, an optimal coordinated operating and financial policy maximizes \( E(B_0) \) where

\[
E(B_0) = E \left[ \sum_{n=1}^{T-1} \beta^{n-1} \left( \beta r (y_n - D_n) + \beta g(y_n, D_n) + \beta (r - c) D_n - (1 - \beta) s_n - [\beta r + (1 - \beta)c] y_n - \rho b_n \right) - \beta^{T-1} (s_T + c y_T + \rho b_T) \right]
\]

This model can be regarded as a generalization of an inventory process with a stopping time. Therefore, the results in Lovejoy (1992) yield bounds on the error that would result from using the policy identified in §3 rather than a policy that optimizes (49). However, we avoid the need for an approximation by showing that the model with wipeout bankruptcy satisfies the condition in Sobel (1981) and, therefore, has an optimal myopic solution.

Recall the notation (early in §4) \( \Gamma(s, y) \) for the set of values of demand \( D_n \) which do not precipitate bankruptcy in period \( n + 1 \) if \( s_n = s \) and \( y_n = y \), namely \( \Gamma(s, y) = \{ d : s + g(y, d) > 0 \} \). Also, recall the notation \( q(s, y) \) for the probability that bankruptcy does not occur in period \( n + 1 \) if \( s_n = s \) and \( y_n = y \), i.e., \( q(s, y) = P\{ D \in \Gamma(s, y) \} \). Then

\[
E(B_0) = \sum_{n=1}^{\infty} \beta^{n-1} \left( -P[T = n] E[s_T + c y_T + \rho b_T | T = n] + P[T > n] E \left[ \beta r (y_n - D_n) + \right] \right)
\]
\[ \begin{align*}
&+ \beta g(y_n, D_n) + \beta (r - c)D_n - (1 - \beta)s_n - [\beta r + c(1 - \beta)]y_n - \rho b_n \vert T > n \bigg] \\
= & \sum_{n=1}^{\infty} \beta^{n-1} \left\{ - \prod_{k=1}^{n-1} q(s_k, y_k)[1 - q(s_n, y_n)](s_n + cy_n + \rho b_n) + \prod_{k=1}^{n} q(s_k, y_k)E[\beta r(y_n - D_n)^+] \\
&+ \beta g(y_n, D_n) + \beta (r - c)D_n - (1 - \beta)s_n - [\beta r + c(1 - \beta)]y_n - \rho b_n \vert T > n \right\} \\
= & \sum_{n=1}^{\infty} \beta^{n-1} \prod_{k=1}^{n-1} q(s_k, y_k)K_0(b_n, s_n, y_n)
\end{align*} \tag{50} \]

where

\[ K_0(b, s, y) = q(s, y)E[\beta r(y - D)^+] + \beta g(y, D) + \beta (r - c)D - (1 - \beta)s \\
- [\beta r + c(1 - \beta)y] - \rho b[D \in \Gamma(s, y)] - (s + cy + \rho b)(1 - q(s, y)). \tag{51} \]

When bankruptcy signified reorganization, the objective was (11). The only difference between (50) and (11) is that the single-period discount factor has been reduced from \( \beta \) to \( \beta q(s, y) \). It follows that dynamic program (13) remains valid when (14) is replaced with

\[ J(b, s, y) = K_0(b, s, y) + \beta q(s, y)E[\psi(y - D) \vert D \in \Gamma(s, y)]. \tag{52} \]

Therefore, \textit{the myopic optimum property and Propositions 3.1, 3.4, 3.5, and 3.7 remain valid (with minor changes) when bankruptcy signifies dissolution of the firm.}

The preceding observation leads to a testable hypothesis, namely a geometric probability distribution for the lifetime of the firm. Let \((s^*, y^*)\) globally maximize \( K_0((-s)^+, s, y) \) subject to \( y \geq 0 \).

If \( x_1 \leq y^* \), it is optimal for \((b_n, s_n, y_n) = ((-s^*)^+, s^*, y^*)\), \( n = 1, \ldots, T \). Therefore,

\[ P\{T = n\} = [1 - q(s^*, y^*)]q(s^*, y^*)^{n-1}. \]

\textbf{Proposition 5.1} \textit{If} \( x_1 \leq y^* \), \textit{it is optimal for} \((b_n, s_n, y_n) = ((-s^*)^+, s^*, y^*)\), \( n = 1, \ldots, T \), \textit{and, as a consequence,} \( T \) \textit{has a geometric distribution with parameter} \( q(s^*, y^*) \).

\subsection*{5.2 The Option to Declare Bankruptcy}

The declaration of bankruptcy in reality is complex due to the minutia of judicial decisions based on the U.S. Bankruptcy Code, the relative seniority of the firm’s various financial obligations, the status and liquidity of the markets in which the firm’s assets would be disposed, and other considerations.
So a serious model of the decision to declare bankruptcy should be structured very differently than the models in §2 or §5.1. In this subsection we take a small step in that direction in order to consider the effects on the coordination of operations and finance in a setting where the option to declare bankruptcy lurks in the background. We certainly do not propose its use to plan the declaration of bankruptcy.

Let the decision variable \( \delta_n = 0 \) if the firm chooses to continue operations in period \( n \) (if it has persisted until then), and let \( \delta_n = 1 \) if it chooses to declare bankruptcy in period \( n \). We redefine \( T \) as the last period in which the firm operates “normally” in the sense that \( \delta_n = 0 \) and \( w_n > 0 \), \( n = 1, \ldots, T \), and either \( w_{T+1} \leq 0 \) or \( \delta_{T+1} = 1 \): \( T = \sup \{ n : \delta_n = 0 \text{ and } w_n > 0 \} \).

“Looting the till” here would consist of making a large loan and distributing a large (positive) dividend in anticipation of declaring bankruptcy the following period and “walking away” from the default. In order to preclude this behavior, we impose an upper bound \( U + cx^- \) on the loan. The rationale for \(+cx^-\) is that the firm is obliged (in the model) to produce at least enough goods in period \( n \) to meet its backlogged demand.

Let \( \mathcal{L} \) be the dollar amount of obligations at bankruptcy that are senior to the equity interests, and recall the notation \( w_n \) for the retained earnings at the beginning of period \( n \). If bankruptcy is declared in period \( n \), the shareholders’ limited liability implies that they receive \((w_n - \mathcal{L})^+\). Thus, we maximize \( E[\sum_{n=1}^{T} \beta^{n-1}v_n + \beta^T (w_n - \mathcal{L})^+] \). For simplicity, we forego an expansion and rearrangement analogous to (48) and (49) and observe that the optimization here corresponds to a dynamic program in which the state is a pair consisting of the retained earnings and inventory level, and the action is the four-tuple consisting of the decision to declare bankruptcy or not and the amounts of the short-term loan, the residual retained earnings, and the order-up-to inventory level.

The dynamic program is \( \psi_{N+1}(\cdot, \cdot) \equiv 0 \) and for each \( n = 1, 2, \ldots, N, x \in \mathcal{R}, \) and \( w \in \mathcal{R}, \psi_n(w, x) = 0 \) if \( w \leq 0 \) and

\[
\psi_n(w, x) = \max \{(w - \mathcal{L})^+, \max_{b, s, y} \{ J_n(b, s, y) : y \geq (x)^+, b \geq 0, b + s \geq 0, b \leq U + cx^- \} \} \quad \text{if } w > 0 \tag{53}
\]

\[
J_n(b, s, y) = w - p(w) + rx^- + cx - cy - s - pb + \beta q(s, y) E(\psi_{n+1}[s + g(y, D), y - D] | D \in \Gamma(s, y)) \tag{54}
\]
The term \((w - L)^+\) is the payoff from electing bankruptcy, and (53) is the expected present value of not electing bankruptcy. The first seven terms on the right side of (54) are the dividend if bankruptcy is not elected, because \(v_n = w_n + rx_n - p(w_n) + cx_n - cy_n - s_n - \rho b_n\). The eighth term is the expected present value of continuation, beginning next period; the arguments of \(\psi_{n+1}\) are due to \(w_{n+1} = s_n + g(y_n, D_n)\) and \(x_{n+1} = y_n - D_n\).

This dynamic program can be viewed as an optimal stopping problem in which auxiliary decisions \((b, s, \text{ and } y)\) are made while continuation occurs and there are some forced stopping states (all pairs \((w, x)\) where \(w \leq 0\)). Another optimal stopping problem with auxiliary decisions is Babich and Sobel (2004).

For economy of exposition, in the remainder of this subsection we assume that \(p(\cdot)\) is a decreasing convex function on \(\mathbb{R}\) and \(g(\cdot, d)\) is a concave function on \(\mathbb{R}\) for each \(d \geq 0\). Then an optimal policy in (53) has the following generalization of the monotone optimal stopping property (Derman and Sacks, 1960). For each inventory level \(x\), there is a critical level of retained earnings \(w_n(x)\) such that it is optimal to declare bankruptcy \((\psi_n(w, x) = (w - L)^+)\) at all states \((w, x)\) with \(w \leq w_n(x)\). This property stems from an analogue of Proposition 3.2 that is valid for (53) and (54).

Although (53) does not have a myopic optimal policy as in Proposition 3.4, it shares the important pecking-order property of the model in earlier sections (Proposition 3.7). So it is optimal to let \(b = (-s)^+\) in (53). Also, as in earlier sections, the optimal choice of \(s\) and \(y\) in (53) is determined by two order-up-to levels that are functions, \(s_n(w)\) and \(y_n(w)\), rather than scalars as before. That is, if it is optimal to continue rather than declare bankruptcy, then the production/procurement quantity should be \(z = y - x = y_n(w) - x\) and the residual retained earnings should be \(s = \min\{s_n(w), U + cx^-\}\). The residual retained earnings level is \(s_n(w)\) if it is feasible, i.e., if \(s_n(w) \leq U + cx^-\), and otherwise it is as close to \(s_n(w)\) as feasible.

6 Nonnegative Dividends

Since large publicly traded firms have limited liability stockholders, they cannot ordinarily obtain capital subscriptions. So in this section we briefly analyze the model when capital subscriptions are precluded, i.e., with the additional constraint that dividends must be nonnegative. Since we add the constraint \(v_n \geq 0\) to the formulation of the model in §2, the objective remains (11) with the

6 Nonnegative Dividends

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constraints (12) augmented with

\[ s_n + cy_n + \rho b_n \leq w_n + r x_n - p(w_n) + cx_n \]

because \( v_n = w_n + r x_n - p(w_n) - s_n - cy_n + cx_n - \rho b_n \). One consequence is an additional state variable.

The dynamic program for the model in §2 has a scalar state, the inventory level \( x \), which must be augmented now with the amount of retained earnings at the beginning of a period, \( w \). Instead of (15) and (16), the corresponding dynamic program is the following recursion with \( \psi_{N+1}(\cdot, \cdot) \equiv 0 \):

\[
\psi_n(w, x) = \max_{b, s, y} \{ J_n(b, s, y) : y \geq (x)_{+}, b \geq 0, b + s \geq 0, \\
\text{s.t. } s + cy + \rho b \leq w + rx - p(w) + cx \} 
\]

(55)

\[
J_n(b, s, y) = K(b, s, y) + \beta E(\psi_{n+1}[s + g(y, D), y - D]) 
\]

(56)

for each \( n = 1, 2, \ldots, N \), \( x \in \mathbb{R} \), and \( w \in \mathbb{R} \).

Let \( b_n(w, x) \), \( s_n(w, x) \), and \( y_n(w, x) \) be the optimal values of \( b \), \( s \), and \( y \), respectively in (55). The following result corresponds to Proposition 3.7 and is proved in the same fashion. Again, it is optimal to borrow the smallest amount that satisfies the liquidity constraint

**Proposition 6.1** \( b_n(w, x) = (s_n(w, x))_{+} \) is optimal for all \( n = 1, 2, \ldots \) and \( (w, x) \in \mathbb{R}^2 \).

The next result is analogous to the concavity property in Proposition 3.2 and has a similar proof.

**Proposition 6.2** If \( p(\cdot) \) is a decreasing convex function on \( \mathbb{R} \) and \( g(\cdot, d) \) is a concave function on \( \mathbb{R} \) for each \( d \geq 0 \), then the value function in (55), \( \psi_n(\cdot, \cdot) \), is a concave function on \( \mathbb{R}^2 \) and \( J_n(\cdot, \cdot, \cdot) \) in (56) is a concave function on \( \mathbb{R}^3 \) for each \( n \).

Even with the nonnegativity of dividends in force, the next result shows that the optimal decision variables remain monotone (cf. Proposition 3.3).

**Proposition 6.3** Under the assumptions of Proposition 6.2, for each \( n \), \( y_n(w, x) \), \( z_n(w, x) = x - y_n(w, x) \), \( v_n(w, x) \) and \( s_n(w, x) \) are increasing with respect to \( w \in \mathbb{R} \) and \( x \in \mathbb{R} \). So \( b_n(w, x) \) is a decreasing function of \( w \) and \( x \).

**Proof.** Adapt Theorem 8-4 in Heyman and Sobel (2003, p.378).  

The following result compares the optimal policy when capital subscriptions are allowable with the optimal policy when dividends are constrained to be nonnegative. Recall the notation $b_n(x), s_n(x),$ and $y_n(x)$ for the optimal amounts of the short-term loan, residual retained earnings, and physical goods base-stock level when the inventory level is $x$ and $n$ periods remain in the planning horizon.

**Proposition 6.4** Under the assumptions of Proposition 6.2, for each $n, x,$ and $w,$

- $y_n(x, w) \leq y_n(x)$ and $s_n(x, w) \leq s_n(x)$ (so $b_n(x, w) \geq b_n(x)$);
- As $w$ grows, $y_n(w, x) \to y_n(x)$ and $s_n(w, x) \to s_n(x)$ (so $b_n(w, x) \to b_n(x)$).

**Proof.** Combine the observation that constraint (56) becomes less binding as $w$ grows with an adaptation of Theorem 8-4 in Heyman and Sobel (2003, p.378).

Therefore, a firm which optimally coordinates its operational and financial decisions but cannot mandate capital subscriptions has lower inventories and higher short-term loans than its counterpart which may obtain capital subscriptions if it wishes. Therefore, each period the former firm has a higher probability of default than the latter. The latter firm can turn to a capital subscription or a short-term loan, whereas the former firm can increase liquidity only with the loan, so its level of residual retained earnings is lower and its short-term loan is higher. Similarly, without recourse to capital subscriptions, the former firm has a lower base-stock level of physical goods because it is less prone to buy or produce goods.

**7 Concluding Remarks**

We formulate and analyze a dynamic stochastic model of the coordination of operational and financial decisions with the criterion of the expected present value of the time stream of dividends received by a representative share owner. The model is surprisingly tractable because it has a myopic policy that is optimal and, therefore, susceptible to analysis. So we can contrast the result with inventory policies based on models without financial considerations. Numerical examples show that the opportunity cost of detaching the two functions, measured in dividends, can be significant. Since most of the paper employs a model of Chapter 11 bankruptcy and admits capital subscriptions, we also explore the


In the new economics of industrial organization there has been a tendency to emphasize the game theoretic study of market competition outside of the firm and agency problems caused by asymmetric information within the firm. In operations research, production and inventory systems are often studied in isolation from the other aspects of firm activity, by cutting out the feedback between production and finance or production and marketing (until recently). In the first instance, this ignores financial constraints and in the second instance it prespecifies the nature of the demand structure as a given of the model. In the actual application of formal mathematical models to the policy of a firm, the optimal value of a game theoretic model of competition remains unknown until we can analyze the associated dynamic game. Meanwhile, in contrast, production/inventory models are of value for many practical problems, even taken in isolation. Nevertheless, we observe here that for firms operating with thin budgets in new or otherwise volatile markets, the joining together of the production/inventory problem with the cash flow and financial problems of the firm may be relevant and worthwhile. Furthermore, we have demonstrated that although to do so is somewhat complicated, it is feasible.

Our results were obtained with a finite-horizon model, but they carry over to the analogous infinite-horizon model. Under reasonable assumptions, as $N \to \infty$ the functions in (15) and (16) (and in (55) and (56)) tend to well-behaved limits that satisfy the obvious infinite-horizon analog of (15) and (16) (and (55) and (56)). If $x_1 \leq y^*$, Proposition 3.5 states that the pairs of dividends and production quantities in successive periods (starting with the second period) are independent and identically distributed random vectors. Within a period, the dividend and production quantity are not independent.

A natural extension of this work is to add the pricing decisions of the firm as an important way of trying to correct inventory and cash flow problems. We leave this extra complication to a future investigation. The modern firm is a highly complex multiproduct institution. Even with modern computational methods, quantitative strategic analysis is at best crude. We suggest here that there is still the potential for considerable value-added investigation internal to the firm and the model investigated here is offered as an example to support this observation.
8 References


