DELTA-SIGMA MODULATION IN SINGLE NEURONS

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ABSTRACT
In this paper we investigate the relationship between the pulse-coded signal representation found in biological neurons and signals represented by first-order delta-sigma bit-streams. We show that a first-order delta-sigma bit-stream is a digital equivalent to a pulse-coded neural signal. By increasing the clock frequency, a delta-sigma bit-stream can approach a pulse-coded representation with arbitrary accuracy.

1. INTRODUCTION

Biological neurons communicate with each other by spikes or pulse trains. The information is not encoded by the shape of the pulses, but by the arrival time and the correlation with other pulses. This is known as pulse-frequency-coding. It is hard to know why evolution have chosen this communication concept but a very strong feature is its low sensitivity to interfering noise. Several papers have pointed out that there may be a connection between pulse-frequency-coding and $\Delta - \Sigma$ noise-shaping. In [1] the authors argue for that an integrate-and-fire neuron posses a behavior similar to a $\Delta - \Sigma$ modulator. The relationship is not shown mathematically, but the authors presents a successfully working Hopfield network where each neuron is implemented by a $\Delta - \Sigma$ modulator. In [2], [3] the concept of spatial noise-shaping is introduced. Here it is shown that by using inhibitory coupling between neighborly neurons, an increase in signal-to-noise ratio results. The resulting noise is shaped and therefore suppressed for lower frequencies.

In this paper we focus on the behavior of a single integrate-and-fire neuron where the accumulator is completely reset after firing. We presents a mathematical analysis showing that, by disregarding the shape of the biological pulse, the output of a first-order $\Delta - \Sigma$ modulator is a digital equivalent to a pulse-coded neural signal. In Chapter 2 we present a new model for pulse frequency coding based on standard frequency modulation. This model is in Chapter 3 compared to a first-order $\Delta - \Sigma$ modulator. In Chapter 4 different features of the model is further analyzed by comparing it to a first-order frequency delta-sigma modulator (FDSM).

2. NEURAL PULSE-CODING MODELED AS STANDARD FREQUENCY MODULATION

Consider an integrate-and-fire neuron where the various synaptic input signals are integrated in the cell body (Fig. 1). When the accumulated cell voltage exceeds the action potential, the cell voltage is reset to zero and a pulse is emitted trough the axon. By looking at a single synaptic input, the axon pulse frequency will be modeled as proportional to $\alpha$ where $\alpha$ represents the input stimulus signal. $\alpha$ could be thought of as a low-pass filtered epresentation of the pulse-coded signal from a synapsing neuron. Since the axon pulse frequency is proportional to the input signal, the mean time between pulses will be given by

$$t_\alpha = \frac{K}{\alpha}, \quad (1)$$

where $K$ is a constant modeling the strength of the synaptic input. In the following we will disregard the phase (initial condition) of the pulse-coded signal and consider Eq. 1 to represent the main feature of the neuron. In general, a frequency modulated signal [4] can be represented by

$$b(t) = \sin(2\pi \int_0^t ka(\tau)d\tau), \quad (2)$$

where $a(\tau)$ is the modulating signal and $k$ is the frequency sensitivity. In this model the carrier frequency is included in $a(\tau)$ assuming $a(\tau) > 0$. By the use of the $\text{sqr}$ function

$$\text{sqr}(x) = \begin{cases} 1 & : x \geq 0 \\ 0 & : x < 0 \end{cases}, \quad (3)$$

the frequency modulated signal is then hard-limited as shown in Fig. 2b-c. By derivating this signal with respect to time and applying the absolute value as illustrated in Fig. 2d-y we have a sequence of impulses (Dirac delta pulses)

$$y(t) = \left| \frac{d}{dt} \text{sqr}(\sin(2\pi \int_0^t ka(\tau)d\tau)) \right|, \quad (4)$$

Fig. 1. Top: A single neuron. Bottom: The pulse frequency coded axon signal.
where the spacing between each impulse is modulated by the input signal. The impulse sequence represents the essence of the pulse-coded signal by representing only the position of each pulse where the pulse shape is of no significance. In Fig. 3 the generation of \( y(t) \) is shown schematically. The time between impulses can be found from Eq. 4 by replacing the input signal \( a(t) \) by the constant \( a \)

\[
y(t) = \left| \frac{d}{dt} \sqrt{\sin(2\pi kat)} \right|.
\]

Each zero-crossing of the term \( \sin(2\pi kat) \) represents an impulse, and the time between impulses will be

\[
x \approx \frac{1}{2ka},
\]

which is in accordance with the main feature of the neuron defined by Eq. 1.

**3. COMPARISON TO FIRST-ORDER \( \Delta - \Sigma \) MODULATION: MODEL A**

In Fig. 4 a standard discrete-time first-order \( \Delta - \Sigma \) modulator [5] is shown. The quantizer is represented by the quantizing function \( q(\cdot) \). The output of this circuit can be expressed as

\[
y_n = q \left( \sum_{k=0}^{n} a_{k-1} - \sum_{k=0}^{n} y_{k-1} \right) \quad (7)
\]

By defining the quantizing function \( q(x) \) as

\[
q(x) \overset{\text{def}}{=} x - \text{frac}(x),
\]

we have

\[
y_n = q \left( \sum_{k=0}^{n} a_{k-1} - \sum_{k=0}^{n} y_{k-1} \right).
\]

By collecting the output terms on the left side, we have

\[
\sum_{k=0}^{n+1} y_{k-1} = q \left( \sum_{k=0}^{n} a_{k-1} \right),
\]

and by inserting this expression into Eq. 9 we arrive with

\[
y_n = q \left( \sum_{k=0}^{n} a_{k-1} - \sum_{k=0}^{n+1} y_{k-1} \right).
\]

By applying the definition of \( q(x) \) we have

\[
y_n = \sum_{k=0}^{n} a_{k-1} - \text{frac} \left( \sum_{k=0}^{n} a_{k-1} \right) - \sum_{k=0}^{n+1} y_{k-1}.
\]

This may be rearranged providing the following description of the \( \Delta - \Sigma \) modulator output signal

\[
y_n = a_n - \left( \text{frac} \left( \sum_{k=0}^{n} a_{k-1} \right) - \text{frac} \left( \sum_{k=0}^{n-1} a_{k-1} \right) \right).
\]

We notice that the output signal is given by the input signal \( a_n \) plus a differentiatized quantization error term. Assuming \( 0 \leq a_n < 1 \), Eq. 13 may be broken into two different cases. **Case I:** The integer part of both sum terms in Eq. 13 are equal. In this case the \( \text{frac}(\cdot) \) functions may be disregarded as they both subtract the same integer from their arguments. The output will always be zero.

\[
y_n = a_n - \left( \sum_{k=0}^{n} a_{k-1} - \sum_{k=0}^{n-1} a_{k-1} \right) = 0
\]

**Case II:** The integer part of the first sum term is given by the integer part of the last sum term plus one. In this case the first \( \text{frac}(\cdot) \) function will reduce its argument by one integer more than the last modulo function. The output will always be one

\[
y_n = a_n - \left( \left( \sum_{k=0}^{n} a_{k-1} - 1 \right) - \sum_{k=0}^{n-1} a_{k-1} \right) = 1.
\]
In other words; the output will be one each time the integer part of the first sum term changes. The mean time between impulses can be found from Eq. 13 by replacing the input signal $a_n$ by the constant $a$

$$y_n = a - \left( \frac{\text{frac}(na) - \text{frac}((n-1)a)}{a} \right)$$  

(16)

The mean number of output zero samples between each output impulse will be $1/a$, and the mean time between each output impulse is

$$\bar{t}_a = \frac{T_s}{a}$$  

(17)

where $T_s$ is the clock period. This is in accordance with the main feature of the neuron defined by Eq. 1. Since the output of the $\Delta - \Sigma$ modulator is sampled (quantized in time), we will have a maximum time-quantization error given by

$$\delta t_{\text{max}} = 2T_s$$  

(18)

By letting the clock frequency approach infinity, this error will approach zero, and by keeping the fraction $T_s/a$ constant by simultaneously decreasing $a$, the $\Delta - \Sigma$ output signal will approach an exact digital representation of the pulse coded signal. In Fig. 5 the input/output signals are illustrated for low clock frequencies. In Fig. 6 (top) the power spectral density of the $\Delta - \Sigma$ modulator is shown for a clock frequency of 8kHz, where the average number of zeros between each impulse is 2. The input signal was a single sinusoidal signal at 12Hz. In Fig. 6 (bottom) the clock frequency is increased to 0.8MHz keeping the fraction $T_s/a$ constant. The excess noise at high frequencies is pattern noise [5]. The modulator was simulated in C++. By doubling the oversampling ratio in a first-order $\Delta - \Sigma$ modulator, the signal-to-quantization-noise-ratio will increase by 9dB. However, by keeping the fraction $T_s/a$ constant, each doubling of the oversampling ratio will only increase the signal-to-quantization-noise-ratio by $9 - 6 = 3\text{dB}$ as the signal power is halved.

This equation can be implemented by the circuit shown in Fig. 7. This circuit is in literature referred to as a frequency

4. COMPARISON TO FREQUENCY $\Delta - \Sigma$ MODULATION: MODEL B

By replacing the accumulator in Eq. 11 with a continuous-time integrator we have

$$y_n = q \left( \frac{nT_s}{T_s} \int a(\tau) d\tau \right) - q \left( \frac{(n-1)T_s}{T_s} \int a(\tau) d\tau \right).$$  

(19)

This equation can be implemented by the circuit shown in Fig. 7. This circuit is in literature referred to as a frequency

Fig. 5. $\Delta - \Sigma$ modulator input/output signals.

Fig. 6. Simulated $\Delta - \Sigma$ modulator output power spectral density. Top: $1/T_s=8kHz$. Bottom: $1/T_s=0.8MHz$

$\Delta - \Sigma$ modulator (FDSM) [6], [7], [8], and is mathematically equivalent to a first-order $\Delta - \Sigma$ modulator where the loop accumulator is replaced with a continuous-time integrator. The

Fig. 7. An implementation of Eq. 19.

FDSM utilize the fact that the phase of a FM signal is the integral of the modulating signal (Eq. 2). By the use of a zero-cross (ZC) counter the FM phase is detected and quantized providing the first term in Eq. 19. The result is then digitally differentiated producing $y_n$. In general the ZC counter word-length must be infinity, but by the use of modulo wrap-around arithmetic a one-bit word-length may be used [6]. A one-bit ZC counter can be implemented by a D flip-flop, and a one-bit modulo subtractor by an XOR gate resulting in the simple

Fig. 8. A practical FDSM.
circuit shown in Fig. 8. This circuit is a well documented

![Fig. 9. A mathematical FDSM equivalent.](image)

\( \Delta - \Sigma \) modulator, and it will for a proper input signal provide
the same output signals as shown in Chapter III. The FDSM

![Fig. 10. \( \Delta - \Sigma \) signals.](image)

operates by first integrating the input signal by the use of a
frequency modulator (Fig. 9, Fig. 10a-b). Then the FM signal
is fed through a D flip-flop which is equivalent to a sqf function
followed by a signal sampler (Fig. 9, Fig. 10c-d). The
resulting digital sequence is then differentiated and rectified
as illustrated in Fig. 9 and Fig. 10e-y. By comparing Fig. 10
to Fig. 2 we see that the FDSM is a digital equivalent to the
neural pulse code model presented in Chapter 2. The mean
time between impulses will be given by Eq. 6, and as a sampled
system, the maximum time-quantization error is given by
Eq. 18. In Fig. 11 the output signal is shown for two different
clock frequencies. We notice that as the clock frequency is
increased the output signal approaches an exact digital repre-
sentation of the continuous-time impulse sequence given by
Eq. 4.

![Fig. 11. \( \Delta - \Sigma \) modulator signals.](image)

We have presented a new model for pulse-frequency-coding.
Based on this model we have shown that a first-order \( \Delta - \Sigma \)
bit-stream is a digital equivalent to a pulse-coded neural sig-
mal. By increasing the clock frequency, a \( \Delta - \Sigma \) bit-stream

5. CONCLUSION

We have presented a new model for pulse-frequency-coding.
Based on this model we have shown that a first-order \( \Delta - \Sigma \)
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mal. By increasing the clock frequency, a \( \Delta - \Sigma \) bit-stream
can approach a pulse-coded representation with arbitrary ac-
curacy. Additive noise affecting the positions of the pulses is
noise-shaped.

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