A class of nonassociative algebras arising from quadratic ODEs

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Abstract. We present an algebraic part of our research on quadratic differential equations in 3- dimensional space, which posses a plane of critical points. Our goal is to provide a classification, up to an isomorphism, of those algebras which arise from above mentioned systems via Markus construction.

1. Introduction

It is well known, an even taught at undergraduate courses, that the stability of non-linear flow $\vec{x}' = \vec{f}(\vec{x})$ near the critical point $\vec{x}_0$ can often be determined from its linearization $D\vec{f}(\vec{x}_0)$. There exist however many system of ODEs, used in natural sciences, where this classical result can not be applied. In a homogeneous quadratic system $\vec{x}' = K(\vec{x})$ the origin $\vec{x}_0 = \vec{0}$ is always a critical point, but $DK(\vec{0}) = 0$ and so the stability of origin in such systems must be studied with new methods.

The first author who realized that the ideas of abstract algebra, ring theory in particular, can be used to study the solution of quadratic ODEs was Markus [7]. Applications of his ideas to the study of stability started with Kinyon and Sagle [6] who proved a fundamental Lemma: if a (real) algebra has a nonzero idempotent, the corresponding system of ODEs has a non-stable origin. Moreover, because of the result of Kaplan and York [4] who proved that a real algebra contains either a nonzero idempotent or nonzero nilpotent, it follows that any algebra with a stable origin must contain at least one nonzero nilpotent n of index 2 and hence the corresponding system of ODEs has a line of critical point, i.e. $\mathbb{R}^n$.

In two dimensional case this information is sufficient to classify all quadratic systems with stable origin (see [10] for example) just with a case by case inspection. Already in dimension 3 just the determination of all equivalence classes of systems with a line of critical points requires enormous amount of computational work and is not really feasible. Because of this reason, the study of one step simpler problem, namely systems with a plane of critical points, was tackled in [10, 11] in order to see how large percentage of algebras not satisfying the condition of Kinyon-Sagle Lemma give rise to stable origin. The present paper contains the algebraic analysis needed in [10,11] but omitted there.

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Our research seems to imply that quadratic systems with stable origin are very few in number. For this reason we believe that the next step in the research of stability of quadratic ODEs should be a second ring-theoretic lemma, in the spirit of Kinyon and Sagle, which would eliminate most algebras (containing nilpotents elements) in advance. Such result might keep the final case by case $\epsilon - \delta$ analysis to a manageable size that could be handled by a human.

The connection between real algebras and quadratic ODEs, given by Markus in [**7**] can be briefly described as follows. Every system of homogeneous quadratic differential equations in $\mathbb{R}^n$ can be written as $\vec{x}' = K(\vec{x})$, where $K : \mathbb{R}^n \to \mathbb{R}^n$ is a quadratic form. There exists a unique symmetric bilinear form $B : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ such that $K(\vec{x}) = B(\vec{x}, \vec{x})$ for every $\vec{x} \in \mathbb{R}^n$.

It is possible to equip $\mathbb{R}^n$ with a structure of a (nonassociative in general) commutative algebra $(\mathcal{A}, *)$ by defining $\vec{x} * \vec{y} = B(\vec{x}, \vec{y})$. For example a quadratic system

$$\begin{matrix} \dot{x} = a_1x^2 + 2b_1xy + c_1y^2 \\ \dot{y} = a_2x^2 + 2b_2xy + c_2y^2 \end{matrix} \; ; \; a_i, b_i, c_i \in \mathbb{R} \text{ for } i = 1, 2$$

gives rise to the following algebra

$*$	e_1	e_2
e_1	$a_1e_1 + a_2e_2$	$b_1e_1 + b_2e_2$
e_2	$b_1e_1 + b_2e_2$	$c_1e_1 + c_2e_2$

.

For a full survey of this theory the reader can consult for example [**16**], [**5**], [**6**] and [**12**]. Walcher's monograph [**16**] is also a standard reference for the state of art in 1990, with many references to older papers. Let us just mention that for the homogeneous systems of degree $m > 2$ one can also apply the Markus construction but the corresponding algebra is actually a $m-$ary algebra (see [**13**], [**14**] and [**16**]).

Since $K(\vec{0}) = \vec{0}$, the origin is always a critical point of system $\vec{x}' = K(\vec{x})$ and it is one of the most interesting questions to study its stability. The stability in the sense of Lyapunov roughly means that solutions which start near the origin remain near the origin for all subsequent times.

Applications of algebra to this problem are based on the following result of Markus (see [**7**, Th. 1], [**16**, p. 20]):

If $(\mathcal{A}_1, *)$ and $(\mathcal{A}_2, \star)$ are two commutative algebras modelled on $\mathbb{R}^n$ and $\Phi : \mathcal{A}_1 \to \mathcal{A}_2$ an isomorphism, then Φ maps the solutions of the system $\vec{x}' = \vec{x} * \vec{x}$ onto the solutions $\vec{x}' = \vec{x} \star \vec{x}$. Since Φ is a bounded linear map this implies that the qualitative properties, such as the stability of the origin, of both systems are the same.

This observation is a base for the classification of the systems of quadratic ODEs with a stable origin within a given class $\mathcal{C}_{\text{diff}}$. The approach is the following:

(i) Determine the class $\mathcal{C}_{\text{alg}}$ which corresponds to the class of $\mathcal{C}_{\text{diff}}$ via the Markus construction.

(ii) Classify the algebras from $\mathcal{C}_{\text{alg}}$ up to an isomorphism.

(iii) For every isomorphism class in $\mathcal{C}_{\text{alg}}$ pick the representative with the simplest multiplication table and form some estimates, of the $\varepsilon - \delta$ type, in order to show whether the origin is stable or not.

In our previous paper [**10**] we used the classification results but we did not provide any proofs, which are presented separately in our present paper.

In order to identify an algebraic problem, which we are to solve, we denote by $\mathcal{C}$ the class of homogeneous quadratic systems of ODEs in $\mathbb{R}^3$ which have a plane of critical point. The identification of the corresponding algebras, via the Markus construction, is given in the following result.

PROPOSITION 1. *A three dimensional real commutative algebra $\mathcal{A}$ corresponds to some system of quadratic ODEs from the class $\mathcal{C}$ if and only if there exists a basis $\{N_1, N_2, E\}$ in which the multiplication table is given by*

$\cdot$	N_1	N_2	E
N_1	0	0	$aN_1 + bN_2 + cE$
N_2	0	0	$\alpha N_1 + \beta N_2 + \gamma E$
E	$aN_1 + bN_2 + cE$	$\alpha N_1 + \beta N_2 + \gamma E$	$dN_1 + eN_2 + fE$

Table 1.

PROOF. From the above table it is obvious that all elements of the form $x = \alpha N_1 + \beta N_2$ are nilpotents, i.e. $x^2 = 0$. Since we are dealing with the system $\dot{x} = f(x) = x^2$, it is obvious that $f(\alpha N_1 + \beta N_2) = 0$ and therefore x is a critical point.

In order to prove the converse, we note that if N is a critical point, then $f(N) = N^2 = 0$. If N_1 and N_2 form a base of a plane of critical points, whose existence we assumed, then all elements of the form $\alpha N_1 + \beta N_2$ satisfy $(\alpha N_1 + \beta N_2)^2 = 0$. From this it follows easily that $N_1 \cdot N_1 = N_2 \cdot N_2 = N_1 \cdot N_2 = N_2 \cdot N_1 = 0$. □

The multiplication table from Proposition 1 appears to be very simple, but it still has enough room for 44 equivalence classes as we shall see in the sequel.

2. A first classification step

The classification of algebras from Proposition 1 will be carried on in three steps, because there are three unknown products. At the first step we form classes I, II, III and IV. At the second step each of them is further subdivided into I.1, I.2 etc. At the final step we use notation I.1.a, I.1.b etc.

In our first step we concentrate on the product $N_1 \cdot E$. We want to change a basis $\{N_1, N_2, E\}$ for a basis $\{N_1', N_2', E'\}$ such that $N_1' \cdot N_1'$, $N_2' \cdot N_2'$ and $N_1' \cdot N_2'$ would still be zero, while $N_1' \cdot E'$ would take a simpler form than $N_1 \cdot E$. Note again that all our algebras are commutative, thus $E' \cdot N_1' = N_1' \cdot E'$ automatically. It is obvious that the most general family of linear transformations preserving the conditions $N_1 \cdot N_1 = N_1 \cdot N_2 = N_2 \cdot N_2 = 0$ is defined by

$$
\begin{aligned}
N_1' &= AN_1 + BN_2, \\
N_2' &= CN_1 + DN_2, \\
E' &= FN_1 + GN_2 + HE,
\end{aligned}
\tag{2.1}
$$

where $H(AD - BC) \neq 0$. Let us begin with a systematic determination of products $N_1' \cdot E'$, $N_2' \cdot E'$ and finally $E' \cdot E'$. On every step just several calculations will be carried in full details. The omitted ones are very similar to those we perform below.

PROPOSITION 2. *For any choice of parameters a, b and c in Table 1 there exists a new basis N_1', N_2', E' in which the product $N_1' \cdot E'$ has exactly one of the following values*

- *I)* $N_1' \cdot E' = E'$,

- *II)* $N_1' \cdot E' = N_1'$,
- *III)* $N_1' \cdot E' = N_2'$,
- *IV)* $N_1' \cdot E' = 0$.

Proof. Let us begin with a new basis of type (2.1) for $B = 0$. Therefore $ADH \neq 0$. Let us compute, taking Table 1 into account,

$$\begin{aligned} N_1'E' &= AN_1(FN_1 + GN_2 + HE) \\ &= AH \cdot N_1E \\ &= AHaN_1 + AHbN_2 + AHcE. \end{aligned}$$

Then we have the following possibilities

If $c \neq 0$ then $N_1'E' = E'$ where $F = aAH$, $G = bAH$ and $A = \frac{1}{c}$.

If $c = 0$ and $b \neq 0$ then $N_1'E' = N_2'$ where $C = aAH$, $D = bAH$ and $A = H = 1$.

If $c = 0$ and $b = 0$ and $a \neq 0$ then $N_1'E' = N_1'$ where $A = 1$ and $H = \frac{1}{a}$. And finally, if $a = b = c = 0$ obviously $N_1'E' = 0$. □

We shall use the notation from the above theorem in the sequel and shall therefore speak about algebras of type I, II, III and IV.

3. A second classification step

In this section we intend to further subdivide all four types by studying the product $N_2 \cdot E$. We start with algebras of type I. From the previous section we know that Table 1 for them is the following

$\cdot$	N_1	N_2	E
N_1	0	0	E
N_2	0	0	$\alpha N_1 + \beta N_2 + \gamma E$
E	E	$\alpha N_1 + \beta N_2 + \gamma E$	$dN_1 + eN_2 + fE$

We again look for a basis in which N_2E will be simplified.

Proposition 3. *Let us consider an algebra of type I. For any choice of parameters α, β and γ there exists a new basis N_1', N_2', E' in which the product $N_2' \cdot E'$ has exactly one of the following values*

- *I.1)* $N_2' \cdot E' = N_1'$,
- *I.2)* $N_2' \cdot E' = N_2'$,
- *I.3)* $N_2' \cdot E' = 0$.

Proof. Let us begin with a new basis of type (2.1). First we ensure that the product $N_1E = E$ is preserved. Let us compute

$$\begin{aligned} N_1'E' &= AN_1(FN_1 + GN_2 + HE) + BN_2(FN_1 + GN_2 + HE) \\ &= AH \cdot N_1E + BH \cdot N_2E \\ &= AH \cdot E + BH \cdot (\alpha N_1 + \beta N_2 + \gamma E) \\ &= \alpha BHN_1 + \beta BHN_2 + (\gamma BH + AH)E. \end{aligned}$$

Hence the following equations must be satisfied

$$\begin{aligned} \alpha BH &= F \\ \beta BH &= G \\ \gamma BH + AH &= H \end{aligned}$$

So we are left with the following family of linear transformations:

$$\begin{aligned} N_1' &= (1-\gamma B)N_1 + BN_2, \\ N_2' &= CN_1 + DN_2, \\ E' &= \alpha BHN_1 + \beta BHN_2 + HE, \end{aligned}$$

where $H(D(1-\gamma B) - BC) \neq 0$. Let us compute $N_2'E'$:

$$\begin{aligned} N_2'E' &= CH \cdot E + DH \cdot (\alpha N_1 + \beta N_2 + \gamma E) \\ &= \alpha DH \cdot N_1 + \beta DH \cdot N_2 + (\gamma DH + CH) \cdot E. \end{aligned}$$

Let us first choose $C = -\gamma D$. Then we have:

If $\alpha + \beta\gamma \neq 0$ then $N_2'E' = N_1'$ where $B = \frac{\beta}{\alpha+\gamma\beta}$ and $D = \frac{1}{H(\alpha+\gamma\beta)}$. (We choose $H = 1$ to ensure $H(D(1-\gamma B) - BC) = \frac{1}{\alpha+\gamma\beta} \neq 0$.)

If $\alpha + \beta\gamma = 0$ and $\beta \neq 0$ then $N_2'E' = N_2'$ where $H = \frac{1}{\beta}$.

If $\alpha + \beta\gamma = 0$ and $\beta = 0$ then obviously $N_2'E' = 0$. □

Note, that by choosing $C = -\gamma D$ we can always avoid the case $N_2'E' = \mu E'$, even in the case $\alpha = \beta = 0$, $\gamma \neq 0$. For algebras of type II we get:

PROPOSITION 4. *Let us consider an algebra of type II. For any choice of parameters α, β and γ there exists a new basis N_1', N_2', E' in which the product $N_2' \cdot E'$ has exactly one of the following values*

- *II.1)* $N_2' \cdot E' = E'$,
- *II.2)* $N_2' \cdot E' = \beta N_2'$, *where* $\beta \neq 0$, $\beta \neq 1$,
- *II.3)* $N_2' \cdot E' = N_1' + N_2'$
- *II.4)* $N_2' \cdot E' = N_1'$.

PROOF. Let us begin with a new basis of type (2.1). First we ensure that the product $N_1E = N_1$ is preserved. Let us compute

$$\begin{aligned} N_1'E' &= AH \cdot N_1E + BH \cdot N_2E \\ &= AH \cdot N_1 + BH \cdot (\alpha N_1 + \beta N_2 + \gamma E) \\ &= (AH + \alpha BH) \cdot N_1 + \beta BHN_2 + \gamma BH \cdot E. \end{aligned}$$

Hence the following equation must be satisfied

$$\left.\begin{aligned} AH + \alpha BH &= A \\ \beta BH &= B \\ \gamma BH &= 0 \end{aligned}\right. \Rightarrow B = 0,\ H = 1.$$

So we are left with the following family of linear transformations:

$$\begin{aligned} N_1' &= AN_1, \\ N_2' &= CN_1 + DN_2, \\ E' &= FN_1 + GN_2 + E, \end{aligned}$$

where $AD \neq 0$. Let us compute $N_2'E'$:

$$\begin{aligned} N_2'E' &= C \cdot N_1 + D \cdot (\alpha N_1 + \beta N_2 + \gamma E) \\ &= (C + \alpha D) \cdot N_1 + \beta DH \cdot N_2 + \gamma D \cdot E. \end{aligned}$$

If $\gamma \neq 0$ then for $D = \frac{1}{\gamma}$, $G = \frac{\beta}{\gamma}$, $F = C + \frac{\alpha}{\gamma}$ we have $N_2'E' = E'$.

If $\gamma = 0$ and $\beta \neq 0$ and $\beta \neq 1$ then for $D = \beta - 1$, $C = \alpha$ we have $N_2'E' = \beta N_2'$.

If $\gamma = 0$ and $\beta = 1$ and $\alpha \neq 0$ then for $A = \alpha$, $D = 1$ we have $N_2'E' = N_2' + N_1'$. If $\gamma = 0$ and $\beta = 0$ then for $A = C + \alpha D$, we have $N_2'E' = N_1'$ for every α. $\square$

For algebras of type III we obtain the following result.

PROPOSITION 5. *Let us consider an algebra of type III. For any choice of parameters α, β and γ there exists a new basis N_1', N_2', E' in which the product $N_2' \cdot E'$ has exactly one of the following values*

- *III.1)* $N_2' \cdot E' = E'$
- *III.2)* $N_2' \cdot E' = kN_1' + N_2'; k \in \mathbb{R}$,
- *III.3)* $N_2' \cdot E' = N_1'$
- *III.4)* $N_2' \cdot E' = -N_1'$
- *III.5)* $N_2' \cdot E' = 0$

Its proof is rather similar to the proof of Proposition 3 and Proposition 4, so we omit details. For algebras of type IV we obtain the following result, which can also be proved in a similar way to Proposition 4.

PROPOSITION 6. *Let us consider an algebra of type IV. For any choice of parameters α, β and γ there exists a new basis N_1', N_2', E' in which the product $N_2' \cdot E'$ has exactly one of the following values*

- *IV.1)* $N_2' \cdot E' = E'$
- *IV.2)* $N_2' \cdot E' = N_2'$
- *IV.3)* $N_2' \cdot E' = N_1'$
- *IV.4)* $N_2' \cdot E' = 0$

REMARK 1. *It is quite obvious that (families of) algebras I.2 and II.1 are isomorphic. The same is true for algebras III.1 and I.1. Algebras I.3 and IV.1 are also isomorphic. The same is true for algebras III.5 and IV.3. The corresponding isomorphism in all four cases is the same. For example, let us consider the isomorphism Φ between algebra(s) I.2 (i.e. $\mathcal{A}_* = (\mathbb{R}^3, *)$) and II.1 (i.e. $\mathcal{A}_\star = (\mathbb{R}^3, \star)$) defined with the following multiplication tables:*

$*$	n_1	n_2	e
n_1	0	0	n_1
n_2	0	0	e
e	n_1	e	e^2

$\mathcal{A}_* = (\mathbb{R}^3, *)$

$\overset{\Phi}{\mapsto}$

$\star$	N_1	N_2	E
N_1	0	0	E
N_2	0	0	N_2
E	E	N_2	E^2

$\mathcal{A}_\star = (\mathbb{R}^3, \star)$,

where $e^2 = dn_1 + \varepsilon n_2 + fe$, $E^2 = \varepsilon N_1 + dN_2 + fE$ and d, ε, $f \in \mathbb{R}$. The isomorphism Φ is defined by the following:

$$N_1 \mapsto n_2$$
$$N_2 \mapsto n_1$$
$$E \mapsto e.$$

So, on the next step we will not consider cases II.1, III.1, IV.1 and IV.3 in order to avoid repetitions.

4. A final classification step

At the final classification step we intend to determine the simplest possible form of the last remaining product $E^2 = dN_1 + eN_2 + fE$ for all types from I.1 to IV.4.

Of course, the above constructed structure must be preserved in all cases, i.e. the products $N_1'N_1'$, $N_2'N_2'$ and $N_1'N_2'$ must remain zero in the new basis $\{N_1', N_2', E'\}$, and the products $N_1'E'$ and $N_2'E'$ must remain unchanged for all cases from I.1 to IV.4.

4.1. Algebras of class I. In the next three propositions we are going to determine the simplest possible form of product $E^2 = dN_1 + eN_2 + fE$ providing the products $N_1E = E$ and $N_2E \in \{N_1, N_2, 0\}$ are known and are of type I.

PROPOSITION 7. *Let $N_1^2 = N_2^2 = N_1N_2 = 0$, $N_1E = E$ and $N_2E = N_1$, i.e. the algebra is of type I.1. For any choice of parameters d, e and f there exists a new basis N_1', N_2', E' in which the product $E' \cdot E'$ has exactly one of the following values*

- *I.1.a)* $(E')^2 = mN_1' + nN_2' + E'$ $(m, n \in \mathbb{R})$,
- *I.1.b)* $(E')^2 = mN_1' + N_2'$ $(m \in \mathbb{R})$,
- *I.1.c)* $(E')^2 = N_1'$,
- *I.1.d)* $(E')^2 = -N_1'$,
- *I.1.e)* $(E')^2 = 0$.

PROOF. Let us begin with a new basis of type (2.1). First we ensure that the product $N_1E = E$ is preserved. The equation $N_1'E' = E'$ reads as

$$BH \cdot N_1 + 0 \cdot N_2 + AH \cdot E = F \cdot N_1 + G \cdot N_2 + H \cdot E$$

and yields $F = BH$, $G = 0$ and $A = 1$ (hence $H \neq 0$). So we are left with the following family of linear transformations:

$$\begin{aligned} N_1' &= N_1 + BN_2, \\ N_2' &= CN_1 + DN_2, \\ E' &= BHN_1 + HE. \end{aligned}$$

The condition $N_2'E' = N_1'$ reads as

$$DH \cdot N_1 + 0 \cdot N_2 + CH \cdot E = N_1 + B \cdot N_2 + 0 \cdot E$$

and yields $H = \frac{1}{D}$ $(D \neq 0)$, $C = 0$, $B = 0$. So we are left now with only one-parameter-family of linear transformations:

$$\begin{aligned} N_1' &= N_1, \\ N_2' &= DN_2 \\ E' &= \frac{1}{D}E, \quad \text{for } D \neq 0. \end{aligned}$$

Let us compute now $(E')^2$:

$$\begin{aligned} (E')^2 &= \frac{1}{D^2}E^2 \\ &= \frac{1}{D^2}(dN_1 + eN_1 + fE) \\ &= \frac{d}{D^2}N_1 + \frac{e}{D^2}N_2 + \frac{f}{D^2}E. \end{aligned}$$

We can conclude:

- if $f \neq 0$, $e \neq 0$ and $d \neq 0$ where $D = f$, $n = \frac{e}{f^3}$ and $m = \frac{d}{f^2}$ we have $(E')^2 = mN_1' + nN_2' + E'$,

- if $f \neq 0$, $e \neq 0$ and $d = 0$ where $D = f$ and $m = \frac{e}{f^3}$ we have $(E')^2 = mN_2' + E'$ and $m \neq 0$,
- if $f \neq 0$, $e = 0$ and $d \neq 0$ where $D = f$ and $m = \frac{d}{f^2}$ we have $(E')^2 = mN_1' + E'$ and $m \neq 0$,
- if $f \neq 0$, $e = 0$ and $d = 0$ where $D = f$ we have $(E')^2 = E'$,
- if $f = 0$, $e \neq 0$ and $d \neq 0$ where $D = e$ and $m = \frac{d}{e^2}$ we have $(E')^2 = mN_1' + N_2'$,
- if $f = 0$, $e \neq 0$ and $d = 0$ where $D = e$ we have $(E')^2 = N_2'$,
- if $f = 0$, $e = 0$ and $d > 0$ where $D = \sqrt{d}$ we have $(E')^2 = N_1'$,
- if $f = 0$, $e = 0$ and $d < 0$ where $D = \sqrt{-d}$ we have $(E')^2 = -N_1'$,
- and finally, if $f = 0$, $e = 0$ and $d = 0$ we obviously have $E^2 = 0$.

□

In a similar way we can treat types I.2 and I.3, so we omit the details. The results are summarized in the two propositions below.

PROPOSITION 8. *Let $N_1^2 = N_2^2 = N_1N_2 = 0$, $N_1E = E$ and $N_2E = N_2$, i.e. the algebra is of type I.2. For any choice of parameters d, e and f there exists a new basis N_1', N_2', E' in which the product $E' \cdot E'$ has exactly one of the following values*

- *I.2.a)* $(E')^2 = dN_1' + N_2' + fE'$, $d, f \neq 0$,
- *I.2.b)* $(E')^2 = 2N_1' + fE'$, $f \neq 0$,
- *I.2.c)* $(E')^2 = dN_1'$, *for* $d \in \mathbb{R}$,
- *I.2.d)* $(E')^2 = 2N_1' + N_2'$,
- *I.2.e)* $(E')^2 = fE'$, $f \neq 0$,
- *I.2.f)* $(E')^2 = N_2' + 2E'$.

PROPOSITION 9. *Let $N_1^2 = N_2^2 = N_1N_2 = 0$, $N_1E = E$ and $N_2E = 0$, i.e. the algebra is of type I.3. For any choice of parameters d, e and f there exists a new basis N_1', N_2', E' in which the product $E' \cdot E'$ has exactly one of the following values*

- *I.3.a)* $(E')^2 = mN_1' + E'$, $m \neq 0$,
- *I.3.b)* $(E')^2 = N_2' + E'$,
- *I.3.c)* $(E')^2 = E'$,
- *I.3.d)* $(E')^2 = N_1'$,
- *I.3.e)* $(E')^2 = -N_1'$,
- *I.3.f)* $(E')^2 = N_2'$,
- *I.3.g)* $(E')^2 = 0$.

4.2. Algebras of class II. In the next four propositions we are going to determine the simplest possible form of product $E^2 = dN_1 + eN_2 + fE$ providing the products $N_1E = N_1$ and $N_2E \in \{E, \beta N_2, N_1 + N_2, N_1\}$ are known and are of type II. In all four cases we can begin with the linear transformations of the form $N_1' = AN_1$, $N_2' = CN_1 + DN_2$, $E' = FN_1 + GN_2 + E$, which preserves the crucial condition $N_1E = N_1$.

PROPOSITION 10. *Let $N_1^2 = N_2^2 = N_1N_2 = 0$, $N_1E = N_1$ and $N_2E = \beta N_2$; $\beta \neq 0$, $\beta \neq 1$, i.e. the algebra is of type II.2. For any choice of parameters d, e and f there exists a new basis N_1', N_2', E' in which the product $E' \cdot E'$ has exactly one of the following values*

- *II.2.a)* $(E')^2 = fE'$; $f \in \mathbb{R}$
- *II.2.b)* $(E')^2 = N_1' + 2E'$,
- *II.2.c)* $(E')^2 = N_2' + 2\beta E'$.

PROOF. The linear transformation $N_1' = AN_1$, $N_2' = DN_2$, $E' = FN_1 + GN_2 + E$ (where $AD \neq 0$) preserves conditions $N_1'E' = N_1'$ and $N_2'E' = \beta N_2'$. A direct computation yields

$$(E')^2 = (d + 2F)N_1 + (e + 2\beta G)N_2 + fE.$$

Now, if $f \neq 0$ and $f \neq 2$ and $f \neq 2\beta$ then for $F = \frac{d}{f-2}$, $G = \frac{e}{f-2\beta}$ we have $(E')^2 = fE'$.

If $f = 2$ and $d \neq 0$ then for $A = d$ and $G = \frac{e}{2-2\beta}$ we have $(E')^2 = N_1' + 2E'$.

If $f = 2$ and $d = 0$ then for $G = \frac{e}{2-2\beta}$ we have $(E')^2 = 2E'$.

If $f = 0$ we can always choose $F = -\frac{d}{2}$ and $G = -\frac{e}{2\beta}$ to get $(E')^2 = 0$.

If $f = 2\beta$ and $e \neq 0$ then for $D = e$ and $F = \frac{d}{2\beta-2}$ we have $(E')^2 = N_2' + 2\beta E'$.

If $f = 2\beta$ and $e = 0$ then for $F = \frac{d}{2\beta-2}$ we have $(E')^2 = 2\beta E'$.

Note that $2\beta - 2 \neq 0$ since $\beta \neq 1$. □

PROPOSITION 11. *Let* $N_1^2 = N_2^2 = N_1N_2 = 0$, $N_1E = N_1$ *and* $N_2E = N_1 + N_2$, *i.e. the algebra is of type II.3. For any choice of parameters* d, e *and* f *there exists a new basis* N_1', N_2', E' *in which the product* $E' \cdot E'$ *has exactly one of the following values*

- *II.3.a)* $(E')^2 = fE'$; $f \in \mathbb{R}$
- *II.3.b)* $(E')^2 = N_2' + 2E'$.

PROOF. The linear transformation $N_1' = AN_1$, $N_2' = CN_1 + AN_2$, $E' = FN_1 + GN_2 + E$ (where $A \neq 0$) preserves conditions $N_1'E' = N_1'$ and $N_2'E' = N_1' + N_2'$. A direct computation yields

$$(E')^2 = (2F + 2G + d)N_1 + (2G + e)N_2 + fE.$$

Now, if $f \neq 0$ and $f \neq 2$ then for $F = \frac{-2d+2e+fd}{(f-2)^2}$, $G = \frac{e}{f-2}$ we have $(E')^2 = fE'$.

If $f = 0$ then for $G = -\frac{e}{2}$ and $F = \frac{e-d}{2}$ we have $(E')^2 = 0$.

If $f = 2$ and $e \neq 0$ then for $C = 2G + d$ and $A = e$ we have $(E')^2 = N_2' + 2E'$.

If $f = 2$ and $e = 0$ then for $G = -\frac{d}{2}$ we have $(E')^2 = 2E'$. □

PROPOSITION 12. *Let* $N_1^2 = N_2^2 = N_1N_2 = 0$, $N_1E = N_1$ *and* $N_2E = N_1$, *i.e. the algebra is of type II.4. For any choice of parameters* d, e *and* f *there exists a new basis* N_1', N_2', E' *in which the product* $E' \cdot E'$ *has exactly one of the following values*

- *II.4.a)* $(E')^2 = fE'$; $f \in \mathbb{R}$,
- *II.4.b)* $(E')^2 = N_1' + 2E'$,
- *II.4.c)* $(E')^2 = N_2'$.

PROOF. The linear transformation $N_1' = AN_1$, $N_2' = (A - D)N_1 + DN_2$, $E' = FN_1 + GN_2 + E$ (where $AD \neq 0$) preserves conditions $N_1'E' = N_1'$ and $N_2'E' = N_1'$. A direct computation yields

$$(E')^2 = (2F + 2G + d)N_1 + eN_2 + fE.$$

Now, if $f \neq 0$ and $f \neq 2$ then for $F = \frac{2e+fd}{f(f-2)}$, $G = \frac{e}{f}$ we have $(E')^2 = fE'$.

If $f = 0$ and $e \neq 0$ then for $D = e$ and $A = 2F+2G+d+e$ we have $(E')^2 = N_2'$.

If $f = 0$ and $e = 0$ then for $G = 0$ and $F = -\frac{d}{2}$ we have $(E')^2 = 0$.

If $f = 2$ and $e + d = 0$ then for $G = \frac{e}{2}$ we have $(E')^2 = 2E'$.

If $f = 2$ and $e + d \neq 0$ then for $A = e + d$ and $G = \frac{e}{2}$ we have $(E')^2 = N_1' + 2E'$. □

In a similar way we can treat all the remaining types III. and IV. The proofs of the corresponding propositions are rather similar to the proofs of Proposition 12, Proposition 11, etc. So, we can omit them.

4.3. Algebras of class III. In the next five propositions we seek a linear transformations which take the product $E^2 = dN_1 + eN_2 + fE$ into as simple form as possible. Of course these isomorphisms must preserve the form of the already determined products $N_1E = N_2$ and $N_2E \in \{E, kN_1 + N_2, \pm N_1, 0\}$.

Proposition 13. *Let $N_1^2 = N_2^2 = N_1N_2 = 0$, $N_1E = N_2$ and $N_2E = kN_1+N_2$; $k \in \mathbb{R}$, i.e. the algebra is of type III.2. For any choice of parameters d, e and f there exists a new basis N_1', N_2', E' in which the product $E' \cdot E'$ has exactly one of the following values*

- *III.2.a) $(E')^2 = fE'$, for $f \neq 0$ and $k \neq \frac{f^2-2f}{4}$,*
- *III.2.b) $(E')^2 = N_1' + fE'$, for $f \neq 0$ and $k = \frac{f^2-2f}{4}$,*
- *III.2.c) $(E')^2 = N_1'$,*
- *III.2.d) $(E')^2 = 0$.*

Proposition 14. *Let $N_1^2 = N_2^2 = N_1N_2 = 0$, $N_1E = N_2$ and $N_2E = N_1$, i.e. the algebra is of type III.3. For any choice of parameters d, e and f there exists a new basis N_1', N_2', E' in which the product $E' \cdot E'$ has exactly one of the following values*

- *III.3.a) $(E')^2 = N_1' + fE'$, for $f \neq 0$,*
- *III.3.b) $(E')^2 = 0$.*

Proposition 15. *Let $N_1^2 = N_2^2 = N_1N_2 = 0$, $N_1E = N_2$ and $N_2E = -N_1$, i.e. the algebra is of type III.4. For any choice of parameters d, e and f there exists a new basis N_1', N_2', E' in which the product $E' \cdot E'$ has exactly one of the following values*

- *III.4.a) $(E')^2 = N_1 + fE'$, for $f \neq 0$,*
- *III.4.b) $(E')^2 = 0$.*

Proposition 16. *Let $N_1^2 = N_2^2 = N_1N_2 = 0$, $N_1E = N_2$ and $N_2E = 0$, i.e. the algebra is of type III.5. For any choice of parameters d, e and f there exists a new basis N_1', N_2', E' in which the product $E' \cdot E'$ has exactly one of the following values*

- *III.5.a) $(E')^2 = E'$,*
- *III.5.b) $(E')^2 = N_1'$,*
- *III.5.c) $(E')^2 = 0$.*

4.4. Algebras of class IV. In the next two propositions we seek for linear transformations which take the product $E^2 = dN_1 + eN_2 + fE$ into as simple form as possible. Of course these isomorphisms must preserve the form of the already determined products $N_1E = 0$ and $N_2E \in \{E, N_1, N_2, 0\}$.

PROPOSITION 17. *Let $N_1^2 = N_2^2 = N_1N_2 = 0$, $N_1E = 0$ and $N_2E = N_2$, i.e. the algebra is of type IV.2. For any choice of parameters d, e and f there exists a new basis N_1', N_2', E' in which the product $E' \cdot E'$ has exactly one of the following values*

- *IV.2.a)* $(E')^2 = fE'$, *for* $f \neq 0$,
- *IV.2.b)* $(E')^2 = N_1' + N_2'$,
- *IV.2.c)* $(E')^2 = N_2'$,
- *IV.2.d)* $(E')^2 = N_2' + 2E'$.

PROPOSITION 18. *Let $N_1^2 = N_2^2 = N_1N_2 = 0$, $N_1E = 0$ and $N_2E = 0$, i.e. the algebra is of type IV.4. For any choice of parameters d, e and f there exists a new basis N_1', N_2', E' in which the product $E' \cdot E'$ has exactly one of the following values*

- *IV.4.a)* $(E')^2 = E'$,
- *IV.4.b)* $(E')^2 = N_2'$,
- *IV.4.c)* $(E')^2 = 0$.

We summarize the obtained results in the following theorem.

Theorem 1. *Every algebra corresponding to ODEs from the class $\mathcal{C}$ is isomorphic to one of the followings:*

$\cdot$	N_1	N_2	E
N_1	0	0	E
N_2	0	0	N_1
E	E	N_1	mN_1+nN_2+E

$m, n \in \mathbb{R}$

$\cdot$	N_1	N_2	E
N_1	0	0	E
N_2	0	0	N_1
E	E	N_1	mN_1+N_2

$m \in \mathbb{R}$

$\cdot$	N_1	N_2	E
N_1	0	0	E
N_2	0	0	N_1
E	E	N_1	N_1

$\cdot$	N_1	N_2	E
N_1	0	0	E
N_2	0	0	N_1
E	E	N_1	$-N_1$

$\cdot$	N_1	N_2	E
N_1	0	0	E
N_2	0	0	N_1
E	E	N_1	0

$\cdot$	N_1	N_2	E
N_1	0	0	E
N_2	0	0	N_2
E	E	N_2	dN_1+N_2+fE

$d, f \neq 0$

$\cdot$	N_1	N_2	E
N_1	0	0	E
N_2	0	0	N_2
E	E	N_2	$2N_1+fE$

$f \neq 0$

$\cdot$	N_1	N_2	E
N_1	0	0	E
N_2	0	0	N_2
E	E	N_2	dN_1

$d \in \mathbb{R}$

$\cdot$	N_1	N_2	E
N_1	0	0	E
N_2	0	0	N_2
E	E	N_2	$2N_1+N_2$

$\cdot$	N_1	N_2	E
N_1	0	0	E
N_2	0	0	N_2
E	E	N_2	fE

$f \neq 0$

$\cdot$	N_1	N_2	E
N_1	0	0	E
N_2	0	0	N_2
E	E	N_2	N_2+2E

$\cdot$	N_1	N_2	E
N_1	0	0	E
N_2	0	0	0
E	E	0	mN_1+E

$m \neq 0$

$\cdot$	N_1	N_2	E
N_1	0	0	E
N_2	0	0	0
E	E	0	N_2+E

$\cdot$	N_1	N_2	E
N_1	0	0	E
N_2	0	0	0
E	E	0	E

$\cdot$	N_1	N_2	E
N_1	0	0	E
N_2	0	0	0
E	E	0	N_1

$\cdot$	N_1	N_2	E
N_1	0	0	E
N_2	0	0	0
E	E	0	$-N_1$

$\cdot$	N_1	N_2	E
N_1	0	0	E
N_2	0	0	0
E	E	0	N_2

$\cdot$	N_1	N_2	E
N_1	0	0	E
N_2	0	0	0
E	E	0	0

$\cdot$	N_1	N_2	E
N_1	0	0	N_1
N_2	0	0	βN_2
E	N_1	βN_2	fE

$\beta \neq 0, \beta \neq 1, f \in \mathbb{R}$

$\cdot$	N_1	N_2	E
N_1	0	0	N_1
N_2	0	0	βN_2
E	N_1	βN_2	$N_1 + 2E$

$\beta \neq 0, \beta \neq 1$

$\cdot$	N_1	N_2	E
N_1	0	0	N_1
N_2	0	0	βN_2
E	N_1	βN_2	$N_2 + 2\beta E$

$\beta \neq 0, \beta \neq 1$

$\cdot$	N_1	N_2	E
N_1	0	0	N_1
N_2	0	0	$N_1 + N_2$
E	N_1	$N_1 + N_2$	fE

$f \in \mathbb{R}$

$\cdot$	N_1	N_2	E
N_1	0	0	N_1
N_2	0	0	$N_1 + N_2$
E	N_1	$N_1 + N_2$	$N_2 + 2E$

$\cdot$	N_1	N_2	E
N_1	0	0	N_1
N_2	0	0	N_1
E	N_1	N_1	fE

$f \in \mathbb{R}$

$\cdot$	N_1	N_2	E
N_1	0	0	N_1
N_2	0	0	N_1
E	N_1	N_1	$N_1 + 2E$

$\cdot$	N_1	N_2	E
N_1	0	0	N_1
N_2	0	0	N_1
E	N_1	N_1	N_2

$\cdot$	N_1	N_2	E
N_1	0	0	N_1
N_2	0	0	$kN_1 + N_2$
E	N_1	$kN_1 + N_2$	fE

$f \neq 0, k \neq \frac{f^2-2f}{4}$

$\cdot$	N_1	N_2	E
N_1	0	0	N_1
N_2	0	0	$kN_1 + N_2$
E	N_1	$kN_1 + N_2$	$N_1 + fE$

$f \neq 0, k = \frac{f^2-2f}{4}$

$\cdot$	N_1	N_2	E
N_1	0	0	N_2
N_2	0	0	$kN_1 + N_2$
E	N_2	$kN_1 + N_2$	N_1

$\cdot$	N_1	N_2	E
N_1	0	0	N_2
N_2	0	0	$kN_1 + N_2$
E	N_2	$kN_1 + N_2$	0

$\cdot$	N_1	N_2	E
N_1	0	0	N_2
N_2	0	0	N_1
E	N_2	N_1	N_1+fE

$f \neq 0$

$\cdot$	N_1	N_2	E
N_1	0	0	N_2
N_2	0	0	N_1
E	N_2	N_1	0

$\cdot$	N_1	N_2	E
N_1	0	0	N_2
N_2	0	0	$-N_1$
E	N_2	$-N_1$	N_1+fE

$f \neq 0$

$\cdot$	N_1	N_2	E
N_1	0	0	N_2
N_2	0	0	$-N_1$
E	N_2	$-N_1$	0

$\cdot$	N_1	N_2	E
N_1	0	0	N_2
N_2	0	0	0
E	N_2	0	E

$\cdot$	N_1	N_2	E
N_1	0	0	N_2
N_2	0	0	0
E	N_2	0	N_1

$\cdot$	N_1	N_2	E
N_1	0	0	N_2
N_2	0	0	0
E	N_2	0	0

$\cdot$	N_1	N_2	E
N_1	0	0	0
N_2	0	0	N_2
E	0	N_2	fE

$f \neq 0$

$\cdot$	N_1	N_2	E
N_1	0	0	0
N_2	0	0	N_2
E	0	N_2	N_1+N_2

$\cdot$	N_1	N_2	E
N_1	0	0	0
N_2	0	0	N_2
E	0	N_2	N_2

$\cdot$	N_1	N_2	E
N_1	0	0	0
N_2	0	0	N_2
E	0	N_2	N_2+2E

$\cdot$	N_1	N_2	E
N_1	0	0	0
N_2	0	0	0
E	0	0	E

$\cdot$	N_1	N_2	E
N_1	0	0	0
N_2	0	0	0
E	0	0	N_2

$\cdot$	N_1	N_2	E
N_1	0	0	0
N_2	0	0	0
E	0	0	0

5. Analysis of nilpotents and idempotents and applications to ODEs

The systems of ODEs which correspond to the algebras treated in the present paper were treated in [**10**] and [**11**]. First of all, we want to emphasize that the present classification was used in [**10**] to make a systematic case-by-case analysis of stability of the origin in the corresponding quadratic systems. The origin is namely a nonhyperbolic critical point in quadratic systems thus the stability is not trivial to determine. The result on the stability of the origin in homogeneous quadratic systems from [**10**] is resumed in the following theorem.

THEOREM 2. *Every three-dimensional system of quadratic ODEs with a plane of critical points in which the origin is stable is equivalent to one of the following:*

$$\begin{array}{llll} \dot{x} = mz^2 + 2yz & \dot{x} = -z^2 + 2yz & \dot{x} = dz^2 & \dot{x} = dz^2 \\ \dot{y} = 0 & \dot{y} = 0 & \dot{y} = z^2 + 2yz & \dot{y} = 2yz \\ \dot{z} = 2xz + z^2 & \dot{z} = 2xz & \dot{z} = fz^2 + 2xz & \dot{z} = 2xz \\ \quad m < -\frac{1}{8} & & \quad f^2 + 8d < 0 & \end{array}$$

$$\begin{array}{llll} \dot{x} = mz^2 & \dot{x} = -z^2 & \dot{x} = -2yz & \dot{x} = 0 \\ \dot{y} = 0 & \dot{y} = 0 & \dot{y} = 2xz & \dot{y} = 0 \\ \dot{z} = 2xz + z^2 & \dot{z} = 2xz & \dot{z} = 0 & \dot{z} = 0 \\ \quad m < -\frac{1}{8} & & & . \end{array}$$

The above result was obtained also using the fact (see [**5**, Proposition 3.4]) that the presence of idempotents in the corresponding algebra implies the instability of the origin in a homogeneous quadratic system of ODEs. Note also, that every real finite dimensional algebra either has a nonzero idempotent or a nonzero nilpotent of index two (see [**4**, Theorem 1]). It is known that the absence of idempotents is not sufficient to guarantee stability which depends on the nature of nilpotents (see for example [**5**, Corollary 3.8], [**10**] and [**8**]).

For this reason we want to check which algebras from the above classification contains nonzero idempotents and also nilpotents outside of the plane generated by N_1 and N_2. In order to compute the nilpotents (of rank two) and idempotents in an algebra of class $\mathcal{C}$ we have to solve two (quadratic) systems of algebraic equations:

- $x^2 = x \cdot x = 0$, where $x \in \mathbb{R}^3$ and
- $x^2 = x \cdot x = x$, where $x \in \mathbb{R}^3$,

respectively. The result below we actually used in [**10**] in order to prove Theorem 2.

Let us denote the (families of) algebras from I.1.a to IV.4.c by integers: 1 to 44. For algebra no. 1 the computing of nilpotents (of rank two) and idempotents means actually to solve the following equation:

$$\begin{aligned}(x_1N_1 + x_2N_2 + x_3E)^2 &= x_3^2\,(mN_1 + nN_2 + E) + 2x_1x_3E + 2x_2x_3N_1 \\ &= \left(mx_3^2 + 2x_2x_3\right)N_1 + nx_3^2N_2 + \left(x_3^2 + 2x_1x_3\right)E \\ &= 0\end{aligned}$$

or

$$\begin{aligned}(x_1N_1 + x_2N_2 + x_3E)^2 &= x_3^2\,(mN_1 + nN_2 + E) + 2x_1x_3E + 2x_2x_3N_1 \\ &= \left(mx_3^2 + 2x_2x_3\right)N_1 + nx_3^2N_2 + \left(x_3^2 + 2x_1x_3\right)E \\ &= x_1N_1 + x_2N_2 + x_3E,\end{aligned}$$

respectively.

A case-by-case treatment yields a result given in the following tables.

Alg. no.	Conditions	Idempotents	Nilpotents
I.1.a	$n \neq 0, D < 0$	$w_{1,2,3}$	–
I.1.a	$n = -\frac{9m+1\pm(6m+1)^{\frac{3}{2}}}{54} \neq 0$ $m \neq -\frac{1}{6}$	$w_{1,2}$	–
I.1.a	$n = \frac{1}{108}, m = -\frac{1}{6}$	w_1	–
I.1.a	$n \neq 0, D > 0$	w_1	–
I.1.a	$n = 0, 1 + 8m > 0$	$w_{1,2}$	–
I.1.a	$n = m = 0$	E	–
I.1.a	$n = 0, 1 + 8m \leq 0$	–	–
I.1.b	$m > \frac{3}{\sqrt[3]{2}}$	$P_{1,2,3}$	–
I.1.b	$m < \frac{3}{\sqrt[3]{2}}$	P_1	–
I.1.b	$m = \frac{3}{\sqrt[3]{2}}$	$P_{1,2}$	–
I.1.c	–	$N_1 + \sqrt{2}E$	$N_2 - 2E$
I.1.d	–	–	$N_2 + 2E$
I.1.e	–	–	–
I.2.a	$f^2 + 8d > 0$	$Q_{1,2}$	–
I.2.a	$f^2 + 8d = 0, f \neq 4$	Q_1	–
I.2.a	$f^2 + 8d = 0, f = 4$	–	–
I.2.a	$f^2 + 8d < 0$	–	–
I.2.b	–	R	–
I.2.c	$\delta > 0$	$\frac{1}{2}N_1 \pm \frac{1}{\sqrt{2\delta}}E$	–
I.2.c	$\delta \leq 0$	–	–
I.2.d	–	$\frac{1}{2}N_1 + \frac{1}{8}N_2 - \frac{1}{2}E$	–
I.2.e	–	$\frac{1}{f}E$	$N_1 - \frac{2}{f}E$
I.2.f	–	–	$N_1 + \frac{1}{2}N_2 - E$
I.3.a	$1 + 8m > 0$	$S_{1,2}$	–
I.3.a	$1 + 8m = 0$	S_1	–
I.3.a	$1 + 8m < 0$	–	–
I.3.b	–	$N_2 + E$	–
I.3.c	–	E	$N_1 - 2E$
I.3.d	–	$\frac{1}{2}N_1 \pm \frac{1}{\sqrt{2}}E$	–
I.3.e	–	–	–
I.3.f	–	–	–
I.3.g	–	–	E

Remark 2. *In the above table (as well as in the sequel) in the column* ***Nilpotents*** *there are nilpotents of rank two which are not in the span of N_1 and N_2.*

In the above table we also have:

$$w = \rho^2 (m + 2n\rho) N_1 + n\rho^2 N_2 + \rho E$$

where ρ is a root of $4nZ^3 + 2mZ^2 + Z - 1 = 0$. Substituting $M = Z + \frac{2m}{12n}$ we obtain the canonical form $M^3 + 3pM + 2q = 0$ where $q = \frac{-9nm-54n^2+2m^3}{432n^3}$ and

$p = \frac{3n-m^2}{36n^2}$. The discriminant equals

$$D = q^2 + p^3 = -\frac{m^2 - 36nm - 108n^2 + 8m^3 - 4n}{6912n^4}$$

The subcases for $D = 0$:

- $p = q = 0 \Leftrightarrow m = -\frac{1}{6}, n = \frac{1}{108}$
- $p^3 = -q^2 \neq 0 \Leftrightarrow n \neq \frac{1}{108}, m \neq -\frac{1}{6}$

For algebra I.1.b we obtain:

$$P = \rho^2 (m + 2\rho) N_1 + \rho^2 N_2 + \rho E$$

where ρ is a root of $4Z^3 + 2mZ^2 - 1 = 0$. Substituting $M = Z + \frac{m}{6}$ we obtain the canonical form: $M^3 - \frac{1}{12}Mm^2 + \frac{1}{108}m^3 - \frac{1}{4} = 0$. Thus, $p = -\frac{1}{36}m^2$ and $q = \frac{1}{216}m^3 - \frac{1}{8}$. The discriminant equals $D = q^2 + p^3 = -\frac{1}{864}m^3 + \frac{1}{64}$.

For algebra I.2.a we get:

$$Q = \frac{1 - f\rho}{2} N_1 + \frac{1 - f\rho}{2d\,(1 - 2\rho)} N_2 + \rho E$$

where

$$\rho = \frac{-f \pm \sqrt{f^2 + 8d}}{4d}.$$

For algebra I.2.b we obtain:

$$R = \left(\frac{1}{2} + \frac{1}{16}f^2 \pm \frac{1}{16} f \sqrt{f^2 + 16}\right) N_1 - \frac{f \pm \sqrt{f^2 + 16}}{8} E$$

For algebra I.3.a we obtain:

$$S = \frac{4m + 1 \mp \sqrt{1 + 8m}}{8m} N_1 + \frac{-1 \pm \sqrt{1 + 8m}}{4m} E$$

Alg. no.	Conditions	Idempotents	Nilpotents
II.2.a	$f \neq 0$	$\frac{1}{f}E$	$-$
II.2.a	$f = 0$	$-$	$-$
II.2.b	$-$	$-$	$-$
II.2.c	$-$	$-$	$-$
II.3.a	$f \neq 0$	$\frac{1}{f}E$	$-$
II.3.a	$f = 0$	$-$	$-$
II.3.b	$-$	$-$	$-$
II.4.a	$f \neq 0$	$\frac{1}{f}E$	$-$
II.4.a	$f = 0$	$-$	E
II.4.b	$-$	$-$	$-$
II.4.c	$-$	$-$	$-$

Alg. no.	Conditions	Idempotents	Nilpotents
III.2.a	–	$\frac{1}{f}E$	–
III.2.b	–	–	–
III.2.c	–	–	$N_1 - N_2 + 2kE$
III.2.d	–	–	–
III.3.a	$f^2 \neq 4$	T	–
III.3.a	$f^2 = 4$	–	–
III.3.b	–	–	E
III.4.a	–	U	–
III.4.b	–	–	–
III.5.a	–	E	–
III.5.b	–	–	–
III.5.c	–	–	E
IV.2.a	–	$\frac{1}{f}E$	–
IV.2.b	–	–	–
IV.2.c	–	–	$N_2 - 2E$
IV.2.d	–	–	–
IV.4.a	–	E	–
IV.4.b	–	–	–
IV.4.c	–	–	E

For algebra III.3.a we obtain:

$$T = \frac{1}{f^2 - 4}N_1 + \frac{2}{(f^2 - 4)\,f}N_2 + \frac{1}{f}E$$

For algebra III.4.a we obtain:

$$U = \frac{1}{f^2 + 4}N_1 + \frac{2}{(f^2 + 4)\,f}N_2 + \frac{1}{f}E.$$

Since, by the work of Kinyon and Sagle, we know that stable origin can only appear in algebras with no nontrivial idempotents, the above calculations finally give us a result which was directly applied to differential equations in [**10**].

THEOREM 3. *Every algebra corresponding to ODEs from the class $\mathcal{C}$ without nonzero idempotents is isomorphic to one of the following:*

$\cdot$	N_1	N_2	E
N_1	0	0	E
N_2	0	0	N_1
E	E	N_1	$mN_1 + nN_2 + E$

$n = 0,\ 1 + 8m < 0$

$\cdot$	N_1	N_2	E
N_1	0	0	E
N_2	0	0	N_1
E	E	N_1	$-N_1$

$\cdot$	N_1	N_2	E
N_1	0	0	E
N_2	0	0	N_1
E	E	N_1	0

$\cdot$	N_1	N_2	E
N_1	0	0	E
N_2	0	0	N_2
E	E	N_2	$dN_1 + N_2 + fE$

$f^2 + 8d < 0$ *or* $(f = 4, d = -2)$

$\cdot$	N_1	N_2	E
N_1	0	0	E
N_2	0	0	N_2
E	E	N_2	dN_1

$d \leq 0$

$\cdot$	N_1	N_2	E
N_1	0	0	E
N_2	0	0	N_2
E	E	N_2	$N_2 + 2E$

$\cdot$	N_1	N_2	E
N_1	0	0	E
N_2	0	0	0
E	E	0	$mN_1 + E$

$1 + 8m < 0$

$\cdot$	N_1	N_2	E
N_1	0	0	E
N_2	0	0	0
E	E	0	$-N_1$

$\cdot$	N_1	N_2	E
N_1	0	0	E
N_2	0	0	0
E	E	0	N_2

$\cdot$	N_1	N_2	E
N_1	0	0	E
N_2	0	0	0
E	E	0	0

$\cdot$	N_1	N_2	E
N_1	0	0	N_1
N_2	0	0	βN_2
E	N_1	βN_2	0

$\beta \neq 0, \beta \neq 1$

$\cdot$	N_1	N_2	E
N_1	0	0	N_1
N_2	0	0	βN_2
E	N_1	βN_2	$N_1 + 2E$

$\beta \neq 0, \beta \neq 1$

$\cdot$	N_1	N_2	E
N_1	0	0	N_1
N_2	0	0	βN_2
E	N_1	βN_2	$N_2 + 2\beta E$

$\beta \neq 0, \beta \neq 1$

$\cdot$	N_1	N_2	E
N_1	0	0	N_1
N_2	0	0	$N_1 + N_2$
E	N_1	$N_1 + N_2$	0

$\cdot$	N_1	N_2	E
N_1	0	0	N_1
N_2	0	0	$N_1 + N_2$
E	N_1	$N_1 + N_2$	$N_2 + 2E$

$\cdot$	N_1	N_2	E
N_1	0	0	N_1
N_2	0	0	N_1
E	N_1	N_1	0

$\cdot$	N_1	N_2	E
N_1	0	0	N_1
N_2	0	0	N_1
E	N_1	N_1	$N_1 + 2E$

$\cdot$	N_1	N_2	E
N_1	0	0	N_1
N_2	0	0	N_1
E	N_1	N_1	N_2

$\cdot$	N_1	N_2	E
N_1	0	0	N_1
N_2	0	0	$kN_1 + N_2$
E	N_1	$kN_1 + N_2$	$N_1 + fE$

$f \neq 0, k = \frac{f^2 - 2f}{4}$

$\cdot$	N_1	N_2	E
N_1	0	0	N_2
N_2	0	0	$kN_1 + N_2$
E	N_2	$kN_1 + N_2$	N_1

$\cdot$	N_1	N_2	E
N_1	0	0	N_2
N_2	0	0	kN_1+N_2
E	N_2	kN_1+N_2	0

$\cdot$	N_1	N_2	E
N_1	0	0	N_2
N_2	0	0	N_1
E	N_2	N_1	N_1+fE

$f^2=4$

$\cdot$	N_1	N_2	E
N_1	0	0	N_2
N_2	0	0	N_1
E	N_2	N_1	0

$\cdot$	N_1	N_2	E
N_1	0	0	N_2
N_2	0	0	$-N_1$
E	N_2	$-N_1$	0

$\cdot$	N_1	N_2	E
N_1	0	0	N_2
N_2	0	0	0
E	N_2	0	N_1

$\cdot$	N_1	N_2	E
N_1	0	0	N_2
N_2	0	0	0
E	N_2	0	0

$\cdot$	N_1	N_2	E
N_1	0	0	0
N_2	0	0	N_2
E	0	N_2	N_1+N_2

$\cdot$	N_1	N_2	E
N_1	0	0	0
N_2	0	0	N_2
E	0	N_2	N_2

$\cdot$	N_1	N_2	E
N_1	0	0	0
N_2	0	0	N_2
E	0	N_2	N_2+2E

$\cdot$	N_1	N_2	E
N_1	0	0	0
N_2	0	0	0
E	0	0	N_2

$\cdot$	N_1	N_2	E
N_1	0	0	0
N_2	0	0	0
E	0	0	0

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