A novel two-stage nonrecursive architecture for the design of generalized comb filters

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A novel two-stage class of decimation filters with superior spurious signal rejection performance around the so-called folding bands, i.e., frequency intervals whose signals get folded down to baseband due to decimation. The key idea to enhance signal rejection in the frequency domain lies on an effective way to place the zeros of a classical comb filter in the aforementioned folding bands. On the other hand, the paper provides a mathematical framework for designing two-stage multiplierless and nonrecursive structures of the proposed filters.

Examples are provided to highlight the key steps in the design of the proposed filters. Moreover, the frequency behavior of the proposed filters in both baseband and stopband is compared with classical and generalized comb filters, and a droop compensator is proposed to counteract the passband distortion of the proposed filters.

1. Introduction

Multirate architectures for sampling rate conversion have been under investigation for many years [1]. From a practical point of view, the decimation of a highly oversampled signal is accomplished using a cascade of two (or more) stages of decimation [1]. The first stage is usually a comb filter of order \( N \) decimating by a factor \( M \), while the last stage employs an FIR filter providing the required selectivity on the sampled signal down-converted to baseband. Moreover, the design of a decimation stage in a multistage architecture imposes stringent constraints only on the shape of the filter frequency response over the so-called folding bands.

Let us briefly highlight the meaning of folding bands in connection to the first decimation stage in a multistage architecture. Consider an analog signal \( x(t) \) with baseband bandwidth \([-B_x, B_x]\) sampled by an A/D converter at the frequency \( f_s = \rho 2B_x \) (\( \rho \) is the oversampling ratio; by definition, it is \( \rho \geq 1 \)). The discrete-time sampled signal, \( x[nT_s] \), presents normalized frequency bandwidth in the dimensionless frequency set \([-f_0, +f_0]\), with \( f_0 = B_x/f_s = B_x/(\rho 2B_x) = 1/(2\rho) \). With this setup, the frequency response of the anti-aliasing filter employed in the first decimation stage should attenuate the quantization noise (QN) and any other undesired signal falling inside the frequency intervals defined as

\[
\left[ \frac{k}{M} - f_0, \frac{k}{M} + f_0 \right], \quad \text{for } k = 1, \ldots, \lfloor M \rfloor.
\]

The reason follows upon noting that the signals within these frequency bands will fold down to baseband because of the sampling rate reduction by \( M \) in the first decimation stage, thus irremediably affecting the signal resolution after the multistage decimation chain [1]. These frequency ranges are pictorially shown in Fig. 1.

Classical comb filters with z-transfer function [2]

\[
H(z) = \frac{1}{M} \frac{1 - z^{-M}}{1 - z^{-1}}
\]

have many advantages over other filters. Among them, the main advantage stems from the fact that their system function is extremely simple to implement and it does not require any multiplier. However, the magnitude response exhibits low attenuation...
in the folding bands, as well as a considerable passband droop that
deteriorates the sampled signal.

With this background, let us provide a survey of the recent lit-
erature related to the problem addressed in this paper. Tutorials
focusing on the design of multirate filters can be found in [1,3–7].
The design of optimized multistage decimation and interpolation
filters has been recently addressed by Coffey in [8], while the de-
sign of multistage decimation architectures relying on constituent
cyclotomic polynomial filters has been presented in [9,10]. Imple-
mentation aspects of comb filters with special emphasis on power
consumption and overflow have been addressed in [11,12] where
nonrecursive architectures relying on the polyphase decomposition
were introduced. Methods [13,14] proposed compensation filters
aimed to decrease the passband droop of comb filters. In [15], the
authors proposed novel decimation schemes for $\sum A/D$ convert-
ers based on Kaiser and Hamming sharpened filters. These filters
were improved in [16], then generalized in [17] for higher order
decimation filters, and in [19] for wideband receivers. In [20], the
authors introduced a two-stage architecture presenting better mag-
nitude characteristics than the structure proposed in [16].

A 3rd-order modified decimation sinc filter was proposed in
[21], and developed in [22]. The class of comb filters was gen-
eralized in [24], whereby the authors proposed an optimization
framework for deriving the optimal zero rotations of Generalized
Comb filters (GCFS) for any filter order and decimation factor. GCFS
in [24] were proposed with the aim of increasing the $\sum A/D$ re-
jection around the folding bands with respect to classical comb
filters of equivalent order. In spite of the particular application to
$\sum A/D$ converters, GCFS are pretty general in that they can be
used in place of classical comb filters in multistage rate conversion
architectures as the ones employed for signal extraction in broad-
band and reconfigurable digital receivers. Works [25,26] proposed
compensation filters to decrease the passband droop of GC filters,
while [27] addressed the design of efficient GC filters exploiting
recursive architectures.

The considerations drawn above in connection to the frequency
behavior of a classical comb filter around the folding bands, par-
ticularly around the first folding band, represent the very starting
point toward the ideas proposed in this work. The main ration-
ales are discussed in Section 2 where we also point out the main
contributions of this work. The rest of the paper is organized as
follows. In Section 3, we derive the z-transfer function of the pro-
duced decimation filters. Section 4 presents a variety of design
issues for improving the frequency response of the filters under
investigation. Moreover, a mathematical framework for optimizing
the zero location of the z-transfer function is developed. In Sec-
tion 5, we present some design examples in order to highlight the
choice of the parameters defining the behavior of the proposed fil-
ers. Finally, Section 6 draws the conclusion.

2. Motivation

One of the advantages of the decimation filters proposed in
[24] stems from the optimization of the zeros locations of a clas-
sical Nth-order comb filter within the folding bands defined in (1).
A nice consequence of this optimization consists in an increased
attenuation in the folding bands. Given a decimation factor $M$,
and neglecting the normalization constant ensuring unity gain at
baseband, the z-transfer function of a 3rd-order GC filter can be
written as

$$H_{\text{GCFS}}(z) = \frac{1 - z^{-M}}{1 - z^{-1}} \left( 1 - z^{-M} e^{i\alpha M} + 1 - z^{-M} e^{-i\alpha M} \right),$$

(3)

where $\alpha$ is the extent of the rotation undergone by the zeros
of a classical comb filter. Using the notation summarized in the pre-
vious section about the folding bands, we note that a convenient
choice for $\alpha$ is $\alpha = q2\pi f_0$, with $q \in [-1, +1]$: this choice is such
that the zeros of $H_{\text{GCFS}}(z)$ fall within the folding bands in (1). The
magnitude of a 3rd-order GC filter is compared with the magni-
itude response of a classical 3rd-order comb filter in Fig. 2 for the
following parameters: $M = 16$, $\rho = 64$, $\alpha = 0.03828$, and $q = 0.78$.

The drawback stemming from the zero-optimization is the in-
roduction of multipliers in the first stage of decimation, as well as
the impossibility to implement multiplierless recursive architec-
tures due to the instability arising from imperfect zero-pole can-
cellation when coefficients quantization is exploited. In addition,
GC filters exhibit passband droops of the same extent of classical
comb filters.

The main contributions of this work stem from solving the
aforementioned problems, while retaining the advantages of GC fil-
ters. We propose a novel class of decimation filters featuring the
following advantages hereafter identified as conditions.

1. No need of a programmable FIR filter at low rate to com-
penate the passband droop introduced by comb filters.
2. Improved spurious signal rejection in the folding bands com-
pared to traditional Nth-order comb filters (in this respect the
proposed filters are superior as compared to comb filters).
3. Nonrecursive implementation not suffering from the instability
problems arising from imperfect pole-zero cancel-
cation due to finite precision effects.
4. Overall multiplierless filter design.

Let us briefly discuss in which way the present work differs
from the previous work relevant to this one, developed by the
same authors. Papers [9,10] addressed the design of multistage
decimation filters based on cyclotomic polynomials. The problem
was to solve an optimization problem for finding a proper combi-
nation of cyclotomic polynomials to meet a set of filter specifi-
cations. The design in [9,10] is different from the focus of this work
since this very work proposes improved decimation filters stem-
ing from classical comb filters. Paper [16] presented a sharpened
comb decimator structure consisting of a cascade of a comb-filter
based decimator and a sharpened comb decimator. In the attempt

![Fig. 2. Magnitude response (in dB) of the 3rd-order GC filter in (3) compared with the magnitude response of a 3rd-order comb. Design parameters are as follows: $M = 16$, $\rho = 64$, and $\alpha = 0.03828$.](Image)
to reduce the passband droop of classical comb filters, the sharpened architecture was used. One of the main drawbacks of the sharpening technique is the use of multiple instances of the same basic cell just for reducing the passband droop of the overall filter. This work, on the other hand, attacks the passband droop problem by cascading a very efficient cell at the end of the decimation architecture without increasing the overall filter complexity. Similar considerations can be derived for work [17] where the sharpening technique was employed for the sake of reducing the passband droop of typical comb filters. Unlike this work, notice that the filter architectures proposed in [16] and [20] did not present improved signal rejection around the folding bands. Papers [18,23,24] addressed the design of generalized comb filters for decimation purposes but the proposed architectures included multipliers and did not consider the passband droop compensation. As explained above in this section, the main contributions of this work compared to the previous ones is the proposal of filter architectures featuring the four conditions listed above, which improve the previous work from the same authors.

3. The proposed decimation filters

This section presents the rationales at the very basis of the proposed class of decimation filters, derives the z-transfer function, and discusses the frequency behavior of the proposed two-stage decimation filters.

3.1. Derivation of the two-stage architecture

We assume that the decimation factor \( M \) is an even integer that can be represented as follows:

\[
M = 2M_1.
\] (4)

With this setup, the transfer function in (2) can be rewritten as

\[
H(z) = \frac{1 1 - z^{-M}}{M 1 - z^{-1}} = \frac{1 1 - z^{-M_1} 1 1 - z^{-2M_1}}{M_1 1 - z^{-1} 2 1 - z^{-M_1}} = H_1(z)H_2(z^{M_1})
\] (5)

where

\[
H_1(z) = \frac{1 1 - z^{-M_1} 1 1 - z^{-2M_1}}{M_1 1 - z^{-1} 2}, \quad H_2(z^{M_1}) = \frac{1 - z^{-2M_1}}{1 - z^{-M_1}}.
\] (6)

The magnitude characteristics of the z-transfer functions in (6) can be derived upon replacing \( z \) with \( e^{j\omega} \):

\[
|H_1(e^{j\omega})| = \left| \frac{1 \sin(\omega M_1)}{M_1 \sin(\frac{\omega}{2})} \right|
\]

\[
|H_2(e^{j\omegaM_1})| = \left| \frac{1 \sin(\omega M_1)}{2 \sin(\frac{\omegaM_1}{2})} \right|
\] (7)

As an example, the magnitude characteristics of both \( |H_1(e^{j\omega})| \) and \( |H_2(e^{j\omegaM_1})| \) are shown in Fig. 3 for \( M = 16 \). From Fig. 3 we can recognize that, upon applying zero rotation only to the second filter \( |H_2(e^{j\omegaM_1})| \) in the cascade, rotation would occur in all zeros but the last one.

3.2. Moving zero rotation to lower rate

Let us focus on the benefits deriving from applying zero rotation only to the second stage \( H_2(z^{M_1}) \). Applying zero rotation to \( H_2(z^{M_1}) \), we get:

\[
H_1(z) = \frac{1 1 - z^{-M_1} 1 1 - z^{-2M_1}}{M_1 1 - z^{-1} 2}, \quad H_2(z^{M_1}) = \frac{1 - z^{-2M_1}}{1 - z^{-M_1}}.
\]

\[
H_2(z^{M_1}) = \frac{1 1 - z^{-M_1} 1 1 - z^{-2M_1}}{M_1 1 - z^{-1} 2 1 - z^{-M_1}} = H_1(z)H_2(z^{M_1})
\]

\[
H_2(z^{M_1}) = \frac{1 - z^{-2M_1}}{1 - z^{-M_1}}
\]

\[
|H_2(e^{j\omegaM_1})| = \left| \frac{1 \sin(\omega M_1)}{2 \sin(\frac{\omegaM_1}{2})} \right|
\]

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\[
H_2(z^{M_1}) = \frac{1 - z^{-2M_1}}{1 - z^{-M_1}}
\]

\[
|H_2(e^{j\omegaM_1})| = \left| \frac{1 \sin(\omega M_1)}{2 \sin(\frac{\omegaM_1}{2})} \right|
\] (11)

Some considerations are in order. From (10), it is easy to note the presence of a classical 2nd-order comb filter \( H_2(z) \) with decimation \( M_1 \) in the first stage of the proposed filter. This stage can be implemented in nonrecursive form by using polyphase decomposition as explained in [11].

On the other hand, using the commutative properties of multirate systems [1], and recalling the relation \( M = 2M_1 \), the second stage \( H_2(z^{M_1}) \) can be moved after the decimation by \( M_1 \) yielding \( \{D_{2r}(z)\}^{-1} \) before the decimator by 2 and \( N_{2r}(z) \) after the overall decimator by \( M \). With this said, we notice the presence of two multipliers: the first operates at \( M_1 \) lower rate while the second one at \( M \) times lower rate. The architecture of the decimation filter with z-transfer function in (10) is depicted in Fig. 4. This architecture can also be generalized by cascading an Nth-order comb filter. By doing so, the following decimation filter \( H_1(z) \) easily follows:
is a linear phase filter with only one multiplier. Moreover, unlike more efficient structure depicted in Fig. 6. Note that the filter (16) sive/recursive form, or by resorting to polyphase decomposition. filter that can be implemented in nonrecursive, mixed nonrecur- the architecture shown in Fig. 5. The first stage is a classical comb Eq. (13) becomes:

\[
H_2(z) = \frac{1}{4} \left[ 1 - 2 \cos(2\alpha M_1)z^{-2M} + z^{-4M} \right].
\] (15)

The z-transfer function in (15) is the recursive representation of the FIR filter:

\[
H_2(z) = \frac{1}{4} \left[ 1 + 2 \cos(\alpha M_1)z^{-M} + z^{-2M} \right]
\] (16)

as it can be verified by accomplishing the polynomial division in (15) against the variable z. Upon employing (16), we obtain the more efficient structure depicted in Fig. 6. Note that the filter (16) is a linear phase filter with only one multiplier. Moreover, unlike (15), \(H_2(z^{2M})\) in (16) does not suffer from any instability problem since there is no pole-zero cancellation involved in it.

3.3. Nonrecursive implementation of the zero rotation term

Upon considering (4), (9) can be rewritten as

\[
H_2(z) = \frac{1}{4} \left[ 1 - 2 \cos(2\alpha M_1)z^{-2M} + z^{-4M} \right].
\] (13)

Using the well-known trigonometric identity

\[
\cos(2\alpha M_1) = 2 \cos^2(\alpha M_1) - 1,
\] (14)

Eq. (13) becomes:

\[
H_2(z) = \frac{1}{4} \left[ 1 - 2 \cos(\alpha M_1)z^{-M} + z^{-2M} \right].
\] (15)

The z-transfer function of the proposed compensated filter is

\[
H(z) = H_1(z)H_2(z)z^{-N} = H_1(z)H_2(z)H_C(z)
\] (20)

where \(H_1(z)\) and \(H_C(z)\) are given in (12) and (18), respectively. We notice in passing that the use of the droop compensator allows the condition 1 to be fulfilled.

4. Optimization of the proposed decimation filters

In this section we propose a theoretical framework for optimizing the zero locations of the proposed decimation filters in such a way to increase spurious signal attenuation around the folding bands compared to classical comb filters.

To get started, consider the z-transfer function \(H_{tr}(z)\) of the overall decimation filter

\[
H_{tr}(z) = H_1(z)H_2(z)H_C(z).
\] (21)

The frequency response of this filter is:

\[
H_{tr}(e^{i\omega}) = H_1(e^{i\omega})H_2(e^{i\omega})H_C(e^{i\omega}).
\] (22)

Table 1

<table>
<thead>
<tr>
<th>Parameters (N+2)</th>
<th>Parameter (b)</th>
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<tbody>
<tr>
<td>2</td>
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<tr>
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<td>9</td>
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droop can be evaluated as the value of the filter frequency response in \(f_0\). The goal is to cascade a computationally efficient, yet simple, multiplierless filter compensating the value of the filter frequency response \(H_{tr}(e^{i\omega})\) evaluated in \(f_0\). To achieve this goal, we adopt the simple compensator proposed for comb compensation in [13]:

\[
H_C(e^{i\omega}) = 1 + 2^{-b}\sin^2(\omega M/2),
\] (17)

where \(b\) is an integer that depends only on the value \(N+2\). However, for the GCF compensation we have to define the new values of \(b\). The values of \(b\) for different values of \(N+2\), obtained by the computer simulations, are given in Table 1.

From (17), we have the following z-transfer function of the passband droop compensator

\[
H_C(z^{-M}) = A[1 + Bz^{-M} + z^{-2M}],
\] (18)

where

\[
A = -2^{-b-2}, \quad B = -(2^{b+2} + 2).
\] (19)
Notice that the normalization constant of the 2nd-order rotated cell is:

$$H_{rr}(e^{i\alpha M}) = H_C(e^{i\alpha M}) \frac{1 - \cos(\alpha M)}{1 - \cos(\alpha M_1)}.$$  \hspace{1cm} (26)

Therefore, it is:

$$H_{rr}(e^{i\alpha M}) = H_C(e^{i\alpha M}) \frac{1 - \cos(\alpha M)}{1 - \cos(\alpha M_1)}$$

$$\times e^{-j\Omega(M - 1)N}$$

$$\times \frac{\sin(\alpha M/2)}{\sin(\pi/2)}$$

$$\times \cos(\omega M) - \cos(\alpha M)$$

$$\times \cos(\omega M_1) - \cos(\alpha M_1).$$ \hspace{1cm} (27)

where

$$\frac{1}{M^2 M^N} \frac{1 - \cos(\alpha M_1)}{1 - \cos(\alpha M)}$$

is the normalization constant of the overall decimation filter in (21) assuring unity gain at baseband. The magnitude squared of the frequency response is

$$|H_{rr}(e^{i\alpha M})|^2 = \left|H_C(e^{i\alpha M})\right|^2 \frac{1 - \cos(\alpha M_1)}{1 - \cos(\alpha M)}$$

$$\times \frac{\sin(\alpha M/2)}{\sin(\pi/2)}$$

$$\times \cos(\omega M) - \cos(\alpha M)$$

$$\times \cos(\omega M_1) - \cos(\alpha M_1)^2.$$ \hspace{1cm} (29)

where we have omitted the magnitude of the Euler exponential $e^{-j\Omega}$ in (27) being equal to one regardless of $\Omega$.

Let us address the choice of the parameter $\alpha$ in (29). The main objective of the proposed class of decimation filters is the increase of the filter attenuation around the folding bands; therefore, a convenient choice for $\alpha$ is

$$\alpha = q2\pi f_0, \quad q \in [-1, +1].$$ \hspace{1cm} (30)

This definition is such that the couple of zeros belonging to $H_{2r}(z)$ falls inside each folding band. We notice in passing that the independent parameter appearing in the definition of $\alpha$ is $q$.

The optimal $\alpha$, identified by $\alpha_{opt}$, can be evaluated by solving the optimization strategy discussed in the following. Find $\alpha_{opt}$ as the argument of the optimization problem:

$$\alpha_{opt} = \arg \min_{\alpha} \int_{B_w} S_N(f) |H_{rr}(e^{i2\pi f})|^2 \, df,$$ \hspace{1cm} (31)

where the integral is extended over the digital bandwidth $B_w$ corresponding to the first folding band defined as

$$B_w = \left[ 1 - \frac{1}{2\rho} \right] M + 1.$$ \hspace{1cm} (32)

with lower edge $f_{0l} = \frac{1}{3}$ and $f_{0u} = \frac{1}{2\rho}$. The function $S_N(f)$ is the power spectrum density of the quantization noise. For oversampled A/D converters without noise shaping, $S_N(f)$ is a constant function over the whole frequency range; therefore, it does not have any effect on the optimization problem and it can be neglected. On the other hand, for noise-shaped A/D converters, such as sigma-delta, it is a function of the digital frequency $f$, but it can be very well considered constant over the first folding band [28]; therefore, it can also be neglected since multiplicative constant terms do not affect
the optimization problem derived above. Therefore, the optimization problem (31) can be solved as follows:

\[
\frac{d}{d\alpha} \int_{B_{\omega}} S_n(f) |H_{tr}(e^{j2\pi f})|^2 \, df = 0.
\]  

Upon omitting the constant function \(S_n(f)\), it is

\[
\int_{B_{\omega}} \frac{d}{d\alpha} |H_{tr}(e^{j2\pi f})|^2 \, df = 0,
\]  

where \(|H_{tr}(e^{j2\pi f})|^2\) is defined in (29) with \(\omega = 2\pi f\). Therefore, (34) can be rewritten as

\[
\int_{B_{\omega}} \frac{d}{d\alpha} \left[ \frac{1 - \cos(\alpha M_1)}{M_1^2 M^N (1 - \cos(\alpha M))^2} \right] \, df = 0.
\]

which simplifies to

\[
\left\{ \frac{1}{M_1^2 M^N (1 - \cos(\alpha M))^2} \right\} \times \left\{ \frac{\sin(\pi f M)}{\sin(\pi f)} \right\}^{2N} \times \left\{ \frac{\sin(\pi f M_1)}{\sin(\pi f)} \right\}^4 \times \left[ \frac{\cos(2\pi f M_1) - \cos(\alpha M)}{\cos(2\pi f M_1) - \cos(\alpha M)} \right] \, df = 0.
\]  

Upon neglecting the constant multiplicative terms, after some algebra to find the derivative against the variable \(\alpha\) in the integral, the following equation can be obtained:

\[
\int_{B_{\omega}} F(f) \left[ \frac{\sin(\pi f M)}{\sin(\pi f)} \right] \times \left[ \frac{\cos(2\pi f M_1) - \cos(\alpha M)}{\cos(2\pi f M_1) - \cos(\alpha M)} \right] \, df = 0
\]

where \(F(f)\) is defined as follows

\[
F(f) = |H_C(e^{j2\pi f M})| \left[ \frac{\sin(\pi f M)}{\sin(\pi f)} \right]^{2N} \times \left[ \frac{\sin(\pi f M_1)}{\sin(\pi f)} \right]^4.
\]  

The integral in (37) cannot be solved in closed-form, and numerical integration has to be employed to find the optimal solution (we have used trapezoidal numerical integration implemented in Matlab). The optimal values of \(\alpha\) (and \(q = \alpha / 2\pi f_0\)) are noted in Table 2 for the parameters shown in the first column. We chose \(N = 3\) since such value let \(H_{tr}(z)\) in (21) be equivalent to a 5th-order comb filter embedded in practical Digital Down Converters (DDCs).

The term Denoising in the last column of Table 2 is defined as follows:

\[
10 \log_{10} \left[ \int_{B_{\omega}} S_n(f) \left| H_{comb,N+2}(e^{j2\pi f}) \right|^2 \, df \right]
\]

\[
- 10 \log_{10} \left[ \int_{B_{\omega}} S_n(f) \left| H_{tr}(e^{j2\pi f}) \right|^2 \, df \right].
\]

| Table 2 |
|------------------|------------------|------------------|
| \(\rho\) = 128, \(N = 3\) | \(M_1 = 32; M_2 = 2\) | 0.911 0.02235930396422 10.73 |
| \(M_1 = 16; M_2 = 2\) | 0.896 0.02199114857513 12.57 |
| \(M_1 = 8; M_2 = 2\) | 0.887 0.02177025534167 12.95 |
| \(M_1 = 4; M_2 = 2\) | 0.884 0.0216966246385 13.03 |
| \(M_1 = 2; M_2 = 2\) | 0.883 0.02167208057125 13.05 |

Therefore, it is the noise power rejection gain of the proposed filter with compensator compared to a classical comb filter.

From the results shown in the table above, we notice a rejection gain of more than 10 dB compared to classical 5th-order comb filters. Therefore, condition 2 is met.

### 4.3. Multiplierless, nonrecursive architecture for implementing the filter cell \(H_{2r}(z)\)

This section presents an effective algorithm guaranteeing multiplierless architectures of the proposed class of decimation filters. Let us quantize the multiplier 2\(\cos(\alpha M_1)\) in (16) as

\[
\begin{align*}
N & = 2^{-k}, \\
I & = 1
\end{align*}
\]

where \(I\) is an integer that can be represented as a sum of power-of-2 terms. Upon using (40), \(H_{2r}(z) = 0.25(1 + 2\cos(\alpha M_1)z^{-1} + z^{-2})\) can be rewritten as

\[
H_{2r}(z) = 2^{-2}\left[ 1 + 2^{-k+1}iz^{-1} + z^{-2} \right].
\]

Next line of pursuit consists in defining the integer \(k\) used above. This is addressed in the next subsection.

### 4.4. Choice of the integer \(k\)

We choose a tolerance \(\Delta\) in such a way that the actual value of \(\alpha\) called \(\alpha_1\), differs from \(\alpha_{opt}\) by the chosen value \(\Delta\), i.e., \(\alpha_1 = \alpha_{opt} + \Delta\). Basically, \(\alpha_1\) is the actual value of \(\alpha_{opt}\) resulting from the quantization of \(2\cos(\alpha M_1)\), while \(\Delta\) is the maximum error tolerable.

A convenient choice justified by simulation results is the value \(\Delta = 0.001\). By this setup, we have \(\cos(\alpha_1 M_1) = 2^{-k}I\), where \(I_{max} = 2^k - 1\). Therefore, it is

\[
\cos(\alpha_1 M_1) \leq 2^{-k} (2^k - 1) = 1 - 2^{-k}.
\]

From the previous relation, the following inequality easily follows:

\[
2^k \geq 1/(1 - \cos(\alpha_1 M_1)).
\]

Upon solving for \(k\), we obtain:
\[ k \geq \log_2 \left( \frac{1}{1 - \cos(\alpha_1 M_1)} \right) \rightarrow k \geq \frac{\log_{10} \left( \frac{1}{1 - \cos(\alpha_1 M_1)} \right)}{\log_{10}(2)}. \tag{44} \]

The latter can be rewritten as follows:
\[ k = \left\lceil \frac{\log_{10} \left( \frac{1}{1 - \cos(\alpha_1 M_1)} \right)}{\log_{10}(2)} \right\rceil, \tag{45} \]
where the function \( \lceil x \rceil \) returns the integer greater than or equal to the real number \( x \).

Next section presents two design examples to clarify the choice of the aforementioned parameters.

5. Design examples and comparisons

This section presents two design examples in order to emphasize the steps for the choice of the parameters appearing in the proposed decimation filters. Comparisons are also provided with respect to classical comb filters.

5.1. Design example 1

In this design example we consider a decimation factor \( M = 16 \) and an oversampling factor \( \rho = 64 \). This example accounts for moderately oversampled signals.

The normalized bandwidth of the sampled signal is \( f_0 = \frac{1}{2\pi} = 0.0078125 \) (this means that the analog bandwidth \( B_x \) of the signal is mapped to the digital frequency interval \([-f_0, +f_0] \) after A/D conversion with sample rate \( f_s \)), while the first folding band (from (1)) is \([0.0546875, 0.0703125]\).

Let us discuss the design steps of the proposed decimation filter.

1. Given \( M = 16 \), it is \( M_1 = 8 \). The order \( N = 3 \) is chosen in order to compare the proposed filter to a 5th-order comb filter employed in commercial products.
2. The parameter \( b \) belonging to the droop compensation filter is \( b = 0 \) from Table 1.
3. From Table 2, the optimal values of the zeros rotations are \( q_{\text{opt}} = 0.896 \) and \( \alpha_{\text{opt}} = 0.0439823 \), thus yielding an additional noise power rejection of 12.56 dB compared to a 5th-order comb filter.
4. Considering \( \Delta = 0.001 \) in \( \alpha_1 = \alpha_{\text{opt}} + \Delta = 0.0449823 \), and using (46), the value \( k = 4 \) easily follows.
5. Finally, \( I = 15 \) from \( I = 2^k - 1 \).

Fig. 9 compares the magnitude responses of the proposed filter \( H_{rr}(z) \) with that of a 5th-order comb filter. A key observation here is the presence of the rotated zeros spanning the folding band \([0.0109375, 0.140625]\) as clearly highlighted in Fig. 10 around the first folding band. The passband behavior of the proposed filter \( H_{rr}(z) \) is compared to the one of a classical 5th-order comb filter in Fig. 11.

Notice that the proposed filter introduces a maximum droop of 0.06 dB, while the 5th-order comb filter presents a signal distortion as high as 1.1 dB. The latter value must be compensated by the FIR filter following the comb filter in any multistage decimation chain, thus increasing the computational complexity of the overall decimation filter.
5.2. Design example 2

In this design example, we consider a decimation factor $M = 16$ and an oversampling factor $\rho = 128$. This setup accounts for highly oversampled digital signals. The normalized bandwidth of the sampled signal is $f_0 = \frac{1}{2\rho} = 0.0039$, while the first folding band is:

$$\left[ \frac{1}{M} - f_0, \frac{1}{M} + f_0 \right] = [0.0586, 0.0664].$$

Since $\omega = 2\pi f$, we notice in passing that the aforementioned frequencies are mapped to the frequencies $\frac{\omega_0}{\pi} = \frac{\rho}{\rho} = 0.0078$.

$$\left[ \frac{2}{M} - 2f_0, \frac{2}{M} + 2f_0 \right] = [0.1172, 0.1328],$$

when the x-axis represents the variable $\omega/\pi$.

Let us discuss the design steps of the proposed decimation filter.

1. Given $M = 16$, it is $M_1 = 8$. The order $N = 3$ is once again chosen in order to compare the proposed filter to 5th-order comb filters.
2. The parameter $b$ belonging to the droop compensation filter is $b = 0$ from Table 1.
3. From Table 2, the optimal values of the zeros rotations are $\alpha_{opt} = 0.887$, and $\alpha_{opt} = 0.02177025534167$, thus yielding an additional noise power rejection of 12.95 dB compared to a 5th-order comb filter.
4. Given $A = 0.001$, it is $\alpha_1 = \alpha_{opt} + A = 0.02212596078600$.

From (46), it is $k = 6$ and $I = 63$ (from $I = 2^k - 1$).

The magnitude response of the proposed filter $H_{rr}(z)$ is compared with the one of a 5th-order comb filter in Fig. 13. Notice the presence of the rotated zeros spanning the first folding band $[0.1172, 0.1328]$, as clearly emphasized in Fig. 14.

The passband behavior of the magnitude responses of the proposed filter $H_{rr}(z)$ and the one of a 5th-order CIC filter is depicted in Fig. 15. Notice that the proposed filter introduces a maximum droop lower than $0.05$ dB, while the 5th-order comb filter presents a signal distortion as high as $0.28$ dB.

Finally, Fig. 16 compares the frequency behavior of the proposed filter $H_{rr}(z)$ employing real multipliers with the frequency response of the filter embedding approximated multipliers in the first folding band.

5.3. Design example 3

Unlike the previous two design examples, the aim of this last example is to apply the proposed decimation filters to $\Sigma\Delta$ A/D converters. Upon using Matlab, we have simulated a 2nd-order $\Sigma\Delta$ A/D converter with a two-level quantizer and a sampling frequency $f_s = 25.6$ kHz. The input signal is a band-limited signal with bandwidth $B_x = 100$ Hz. With this setup, it is $\rho = f_s/2B_x =$
Fig. 15. Magnitude response of the proposed filter with quantized multipliers compared to a 5th-order comb filter at baseband. The plot also shows the droops introduced by the two filters in the frequency domain.

Fig. 16. Magnitude response of the proposed filter with quantized multipliers compared to the one embedding real multipliers around the first folding band. While the normalized frequency bandwidth of the input signal is $f_0 = 1/2\rho = 1/256$. For decimating oversampled $\Sigma\Delta$ A/D converters, the parameter $N$ must be chosen in such a way that $N + 2$ equals the $\Sigma\Delta$ A/D order plus one [28]. Therefore, for a 2nd-order $\Sigma\Delta$ A/D modulator, it is $N + 2 = 2 + 1$, or $N = 1$. The oversampled signal at the output of the $\Sigma\Delta$ A/D modulator is decimated through the filter in Fig. 8 using $M_1 = 16$, $M = 32$. Notice that this kind of filters can be used to decimate down to 4 times the Nyquist frequency [28] as for regular comb filters. Therefore, a more frequency selective filter must be employed after the cascade in order to reduce the sampling rate to the Nyquist frequency and select the useful signal. This filter is usually a cascade of two half-band filters each decimating by 2 [1,3]. The power spectrum of the oversampled digital signal at the output of the $\Sigma\Delta$ A/D modulator is shown in the upper subplot of Fig. 17. As expected, this plot shows the presence of the useful signal with bandwidth $f_0 = 1/2\rho = 1/256$ shrunk at baseband, and the noise spectrum moved outside the useful signal bandwidth. The power spectrum of the decimated signal at the output of the proposed filter in

Fig. 17. Power spectrum of the digital signal at the output of a 2nd-order $\Sigma\Delta$ A/D converter (upper subplot), and power spectrum of the signal decimated by $M = 32$ with the proposed filter employing $N = 1$ (lower subplot).

6. Conclusion and discussion

This paper proposes novel reprogrammable, multiplierless, multirate filter architectures, especially tailored to Digital Sample Rate Conversion in Oversampled Transceivers, Software Radio Transceivers, and Sigma-Delta Converters. The proposed solution consists of novel two-stage multiplierless and nonrecursive decimation architectures with improved performance compared to classical comb filters employed in commercial products. Like comb filters, the proposed architecture presents a reprogrammable order $N$ in order to attain the desired specifications for the intended signals, as well as a reprogrammable decimation factor $M$. Compared to comb filters, the proposed filters enjoy a number of advantages. Among them, we mention the improved spurious signal rejection in the folding bands compared to classical comb filters, and a reduced passband distortion. When compared to GC Filters, the proposed filters are multiplierless, present reduced passband distortion and, additionally, do not have problems of pole-zero cancellation.

One key application of the proposed filters is for decimating oversampled $\Sigma\Delta$ A/D converters. In this scenario, the parameter $N$ must be chosen in such a way that $N + 2$ equals the $\Sigma\Delta$ A/D order plus one.

We have provided a number of design examples to show the way the proposed filters can be applied to meet the required specifications in general multirate applications as well as for $\Sigma\Delta$ A/D converters.

References


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