A procedure for solving some second-order linear ordinary differential equations

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ABSTRACT

The purpose of this work is to introduce the concept of pseudo-exactness for second-order linear ordinary differential equations (ODEs), and then to try to solve some specific ODEs.

1. Introduction

Many of the differential equations that describe physical phenomena are linear differential equations, and among these, the second-order differential equation is the most common and the most important special case. They arise in the fields of mechanics, heat, electricity, aerodynamics, stress analysis, and so on. In general, a second-order linear differential equation with variable coefficients cannot be solved directly, and in most cases this is impossible. Therefore, the problem of finding the solution of a second-order homogeneous ODE

\[ Py'' + Qy' + Ry = 0, \]  

where \( y = f(x) \) is an analytical solution of (1) and \( P, Q, R \) are known functions of \( x \) with no restrictions placed on their nature, arises frequently in science and engineering. In practice, equations of the form (1) do not usually have closed form solutions; and even when they do, it may be very difficult to find them. Hence the exploitation of analytical methods for solving such equations becomes a main subject of considerable interest in most textbooks [1–3]. In efforts to overcome this problem several methods have been introduced [1], but all were for specific forms or needed to have a precondition. Probably the most well-known and widely used algorithm for finding a solution is the method of reduction of order [1–3]; and if the equation is exact, methods are known. In the next section, we introduce a new concept for a second-order linear ODE, and using this concept we try to find closed form solutions of (1) if they exist, and then we extend the consideration to finding a particular solution of the corresponding non-homogeneous case of (1).

2. The definition and procedure

Definition. We call Eq. (1) a pseudo-exact equation if

\[ P'' - Q' + R = 1. \]
If Eq. (1) is not exact, it can be made exact by multiplying it by an appropriate integrating factor \( \mu(x) \). Thus we require that \( \mu(x) \) be such that

\[
\mu(x) \frac{d^2y}{dx^2} + \mu(x) \frac{dy}{dx} + \mu(x) y = 0,
\]

is exact. Since, (3) is exact then we will have

\[
(\mu(x) \frac{d^2y}{dx^2}) - (\mu(x) \frac{dy}{dx}) + R \mu(x) = 0.
\]

This leads to

\[
P \mu'' + (2P' - Q) \mu' + (P'' - Q' + R) \mu = 0.
\]

This equation is known as the **adjoint** equation. It plays a very important role in the advanced theory of differential equations [1]. In general the problem of solving the adjoint differential equation is as difficult as that of solving the original equation. But if we choose \( R = 1 \) it can be seen that (4) is a pseudo-exact equation and we may solve it. Therefore, we will have two options: change the original equation to a pseudo-exact form, or find \( \mu(x) \) in (4) with \( R = 1 \).

How can we change Eq. (1) to a pseudo-exact equation? By a simple procedure Eq. (1) can be inverted into a pseudo-exact form.

**Procedure.** Divide Eq. (1) by \( R \) and rewrite (1) as

\[
\frac{P}{R} \frac{d^2y}{dx^2} + \frac{Q}{R} \frac{dy}{dx} + y = 0.
\]

Now if we choose \( \frac{P}{R} = p \) and \( \frac{Q}{R} = q \), then Eq. (5) can be written as

\[
p \frac{d^2y}{dx^2} + q \frac{dy}{dx} + y = 0.
\]

Differentiating (6) leads to

\[
p \frac{d^3y}{dx^3} + (q + p') \frac{d^2y}{dx^2} + (q' + 1) \frac{dy}{dx} = 0.
\]

Now if we let \( y' = z \), then we obtain

\[
pz'' + (q + p')z' + (q' + 1)z = 0.
\]

which is a pseudo-exact equation. Eq. (8) may be solved easily if Eq. (1) has closed form solutions.

The idea of a pseudo-exact equation can be extended to non-homogeneous equations. Consider

\[
y'' + py' + qy = 0,
\]

which is a pseudo-exact equation. Hence \( q - p' = 1 \). That is, \( q = p' + 1 \). Therefore (9) becomes

\[
y'' + py' + (1 + p')y = 0.
\]

Using Eq. (10) we write

\[
y'' + py' + (1 + p')y = e^{- \int pdx}.
\]

It is not difficult to show that the particular solution of Eq. (11) is given by \( y_p = e^{\int pdx} \).

This idea can also be extended to any second-order linear ODE in the form of

\[
y'' + py' + qy = (q - p')e^{- \int pdx},
\]

where its homogeneous equation is not pseudo-exact, and it can easily be shown that its particular solution is \( y_p = e^{- \int pdx} \). For example, \( y_p = \sin^2 x \) is a particular solution of

\[
y'' - 2 \cot x y' + (2 \cot^2 x + 1)y = - \sin^2 x.
\]

But in this work we focus on homogeneous equations and leave the non-homogeneous case to the reader. In the next section, by considering some examples we can obtain a better understanding about the application of pseudo-exact form equations.

### 3. Examples

This section deals with some tests on a few examples that been considered in several ODE textbooks. Here we resolved them by using the above procedure and one can thus compare the methods.
Example 1. Consider

\[(1 - x \cot x)y'' - xy' + y = 0.\]

The general solution of this equation is \(y = c_1 x + c_2 \sin x\), and is taken from [1]. The author in this book assumed that \(y_1 = x\), and then found \(y_2\) by method of reduction of order. But if we apply the procedure introduced here, first we differentiate, and then put \(y' = z\). We obtain

\[(1 - x \cot x)z'' - \cot x(1 - x \cot x).z' = 0.\]

That is, \(z'' - \cot x.z' = 0\).

It is not difficult to show that \(z = c_1 + c_2 \cos x\). Therefore, \(y' = c_1 + c_2 \cos x\). By a simple integration we obtain \(y = c_1 x + c_2 \sin x + c_3\). It is easy to show that \(c_3 = 0\), unless we are dealing with a non-homogeneous case.

Example 2. Consider

\[x(x - 2)u'' - 2(x - 1)u' + 2u = 0, \quad 0 < x < 2.\]

This equation was chosen from [2] and its general solution is \(u = c_1 x^2 + c_2(x - 1)\).

The author did not solve it. He just asked the reader to show that \(u_1 = x^2\) and \(u_2 = x - 1\) are two solutions of this equation. But if we follow the above procedure we come to \(\frac{1}{2}x(x - 2)u''' = 0\). That is, \(u''' = 0\). Integrating three times, successively, we obtain \(u = c_1 x^2 + c_2 x + c_3\). By a simple substitution, we come to \(c_3 = -c_2\).

Example 3. Consider

\[(1 + x^2)y'' - 4xy' + 6y = 0.\]

This equation was chosen from [3]. The author offered to solve it by a power series method. Here we inverted it to a pseudo-exact form, i.e.,

\[
\frac{1}{6}(1 + x^2)z'' - \frac{1}{3}xz' + \frac{1}{3}z = 0.
\]

Obviously, \(z_1 = x\), and by method of reduction \(z_2 = x^2 - 1\). Thus, \(y_1 = x\), whence \(y_1 = \frac{x^2}{2} + c_1\), and \(y_2 = x^2 - 1\), that is \(y_2 = \frac{x^2}{2} - x + c_2\). Substituting in the given equation leads to \(c_1 = 6\), and \(c_2 = 0\). Therefore, the general solution will be

\[y = b_1 \left(\frac{x^2}{2} - \frac{1}{6}\right) + b_2 \left(\frac{x^3}{3} - x\right) = a_0(1 - 3x^2) + a_1 \left(x - \frac{x^3}{3}\right),\]

where \(a_0 = -\frac{1}{6} b_1\) and \(a_1 = -b_2\), respectively.

We end this section by considering the Hermite equation.

Example 4. The Hermite equation is given by

\[y'' - 2xy' + 2\lambda y = 0.\]  \hspace{1cm} (13)

It is clear that for \(\lambda = 1\), \(y_1(x) = x\). Let \(\lambda = 2\), that is,

\[y'' - 2xy' + 4y = 0.\]  \hspace{1cm} (14)

Writing this equation as

\[
\frac{1}{4}y'' - \frac{1}{2}xy' + y = 0,
\]

and differentiating, we can write

\[
\frac{1}{4}z'' - \frac{1}{2}xz' + \frac{1}{2}z = 0,
\]

where \(z = y'\). Obviously \(z_2(x) = x\); hence \(y_2(x) = \frac{x^2}{2} + c\). Substituting in (13) leads to

\[y_2(x) = 1 - 2x^2.\]

If we choose \(\lambda = 3\) in (12), we get

\[y'' - 2xy' + 6y = 0.\]
In the same manner we obtain
\[ \frac{1}{6}z'' - \frac{1}{3}xz' + \frac{1}{3}z = 0, \]
which has the same solution as (14). That is, \( z_3(x) = 1 - 2x^2 \). Since \( z_3(x) = y_3'(x) \), then it is easy to show that \( y_3(x) = x - \frac{2}{3}x^3 \).

If we carry through this implementation we can obtain the following results:
\[
\begin{align*}
y_4(x) &= 1 - 4x^2 + \frac{4}{3}x^4, \\
y_5(x) &= x - \frac{4}{3}x^3 + \frac{4}{15}x^5,
\end{align*}
\]

This implementation can also be used to solve the Legendre equation.

4. Conclusions

The results from all the examples in this work show the efficiency of this procedure. In spite of the fact that in Examples 1–3, we did not know that the given equation has a closed form solution, with this procedure such solutions may be found. In Example 4 it was shown that the Hermite equation can also be solved by this procedure. The author hopes to extend this method to all second-order linear ODEs, or, working on (12), to get a general solution of (1). Research on this matter is one of my future goals.

References