Disturbance Suppression via Robust MPC using Prior Disturbance Data – Application to Flight Controller Design for Gust Alleviation –

Masayuki Sato, Nobuhiro Yokoyama, and Atsushi Satoh

Abstract—This paper addresses disturbance suppression problem for uncertain plant systems using prior disturbance data which contain some measurement errors. We tackle optimal control input design problem using Model Predictive Control (MPC) scheme in which a priori measured disturbance data are exploited. We show that if the uncertainties of the plant systems are expressed by bounded but time-invariant uncertain delays at the control input, then we only have to consider finitely many plant models instead of the original uncertain plant systems. Furthermore, we also show that if the measurement errors in prior disturbance data are expressed as affine with respect to some constant uncertain vector, whose elements are bounded, then we only have to evaluate the measurement errors at the vertices of the vector. Using these, we propose a robust MPC design with finitely many conditions for our addressed problem. Finally, we apply our proposed method to flight controller design problem for suppressing the vertical acceleration driven by turbulence, i.e., Gust Alleviation (GA) flight controller design problem.

I. INTRODUCTION

Disturbance suppression is one of the very important design specifications for many systems, and many papers on this topic have been reported. As the design problem of disturbance suppressing controllers can be expressed as one of $H_\infty$ problems using fictitious performance block [1], the design problem for disturbance suppression controllers is now easily solved. In this framework, only the states of the systems are exploited. If the disturbance data are obtained a priori and can be exploited for control inputs, it is inferred that disturbance suppression performance is improved. Flight control has a very similar topic, i.e., Gust Alleviation (GA) flight controller design problem, in short GA problem, using prior disturbance data [2], [3], [4]. Roughly speaking, GA problem is to design flight controllers which reduce the vertical acceleration driven by turbulence. In the 1970s, the turbulence is measured at the head of the aircraft; however, the lead time before encountering the turbulence becomes very short as the aircraft speed increases, thus the measured turbulence data cannot be effectively used. On this issue, as electronic and optic technologies have advanced these years, turbulence can be measured several seconds ahead [5], [6].

Considering these backgrounds, design methods for disturbance suppression controllers exploiting a priori measured disturbance data have been desired.

As we suppose that disturbance data are given a priori, if the current state is also available, then controllers based on Model Predictive Control (MPC) scheme will work very well, as illustrated in [7], [8]. However, if there exist uncertainties in the plant model, which occurs in many applications, then MPC must consider all possibilities for the uncertainties. This generally produces infinitely many conditions and consequently leads to unsolvable problems. To circumvent this difficulty, many methods have been proposed, e.g., [9], [10], [11], [12], [13], [14]. Roughly speaking, we have the sole solution for it, i.e., designing conservative controllers. For example, the uncertain plant is expressed as a polytopic system then the controller is designed by solving Linear Matrix Inequalities (LMIs) associated with $H_\infty$ performance or $H_2$ performance using common Lyapunov functions [9], [12], or robust invariant ellipsoid is used to ensure the robust performance [13]. However, common Lyapunov functions and invariant ellipsoids both generally introduce conservatism, which should be avoided.

In general, there are various uncertainties for the plant systems, e.g., parametric uncertainties, frequency-domain uncertainties, uncertain dead time, etc. If the operating ranges of the controlled systems are relatively small and the nominal plant models are relatively well known, then the uncertainty to be considered most is the unmodeled dynamics of the plant, which usually lies in the high frequency range. One of the effective representations of the plant uncertainties existing in the high frequency range is uncertain but bounded delays at the control input channels [15], [16], [17], where the effectiveness of the delay models for representing uncertainties in the high frequency range is demonstrated. Roughly speaking, the reason is as follows: since the delay at control input generally augments phase lag, the controller designed considering the uncertain delay at the control input would have enough stability margin compared to the controller designed considering no uncertain delay. This consequently implies that the uncertain delay at the control input makes the controlled system have sufficiently small gain in the high frequency range, i.e., robust against uncertainties in the high frequency range. Thus, we assume that plant uncertainties are expressed as bounded but uncertain time-invariant delays at the control input channels. In this paper, the disturbance is assumed to be measured a priori. In general, the measured data have measurement errors, such as, bias error, position error, etc. Therefore, when exploiting
the measured disturbance data, the measurement errors must be considered. Considering these backgrounds, we address controller design problem achieving disturbance suppression for plant systems, in which the uncertainties are represented as bounded but time-invariant uncertain delays at the control input, via MPC using prior disturbance data containing some measurement errors. For this problem, we proposed a robust MPC design with finitely many conditions with introducing neither conservatism nor approximations when deriving the associated conditions.

Hereafter, $0_n$, $0_{n,m}$, $0$ and $I_n$ respectively denote an $n$-dimensional zero matrix, an $n \times m$-dimensional zero matrix, an appropriately dimensioned zero matrix and an $n$-dimensional identity matrix, $I_n$ denotes an $n$-dimensional vector with all elements being unities, $Z$, $\mathbb{R}^n$ and $\mathbb{R}^{n \times m}$ respectively denote the set of integers, the set of $n$-dimensional real vectors and the set of $n \times m$ dimensional real matrices, $\otimes$ denotes Kronecker product, and $[p]$ denotes $\min \{n \in \mathbb{Z} | p \leq n \}$. For $n$-dimensional vectors $p = [p_1 \cdots p_n]^T$ and $r = [r_1 \cdots r_n]^T$, $p \leq r$ denotes that $p_i \leq r_i$ holds for all $i$, that is, the inequality holds element-wise.

II. PRELIMINARIES

In this section, we first define our supposed uncertain plant system and derive a family of models representing the uncertain plant system, then define a priori obtained disturbance data with some measurement errors, and finally define our addressed problem.

A. Uncertain Plant System

Let us define the nominal continuous-time plant system $P_c$.

$$
P_c : \begin{cases} 
\dot{x}(t) = A_c x(t) + B_1 w(t) + B_2 u(t) \\
\dot{z}(t) = C_c x(t) + D_1 w(t) + D_2 u(t)
\end{cases}, \tag{1}
$$

where $x(t) \in \mathbb{R}^n$, $w(t) \in \mathbb{R}^{n_w}$, $u(t) \in \mathbb{R}^{n_u}$, and $z(t) \in \mathbb{R}^{n_z}$ respectively denote the state, the disturbance input, the control input, and the performance output. We assume that all states are measurable and available for our controller.

We assume that the uncertainties of the plant system are represented as a delay system with bounded but time-invariant uncertain delay at the control input; that is, the control input $u(t)$ is given as

$$
u(t) = v(t - T_d), \tag{2}
$$

where $v(t) \in \mathbb{R}^{n_u}$ denotes the control input command created by the implemented computer and $T_d$ [s] denotes the uncertain delay time which is assumed to lie in the interval between $T_{d_{\text{min}}}$ and $T_{d_{\text{max}}}$, i.e.

$$
T_d \in [T_{d_{\text{min}}}, T_{d_{\text{max}}}]. \tag{3}
$$

Considering the delay due to the calculation of the control input command with the implemented computer, the minimum delay time $T_{d_{\text{min}}}$ is assumed to be larger or equal to the sampling period of the implemented computer which is given as $T_s$ [s], i.e.

$$
T_s \leq T_{d_{\text{min}}}. \tag{4}
$$

Similarly to usual MPC in the literature [9], [11], [13], etc., we address controller design problem for the discretized plant of $P_c$. Suppose that the discretized plant of $P_c$ is given as $P_d$ using a zero-order hold.

$$
P_d : \begin{cases} 
x_{k+1} = Ax_k + B_1 w_k + B_2 u_k \\
z_k = C x_k + D_1 w_k + D_2 u_k
\end{cases}, \tag{5}
$$

where $x_k \in \mathbb{R}^n$, $w_k \in \mathbb{R}^{n_w}$, $u_k \in \mathbb{R}^{n_u}$, and $z_k \in \mathbb{R}^{n_z}$ respectively denote the state, the disturbance input, the control input, and the performance output of $P_d$ at step $k$. The sampling period for the discretization is assumed to be the same as the sampling period of the implemented computer $T_s$ [s]. From that we have assumed that all states of $P_c$ are measurable and available, it is also assumed that all states of $P_d$ are measurable and available.

Remark 1: In equation (5), the disturbance $w_k$ is assumed to be constant during the sampling period $T_s$, generally speaking, which is not realistic. However, if the sampling period is small then the disturbance can be regarded as constant during the sampling period.

Similarly, the delay system (2) should be discretized in some way. Considering that the control input is applied not continuously but discretely, e.g. only at sampling steps, the delay effect of the control input command to the control input is limited. That is, in a sharp contrast to the continuous-time case, for the discrete-time case, we only have to consider the whole elements of delay step set $T_d$ which is defined as (6), as the uncertain delay $T_d$ lying in the interval $[T_{d_{\text{min}}}, T_{d_{\text{max}}}]$.

$$
T_d = \{d, d + 1, \cdots, d\}, \tag{6}
$$

where $d$ and $d$ are respectively defined as $\lfloor T_{d_{\text{min}}} \rfloor$ and $\lfloor T_{d_{\text{max}}} \rfloor$. The number of the elements in set $T_d$ is denoted by $d$, which is given as $d = d + 1$. That is, if continuous-time systems are considered then all possible delays lying in $[T_{d_{\text{min}}}, T_{d_{\text{max}}}]$ should be considered; however, if discrete-time systems are considered then we only have to consider the delay steps belonging to set $T_d$. Thus, the control input $u_k$ is given as one of the following set:

$$
\{ v_{k-d}, v_{k-d-1}, \cdots, v_{k-d}\} \tag{7}
$$

where $v_m$ denotes the control input command of $P_d$ created at step $m$; however, it is unknown which exactly is $u_k$ (see Fig. 1).

The control input command $v_k$ is factorized into the previous control input command $v_{k-1}$ and the deviation between these commands to consider the rate limit of the control input command, i.e.

$$
v_k = v_{k-1} + \Delta v_k. \tag{8}
$$

Under these preliminaries, our supposed plant $P_u$, which has uncertain delay at the control input, is described as follows.

$$
P_u \in \{P_{ud}, P_{ud+1}, \cdots, P_{ud}\}, \tag{9}
$$
where \( P_u^i (i = 1, 2, \ldots, d) \) is defined as
\[
P_u^i \triangleq \left\{ \begin{array}{l}
\dot{x}_k^{i+1} = A^i \dot{x}_k^{i} + B^i_1 w_k + \hat{B}^i_2 \Delta v_k \\
\dot{x}_k^{i} = C^i \dot{x}_k^{i} + D^i_1 w_k + \hat{D}^i_2 \Delta v_k
\end{array} \right.,
\]
where \( \dot{x}_k^{i} \) denotes the augmented state of \( i \)-th plant model at step \( k \) and is defined as
\[
\dot{x}_k^{i} := \left[ x_k^{i T} \quad v_k^{i-1} \quad \ldots \quad v_k^{i-1} \right]^T,
\]
with \( x_k^{i} \) which denotes the state of \( i \)-th plant model at step \( k \) and the matrices \( A^i \), etc. are defined as in (11), which is at the top of the next page.

**Remark 2:** Although the uncertainty model using the delay (2) with bounded uncertain time-invariant delay (3) generally introduces approximations from the real system, the derivation of a family of plant models (9) from the supposed uncertainties introduces neither assumption nor approximation. Thus, our formulation introduces no further conservatism from our supposed uncertain plant model.

**Remark 3:** Matrices \( A^i \), etc. have no uncertainties. That is, we consider the set defined in (9), whose elements have no uncertainties, instead of considering the uncertainties of our supposed plant system.

As the current state \( x_k \) for \( P_d \) is assumed to be available, and previously created control input commands, \( v_k^{i-1}, \ldots, v_k^{i-1} \), can be memorized in the implemented computer which produces the control input command, the augmented state of \( i \)-th plant model, \( \dot{x}_k^{i} \), is assumed to be available and given as \( \dot{x}_k^{i} \); that is, the following holds.
\[
\dot{x}_k = \dot{x}_k^d = \dot{x}_k^{d+1} = \cdots = \dot{x}_k^{i} \tag{12}
\]

Hereafter, we consider \( P_u \) as our plant model to design control input for \( P_c \) defined in (1) with \( u(t) \) being given as (2).

B. Uncertain Disturbance Data

Using some system, it is assumed that disturbance \( w \) is measured before the disturbance affects the plant \( P_c \). Generally speaking, the measured data have measurement errors even if the calibration was conducted before its use. Thus, we assume that the \( j \)-step ahead real disturbance at step \( k \), which is denoted by \( w_{k+j} \), satisfies the following relation with the \( j \)-step ahead disturbance measured at step \( k \), which is denoted by \( w_{k+j} \).
\[
\tilde{w} := \begin{bmatrix} w_k \\ w_{k+1} \\ \vdots \\ w_{k+N-1} \end{bmatrix} = \begin{bmatrix} w_k^{-1} \\ w_{k+1}^{-1} \\ \vdots \\ w_{k+N-1}^{-1} \end{bmatrix} + \begin{bmatrix} X_0 \\ X_1 \\ \vdots \\ X_{N-1} \end{bmatrix} \Delta_w, \tag{13}
\]
where \( N \), which is given as a constant non-negative integer, denotes the maximum step number for measuring disturbance \( a \ priori \), \( X_j \in \mathbb{R}^{n_w \times n_w} (j = 0, 1, \ldots, N - 1) \) denote the constant given matrices which define the measurement errors in the measured disturbance data with uncertain constant vector \( \Delta_w \) which satisfies (14) (see Fig. 2).
\[
-1_{n_w} \leq \Delta_w \leq 1_{n_w} \tag{14}
\]

We define set \( \Omega \) as the existence region of \( \tilde{w} \).
\[
\Omega = \{ \tilde{w} \in \mathbb{R}^{n_w \times N} : \tilde{w} \text{ given as (13) with } \Delta_w \text{ satisfying (14)} \} \tag{15}
\]

**Remark 4:** Note that matrices \( X_j \) might be different for different \( j \)-s; that is, it is possible that \( X_0 \neq X_1 \neq \cdots \neq X_{N-1} \) holds. Furthermore, note that matrices \( X_j \) might be different for different steps \( k \)-s; that is, each matrix \( X_0, \cdots, X_{N-1} \) at step \( k \) might be different from each matrix \( X_0, \cdots, X_{N-1} \) at step \( k - 1 \). This corresponds to time-varying measurement error case.

**Remark 5:** Note that \( \tilde{w} \) is affine with respect to each element of \( \Delta_w \).

Hereafter, we assume that, at each step, we cannot obtain the real disturbance data \( w_{k+j} \) \( a \ priori \), but instead, we obtain the measured data \( w_{k+j} \) which satisfies (13) with some measurement errors defined as \( X_j \Delta_w \).

C. Problem Definition

Using uncertain plant model \( P_u \) given as (9) using (10) and measured prior disturbance data \( w_{k+j} \) satisfying (13), we define our addressed problem.
Considering that the disturbance data are measured for \( N \) steps ahead, the horizon step number in MPC is also set as \( N \). For \( i \)-th plant model \( P_u^i \), we define the performance index for disturbance suppression as

\[
J_i(\hat{x}^i, \Delta v, z^i) = \sum_{j=0}^{N-1} \left( (\hat{x}^i_{k+j+1|k})^T Q \hat{x}^i_{k+j+1|k} + \Delta v_{k+j|k}^T R \Delta v_{k+j|k} + (z^i_{k+j|k})^T S z^i_{k+j|k} \right),
\]

where matrices \( Q \) and \( S \) are appropriately defined positive semidefinite matrices, matrix \( R \) is an appropriately defined positive definite matrix, \( \hat{x}^i_{k+j|k} \) denotes the \( i \)-th plant’s augmented state at step \( k+j \) predicted at step \( k \), \( \Delta v_{k+j|k} \) denotes the control input command deviation at step \( k+j \) created at step \( k \), and \( z^i_{k+j|k} \) denotes the performance output at step \( k+j \) predicted at step \( k \).

Furthermore, the augmented state \( \hat{x}^i \), the control input command deviation \( \Delta v \) and the performance output \( z^i \) have their constraints, i.e. they should satisfy the following constraints:

\[
\gamma_{\text{min}} \leq \hat{x}^i_{k+j+1|k} \leq \gamma_{\text{max}}, \quad j = 0, \ldots, N - 1,
\]

\[
\delta_{\text{min}} \leq \Delta v_{k+j|k} \leq \delta_{\text{max}}, \quad j = 0, \ldots, N - 1,
\]

\[
\xi_{\text{min}} \leq z^i_{k+j|k} \leq \xi_{\text{max}}, \quad j = 0, \ldots, N - 1,
\]

where \( \gamma_{\text{min}}, \gamma_{\text{max}} \in \mathbb{R}^{n_u} \) are given constant vectors with finite elements, and \( \delta_{\text{min}}, \delta_{\text{max}} \in \mathbb{R}^{n_u} \) and \( \xi_{\text{min}}, \xi_{\text{max}} \in \mathbb{R}^{n_u} \) are given constant vectors with possibly infinite elements.

Considering that if the worst performance among \( P_u^i \) is minimized then the minimized performance is the upper limit of disturbance suppression performance for uncertain plant \( P_u \), we would like to obtain \( \Delta v_{k+j|k} \) which minimizes the maximum of \( J_i(\hat{x}^i, \Delta v, z^i) \). Thus, our problem is defined as follows.

**Problem 1**: Suppose that uncertain plant system is given as \( P_u \) defined as in (9) using \( P_u^i \) in (10), the current augmented state \( \hat{x} \) is available, and \( j (j = 0, \ldots, N - 1) \) step ahead disturbance data at step \( k \) is measured as \( w_{k+j|k} \) which satisfies (13) for the real disturbance \( w_{k+j} \).

Under these assumptions, find \( \Delta v_{k+j|k} \) which satisfies (17), (18) and (19).

If Problem 1 is solved online, then the control input command \( v \) at step \( k \), i.e. \( v_k \), is calculated as \( v_{k-1} + \Delta v_k \). Our control strategy is to obtain the optimal control input command by solving an optimization problem online using the family of plant models; that is, our control scheme is MPC.

Solving Problem 1 is equivalent to solving the following problem.

**Problem 2**: Find \( \Delta v_{k+j|k} \) \( (j = 0, \ldots, N - 1) \) which minimize the following performance index.

\[
\max_{\Delta v_{k+j|k}} \max_{w \in \Omega, i \in \{\bar{d}, \ldots, \bar{d}\}} J_i(\hat{x}^i, \Delta v, z^i)
\]

subject to (17), (18), (19), (10) with (12) and (13).

**Remark 6**: Note that Problem 2 seeks the common control input command deviation for all \( i \) and for all possible \( \bar{w} \in \Omega \). Therefore, solving Problem 2 produces robust control input command deviation \( \Delta v_{k+j|k} \) against the uncertain delays at the control input satisfying (3), and all possible disturbance \( \bar{w} \in \Omega \).

In the next section, we show our proposition to solve Problem 2.

### III. MAIN RESULTS

In this section, we show the method to solve Problem 2. For simplicity, we first consider the case in which the measured disturbance data have no measurement errors, i.e. \( w_{k+j} = w_{k+j|k} \), next consider the case in which the measured disturbance data have some measurement errors.

#### A. No Measurement Error Case

Let us assume that all \( X_j \) in (13) are set as \( 0 \). Then, \( w_{k+j} \) is given as \( w_{k+j|k} \). That is, the following holds.

\[
\bar{w} = \begin{bmatrix} w_{k|k} & \cdots & w_{k+N-1|k} \end{bmatrix}^T
\]

We define the following vectors.

\[
\bar{\Delta}v = \begin{bmatrix} \Delta v_{k|k}^T & \cdots & \Delta v_{k+N-1|k}^T \end{bmatrix}^T
\]

\[
\bar{\hat{x}}^i = \begin{bmatrix} (\hat{x}^i_{k+1|k})^T & \cdots & (\hat{x}^i_{k+N|k})^T \end{bmatrix}^T, \quad i = d, \ldots, \bar{d}
\]

\[
\bar{z}^i = \begin{bmatrix} (z^i_{k|k})^T & \cdots & (z^i_{k+N-1|k})^T \end{bmatrix}^T, \quad i = d, \ldots, \bar{d}
\]

Then, the state equation and the performance output equation of \( P_u^i \) are respectively given as follows:

\[
\tilde{x}^i = \begin{bmatrix} I_N \otimes \hat{A}^i & 0_{(n_u+n_u \bar{d}),n_u+n_u \bar{d}} \end{bmatrix} \begin{bmatrix} \hat{x}^i_{k|k} \end{bmatrix} + \left( I_N \otimes \hat{B}_1^i \right) \bar{w} + \left( I_N \otimes \hat{B}_2^i \right) \bar{\Delta}v, \quad i = d, \ldots, \bar{d}
\]

\[
\tilde{z}^i = \begin{bmatrix} I_N \otimes \hat{C}^i & 0_{n_u,n_u \bar{d}} \end{bmatrix} \begin{bmatrix} \hat{x}^i_{k|k} \end{bmatrix} + \left( I_N \otimes \hat{D}_1^i \right) \bar{w} + \left( I_N \otimes \hat{D}_2^i \right) \bar{\Delta}v, \quad i = d, \ldots, \bar{d}
\]
We define the followings.
\[
\begin{align*}
\tilde{Q} := I_N \otimes Q, & \quad \hat{R} := I_N \otimes R, & \quad \tilde{S} := I_N \otimes S, \\
\tilde{\gamma}_{\min} := I_N \otimes \gamma_{\min}, & \quad \tilde{\gamma}_{\max} := I_N \otimes \gamma_{\max}, \\
\tilde{\delta}_{\min} := I_N \otimes \delta_{\min}, & \quad \tilde{\delta}_{\max} := I_N \otimes \delta_{\max}, \\
\tilde{\xi}_{\min} := I_N \otimes \xi_{\min}, & \quad \tilde{\xi}_{\max} := I_N \otimes \xi_{\max}.
\end{align*}
\]

Using these definitions, the following proposition, which is equivalent to Problem 2, is obtained.

**Proposition 1:** Find \( \hat{v} \) which minimizes \( q \) subject to (22), (23), and (24).

\[
\begin{align*}
q & \geq \begin{bmatrix}
\hat{z}_i^	op & \hat{z}_i^	op
\end{bmatrix} \begin{bmatrix}
\tilde{Q} & 0 & 0 \\
0 & \tilde{R} & 0 \\
0 & 0 & \tilde{S}
\end{bmatrix} \begin{bmatrix}
\hat{z}_i^	op \\
\hat{z}_i^	op
\end{bmatrix}, \quad i = d, \ldots, d (22)
\end{align*}
\]

\[
\begin{align*}
\begin{bmatrix}
\tilde{\gamma}_{\min} \\
\tilde{\xi}_{\min}
\end{bmatrix} & \leq \begin{bmatrix}
\hat{z}_i^	op \\
\hat{z}_i^	op
\end{bmatrix} \leq \begin{bmatrix}
\tilde{\gamma}_{\max} \\
\tilde{\xi}_{\max}
\end{bmatrix}, \quad i = d, \ldots, d (23)
\end{align*}
\]

\[
\begin{align*}
\delta_{\min} & \leq \hat{v} \leq \delta_{\max} (24)
\end{align*}
\]

As Proposition 1 is a Second-Order Cone Programming (SOCP) problem [18], we can easily obtain the global optimum with the aid of some software, e.g. [19].

Thus, if the measured disturbance data have no measurement errors then our addressed problem, i.e. Problem 1, is solved by virtue of Proposition 1 without introducing any conservatism (see Remark 2).

**Remark 7:** If Proposition 1 is solved, then the state is bounded by \( \gamma_{\min} \) and \( \gamma_{\max} \); that is, the boundedness of the state is assured.

**B. Measurement Error Case**

Let us assume that the real disturbance \( w_{k+j} \) cannot be measured and the measured disturbance \( \tilde{w}_{k+j} \) satisfies (13).

We first conduct full rank decompositions for matrices \( \tilde{Q}, \hat{R}, \) and \( \hat{S} \)

\[
\begin{align*}
\tilde{Q} = \tilde{Q}\tilde{T}, & \quad \hat{R} = \hat{R}\hat{T}, & \quad \hat{S} = \hat{S}\hat{T}.
\end{align*}
\]

Then, inequality (22) is equivalently transformed to the following inequality by applying Schur complement.

\[
\begin{bmatrix}
q & \tilde{z}_i \tilde{Q} & \tilde{z}_i \hat{R} & \tilde{z}_i \hat{T} \\
\tilde{Q}^\top & 0 & 0 \\
\hat{R}^\top & 0 & 0 \\
\hat{T}^\top & 0 & 0 & 1
\end{bmatrix} \geq 0 (25)
\]

The state \( \hat{z}_{k+N|k} \) and the performance output \( z_{k+N-1|k} \) are respectively described as in (26) and (27).

\[
\hat{z}_{k+N|k} = \begin{bmatrix}
(\hat{A}^\top)^N \hat{x}_{k|k} \\
\hat{A}^\top \hat{B}_1 \\
\vdots \\
\hat{A}^\top \hat{B}_l
\end{bmatrix} \hat{w} + \begin{bmatrix}
(\hat{A}^\top)^{N-1} \hat{B}_1 \\
\hat{A}^\top \hat{B}_1 \\
\vdots \\
\hat{A}^\top \hat{B}_l
\end{bmatrix} \hat{v} (26)
\]

\[
\hat{z}_{k+N-1|k} = \begin{bmatrix}
(\hat{C}^\top)^{N-1} \hat{x}_{k|k} \\
\hat{C}^\top \hat{B}_1 \\
\vdots \\
\hat{C}^\top \hat{B}_l
\end{bmatrix} \hat{w} + \begin{bmatrix}
(\hat{C}^\top)^{N-2} \hat{B}_1 \\
\hat{C}^\top \hat{B}_1 \\
\vdots \\
\hat{C}^\top \hat{B}_l
\end{bmatrix} \hat{v} (27)
\]

Note that both \( \hat{x}_{k+N_1|k} \) and \( z_{k+N-1|k} \) are affine with respect to each element of \( \Delta_w \), because \( \hat{w} \) is affine with respect to each element of \( \Delta_w \). Similarly, \( \hat{x}_{k+m|k} \) \((m = 1, \ldots, N-1)\) and \( z_{k+m|k} \) \((m = 0, \ldots, N-2)\) are also affine with respect to each element of \( \Delta_w \). Considering these and that (25) is affine with \( \hat{x} \) and \( \hat{z} \), we only have to evaluate the effect of \( \Delta_w \) at its vertices.

Now let \( \Phi \) be defined as the set composed of all the vertices of \( \Delta_w \), that is,

\[
\Phi = \left\{ p = [p_1 \ldots p_{n_w}]^\top \in \mathbb{R}^{n_w} \mid p_i = \pm 1, \quad i = 1, \ldots, n_w \right\} (28)
\]

The number of the elements belonging to \( \Phi \) is \( 2^{n_w} \).

Under these preliminaries, the following proposition, which is equivalent to Problem 2, is obtained.

**Proposition 2:** Find \( \hat{v} \) which minimizes \( q \) subject to (22), (23), and (24) for all \( \Delta_w \in \Phi \)

Similarly to Proposition 1, as Proposition 2 is also a SOCP problem, we can easily obtain the global optimum with the aid of some software, e.g. [19].

Thus, if the actual disturbance data have measurement errors expressed as \( X_j \Delta_w \) and satisfies (13) for the real disturbance, then our addressed problem, i.e. Problem 1, is solved by virtue of Proposition 2 without introducing any conservatism (see Remarks 2 and 5) even if the measured disturbance data has measurement errors.

Similarly to Proposition 1, if Problem 2 is solved, then the state is bounded by \( \gamma_{\min} \) and \( \gamma_{\max} \).

**Remark 8:** The increase of the numbers \( N, n_w \) and \( i \) leads to huge numerical complexity of Proposition 2. Thus, obtaining the delay time bound precisely is very important to reduce \( i \). However, in general, \( n_w \) cannot be reduced, because this number represents the number of channels for disturbance input. The remaining number \( N \) have a great impact to controller performance. On this issue, we have proposed a further result with reducing the numerical complexity of Proposition 2 in [20], and demonstrate the trade-off between controller performance and the numerical complexity.

**IV. APPLICATION TO GA PROBLEM**

In this section, we apply our proposed method to GA problem. The controller design requirement is to minimize the vertical acceleration driven by wind turbulence.

We show a design example to demonstrate that our proposed MPC works well for GA problem under the condition that there exist uncertain bounded delay at the control input and the measurement errors in *a priori* measured disturbance data.

**A. GA Problem Setup**

Let us consider the linearized aircraft longitudinal motions of JAXA’s research aircraft MuPAL-α [17] at altitude 1520[m] and true air speed 66.5[m/s]. Here we assume that only the elevator is used for aircraft control. The transfer function of its actuator dynamics is supposed to be modeled as \( 1/(0.1s + 1) \). Then, the continuous-time
The state-space matrices of $P^i_d$ ($i = 1, \ldots, 4$), which is omitted here for lack of space. The augmented state $\mathbf{x}^i_k$ is given as

$$
\begin{bmatrix}
    u_t, & \theta, & \theta_c(-1), & \delta_c(-2), & \delta_c(-3), & \mathbf{x}_e\end{bmatrix}^T,
$$

where $\delta_c(-t)$ denotes the elevator command created at $t$ step before. We would like to obtain the elevator input command, $\delta_c(0)$, which minimizes the effect of vertical turbulence to vertical acceleration for all possible delays.

The constraints for the augmented state $\mathbf{x}^i_k$ and the control input command are given as follows:

$$
\begin{align*}
    \gamma_{\max} &= \begin{bmatrix} 10 & 10 \pi \frac{180}{180} & 10 \pi \frac{180}{180} & 1 & 3 \end{bmatrix}^T, \\
    \gamma_{\min} &= -\gamma_{\max}, \\
    \delta_{\max} &= \frac{\pi}{180}, \\
    \delta_{\min} &= -\delta_{\max}.
\end{align*}
$$

This means that the rate limit of elevator command is set as $\pm 10$ [deg/s]. The constraints for performance output $\xi_{\min}$ and $\xi_{\max}$ are set as $-\infty$ and $\infty$; that is, performance output has no constraints.

Matrices $Q$ and $S$ in $J_i(\mathbf{x}, \Delta v, z)$ are set as $Q = 0_9$ and $S = 1$ respectively. Matrix $R_i$ will be set later.

We suppose that disturbance, i.e. vertical turbulence, is given as

$$
w_g(t) = \sin(\omega t),
$$

where $t$ denotes time starting from 0, and $\omega$, which will be set later, denotes the frequency of the turbulence.

We set matrices $X_j$ in the measurement error as

$$
X_j = 0.2 + 0.1 \times (66.5/100 \times T_s) j.
$$

This means that the measurement error for $w_g$ is composed of constant bias error 0.2 [m/s] and measurement error which is proportional to distance, the latter has 0.1 [m/s] measurement error at 100 [m] ahead. In our simulations, we consider three possibilities; that is, (i) the real turbulence is the same as the measured turbulence data, (ii) the real turbulence is the upper bound of the assumed turbulence data, i.e. $w_{k+j} = w_{k+j|h} + X_j$ using (31), and (iii) the real turbulence is the lower bound of the assumed turbulence data, i.e. $w_{k+j} = w_{k+j|h} - X_j$ using (31).

B. Simulation Results

We conduct numerical simulations for 20 [s] using continuous-time system (1) composed of MuPAL-α’s linearized longitudinal motions and first-order modeled elevator actuator system, and our proposed MPC in which Proposition 2 is solved online. In the simulations, various constant delay step at the control input $t_d$, various constant turbulence frequency $\omega$, various constant weighting matrix $R$, and various constant receding horizon length $N$ are used from the following sets:

$$
\begin{align*}
    t_d &\in \{1, 2, 3, 4\}, \\
    \omega &\in \{0.1, 0.5, 1.0, 3.0, 4.0, 5.0, 6.0, 7.0\}, \\
    R &\in \{10^1, 10^2, 10^3, 10^4, 10^5\}, \\
    N &\in \{10, 12, 14, 16, 18, 20, 22, 24\}.
\end{align*}
$$

For comparison, we consider the following scenarios.

Scenario A: MPC in which Proposition 2 is solved online is applied.

Scenario B: no control is applied.

Scenario C: MPC in which Proposition 2 is solved online but with measured turbulence data being set as zeros, i.e. MPC without prior disturbance data, is applied.

In all the simulations, our proposed GA flight controller, i.e. MPC solving Problem 2 online, produces the optimal solution; that is, our proposed GA flight controller has robustness against the supposed delay at the elevator input channel and the measurement errors for turbulence.

Fig. 3 shows the performance comparison for scenarios A, B and C. In this figure, $J_A$, $J_B$ and $J_C$ respectively denote the following performance index for the corresponding scenarios, which are obtained from the simulations:

$$
\max_{l_d \in \{1, 2, 3, 4\}} \int_0^{20} \sum_{i=1}^{4} (|a_z|^2 dt.
$$

For comparison, we plot mesh planes at $J_A/J_B = 1$ and $J_A/J_C = 1$.

This figure implies the followings for applying GA control.

1. For turbulences, whose frequencies are less than 0.5 [rad/s], vertical acceleration is increased compared to uncontrolled case.
2. For turbulences, whose frequencies are more than 6 [rad/s], vertical acceleration is hardly reduced even if prior turbulence data are obtained.
3. For turbulences, whose frequencies are between 1 [rad/s] and 4 [rad/s], Proposition 2 using appropriately selected $R$, e.g. $10^3 \sim 10^4$, and $N$ above about 18 reduces turbulence effect.

By comparing the simulation results for no measurement error case and positive/negative measurement error case in prior disturbance data, it is confirmed that the item 1) is caused by the measurement errors in prior disturbance data. The item 2) is reasonable because aircraft motion system has a direct term from vertical turbulence to vertical acceleration and uncertain delays at its control input channel.

Finally, CPU time to solve the conditions in Proposition 2 is shown in Table I, where the simulation is conducted with Matlab® using SeDuMi [19] along with a parser YALMIP [21] with a PC (Dell Precision T7400,
TABLE I

<table>
<thead>
<tr>
<th>ω [rad/s]</th>
<th>K</th>
<th>N</th>
<th>PT7400</th>
<th>P650</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>10^1</td>
<td>10</td>
<td>0.30 – 0.52 (0.38)</td>
<td>0.88 – 1.45 (4.18)</td>
</tr>
<tr>
<td></td>
<td>24</td>
<td>0.47 – 1.36 (0.74)</td>
<td>1.78 – 3.94 (2.52)</td>
<td></td>
</tr>
<tr>
<td>1.0</td>
<td>10^1</td>
<td>10</td>
<td>0.28 – 0.53 (0.37)</td>
<td>0.97 – 1.48 (4.18)</td>
</tr>
<tr>
<td></td>
<td>24</td>
<td>0.45 – 3.62 (0.76)</td>
<td>1.64 – 3.94 (2.52)</td>
<td></td>
</tr>
<tr>
<td>6.0</td>
<td>10^1</td>
<td>10</td>
<td>0.20 – 0.53 (0.38)</td>
<td>0.77 – 1.41 (1.17)</td>
</tr>
<tr>
<td></td>
<td>24</td>
<td>0.51 – 0.96 (0.69)</td>
<td>2.08 – 2.88 (2.38)</td>
<td></td>
</tr>
</tbody>
</table>

CPU TIME [s] (max – min (average))

This paper tackles controller design problem achieving disturbance suppression for uncertain plant systems under the condition that disturbance data are measured a priori with some measurement errors. We adopt Model Predictive Control (MPC) scheme for our problem. Generally speaking, MPC for uncertain systems must solve infinitely many conditions if conservatism is avoided. However, if the uncertainties in plant systems are represented as delays at the control input, in which the delay time is set as bounded but time-invariant uncertain delay, then it is shown that we only have to consider finitely many plant models instead of the original uncertain plant systems. On the measurement errors in a priori measured disturbance data, if the errors are represented as affine with respect to some constant uncertain vector, whose elements are all bounded, then we only have to evaluate the effect with the maxima and minima of the uncertain vector. Using these, for our addressed problem, we propose a MPC-based controller design with finitely many conditions.

We apply our proposed method to Gust Alleviation (GA) flight controller design problem. A numerical example illustrates that our proposed controller with appropriately set controller parameters suppresses the disturbance effect, and implies that high frequency turbulence effect cannot be well suppressed even if prior disturbance data are exploited.

In this paper, we have not tackled the feasibility of Proposition 2 at every step if it has a feasible solution at the initial step, which is now under investigation.

REFERENCES

Fig. 3. Disturbance suppression performance