A Min-Max Model Predictive Control for a Class of Hybrid Dynamical Systems

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Abstract

This paper presents a min-max model predictive control algorithm for a class of hybrid systems by exploiting the equivalence between piecewise linear systems and mixed logical dynamical systems. The control algorithm consists of two control modes which are a state feedback mode and a min-max model predictive control mode. In the min-max model predictive control mode, the constrained positively invariant sets are used as the end set constraint, and an approach based on a min-max model predictive control formulation is employed. This control algorithm guarantees that the state converges to a union of constrained positively invariant sets with no constraint violation.

1 Introduction

In recent years, model predictive control (receding horizon control) has attracted the attention of researchers [1]. Model predictive control can be used for path planning of autonomous vehicles and formation control [2, 3, 4, 5]. Bemporad et al. have proposed the explicit controller for model predictive control problems to reduce the computational complexity of on-line optimization [6]. This result is a breakthrough in the research of model predictive control and the generalization of the problem is reported [7, 8, 9]. Further it is well known that, in the practical applications, physical bounds on the state and control input are present, so the control law is required to guarantee that the closed-loop system fulfills these constraints. The min-max model predictive control is important, since when disturbances or model mismatch are present closed-loop performance can be poor with likely violations of the constraints and no convergence can be guaranteed. For the issue the terminal penalty and constraints play important role [10]. In [11] feedback min-max model predictive control for linear time invariant discrete-time systems is proposed and the control algorithm which guarantees a convergence to the invariant set with no constraint violation. On the other hand, hybrid systems arise in a large number of application areas, and are attracting increasing attention. The hybrid system framework allows to model a broad class of systems arising many applications and to address the reconfigure problems. It is known that a class of hybrid models can be described by the piecewise linear systems which are defined by partitioning the state space in a finite number of polyhedral regions and associating to each region a different linear dynamics. Many mechatronic systems can be modeled as the piecewise linear systems e.g. automotive application [12].

We consider min-max model predictive control based on feedback min-max model predictive control [11] for piecewise linear systems as extending the class of system. However since in [11] the system is restricted to linear time invariant discrete-time systems, the control cannot deal with hybrid systems directly and a method to construct the end set constraint is not given clearly. Then we proposed the min-max model predictive control for piecewise linear systems using the constraint positively invariant sets [13]. However, since this control algorithm includes the min-max optimization problem, the implementation of the algorithm is computationally demanding.

In this paper, we propose a min-max model predictive control algorithm for piecewise linear systems affected by the additional disturbance. We construct the algorithm by two control modes: the state feed back mode for keeping the state in a set and the model predictive control mode for steering the state to the set. In the control algorithm we employ the equivalence [14] of the piecewise linear system form and the mixed logical dynamical (MLD) system form [15] to extend the system form. In model predictive control mode we construct the end set constraint by using the intersection of the constrained positively invariant sets [16, 17]. This control algorithm guarantees convergence to the union of the constrained positively invariant sets and satisfying the constraints in spite of existence of disturbance. Further we employ the min-max approach for the control algorithm to reduce the computation. We use the algorithm using vertices of the polyhedron for calculating the optimal input sequence and the worst disturbance and present the necessary and sufficient condition for the relation between the min-max solution and vertices of the polyhedron. By the algorithm and the condition we can obtain the optimal input sequence as the piecewise affine form with respect to the state and the control algorithm's application range should be increased.
2 Preliminary

2.1 Piecewise Linear Systems with Disturbance

In this paper, piecewise linear systems with disturbances are described by the following equation,

\[ x(t+1) = Ax(t) + Bu(t) + B_w w(t) \quad \text{for} \quad t \in X_i \]

(1)

where \( x(t) \in \mathbb{R}^n \) is the state, \( u(t) \in \mathbb{R}^m \) is the input, \( X_i \) is the partition of the state set which satisfies \( X_i \cap X_j = \emptyset \) and with \( i \neq j \). \( X_0 \) is the origin and we assume that \( (A, B_i) \) is controllable. The vector \( w(t) \in W \subset \mathbb{R}^p \) is an unknown bounded disturbance and the set \( W \) is convex and contains the origin. In addition the system is subject to constraints on either or both the states and the control inputs i.e. \( x(t) \in X, u(t) \in U, \forall t \in N \). We assume \( X \) and \( U \) are convex polyhedral. Consider the output to be constrained

\[ y_c(t) = Cx(t) + Du(t) + D_w w(t). \]

(2)

By an appropriate choice of matrix \( C, D \) and a set \( Y \), all constraints mentioned can be summarized by

\[ y_c(t) \in Y. \]

(3)

Assume that the set \( Y \) is convex and contains the origin.

2.2 The Mixed Logical Dynamical Form of Piecewise Linear Systems

Here the mixed logical dynamical form [15] which is equivalent to piecewise linear systems is introduced. Consider the following general piecewise linear system

\[ x(t+1) = A_i x(t) + B_i u(t) \quad \text{for} \quad t \in X_i \]

\[ y(t) = C_i x(t) + D_i u(t) \]

where \( x(t) \in \mathbb{R}^n \) is the state, \( u(t) \in \mathbb{R}^m \) is the input, \( X_i \) is a partition of the state set which satisfies \( X_i \cap X_j = \emptyset \) and with \( i \neq j \). \( U_i \) is the input set and we assume that \( (A_i, B_i) \) is controllable.

Piecewise affine systems are described by the state space equations

\[ x(t+1) = A_i x(t) + B_i u(t) + f_i \]

\[ y(t) = C_i x(t) + g_i \]

(4)

(5)

for \( x(t) \in X_i \), \( u(t) \in U_i \), \( f_i \), and \( g_i \). Since \( f_i \), \( g_i \) can be thought as generated by integrators with no input, the piecewise linear system and the piecewise affine system are equivalent. The piecewise affine system (6) can be transformed into mixed logical dynamical systems formulation [15]. And it also is reported that Mixed Logical Dynamical Systems are formally equivalent to piecewise affine systems [19]. The key idea of mixed logical dynamical systems is technique which can transform into propositional logic into mixed logic inequalities, i.e. inequalities involving both continuous and 0-1 variables.

The general mixed logical dynamical form [15] is:

\[ x(t+1) = Ax(t) + Bu(t) + B_0 \delta(t) + B_2 \zeta(t) \]

\[ y(t) = Cx(t) + Du(t) + D_0 \delta(t) + D_2 \zeta(t) \]

\[ E \delta(t) + E_2 \zeta(t) \leq E_2 u(t) + E_3 x(t) + E_5 \]

\[ \delta(t) = F(x(t), u(t), w(t)), \zeta(t) = G(x(t), u(t), w(t)). \]

where

\[ x = [x_c \ x_t], x_c \in \mathbb{R}^{n_c}, x_t \in \{0,1\}^{n_t}, n := n_c + n_t \]

is the state of the system, whose components are distinguished between continuous \( x_c \) and 0-1 \( x_t \),

\[ u = [u_c \ u_t], u_c \in \mathbb{R}^{m_c}, u_t \in \{0,1\}^{m_t}, m := m_c + m_t \]

is the command input. \( \delta \in \{0, 1\}^{n_2} \) and \( \zeta \in \mathbb{R}^{n_2} \) represent respectively auxiliary logical and continuous variables.

The piecewise linear system (1) can be transformed mixed logical dynamical system formulation. The mixed logical dynamical systems form is

\[ x(t+1) = Ax(t) + Bu(t) + B_0 \delta(t) + B_2 \zeta(t) + B_0 w(t) \]

\[ E \delta(t) + E_2 \zeta(t) \leq E_2 u(t) + E_3 x(t) + E_5. \]

(8a)

(8b)

Assume that system (8) is completely well-posed [15], which in words means that for all \( x, u, w \) within a bounded set the variables \( \delta, \zeta \) are uniquely determined, i.e. there exist functions \( F, G \) such that, at each time \( t \),

\[ \delta(t) = F(x(t), u(t), w(t)), \zeta(t) = G(x(t), u(t), w(t)). \]

2.3 Constrained Positively Invariant Set

The constrained positively invariant set [16, 17] is explained in order to use it for an end set constraint of model predictive control. Consider the control input \( u = K x \) for the system (1) then the system can be rewritten as

\[ x(t+1) = (A_i + B_i K_i) x(t) + B_w w(t) \]

\[ y_c(t) = (C + D_i K_i) x(t) + D_w w(t). \]

(9)

(10)

For each closed-loop system, we define a state constraint set.

**Definition 1** [16, 17] State constraint set \( X((C + D K), D_w, Y, W) \) is defined by

\[ X_i = \{x | (C + D K_i) x + D_w w \in Y, \forall w \in W \}. \]

(11)

**Remark 1** Necessary and sufficient condition \( y_c(t) \in Y \) for possible disturbance \( w(t) \in W \) is \( x(t) \in X_i \).

**Definition 2** [16, 17] \( \mathcal{O}_i \subset \mathbb{R}^n \) contains origin in its interior. \( \mathcal{O}_i \) is a constrained positively invariant set, if it is a positively invariant set and is contained in \( X_i((C + D K), D_w, Y, W) \).

If a constrained positively invariant set exists, for any initial state \( x(0) \in \mathcal{O}_i \) and \( w(t) \in W \), then \( x(t) \in \mathcal{O}_i \) for all \( t \in \mathbb{Z}^+ \), where \( \mathbb{Z}^+ \) denotes the set of nonnegative integer.

**Definition 3** Maximal constrained positively invariant set is defined as follows

\[ \mathcal{O}_{\infty} = \{x(0) | y_c(t) x(0), w(t) \in Y, \forall t \in \mathbb{Z}^+, \forall w \in W \}. \]

(12)

Maximal constrained positively invariant set \( \mathcal{O}_{\infty} \) can be obtained by recursive process proposed in [16] [17].

Next we define a set which is used for an end set constraint of model predictive control as

\[ \mathcal{P} := \cap_{i=1}^n \mathcal{O}_{\infty}. \]

(13)
3 Min-Max Model Predictive Control Problem

3.1 Min-Max Model Predictive Control Law

Since the state can not be steered to the origin due to existing disturbance \( w(t) \), the control objective is to drive the state of the system to the set which is constructed by invariant sets. In this paper, we propose 2 modes for the control law.

**Mode 1:** The control law is the form \( u = K x \).

**Mode 2:** At time \( t \), predictions for possible disturbance are represented by \( \{ u_{t+1} \} \) and \( \{ u_{t+k} \} \) denotes the input sequence for the disturbance realization. For the sake of simplicity we define \( x_{t+k} := x(t+k, x(t), u_{t}^{k-1}) u_{t+k-n}^{k-1} \) and \( \{ u_{t+k} \} \). \( \{ x_{t+k} \} \) are similarly defined respectively.

The prediction for state \( x_{t+k} \) is defined as follows

\[
\begin{align*}
    x_{t+k+1} &= A x_{t+k} + B_{1} u_{t+k} + B_{2} \delta_{t+k} + E_{1} z_{t+k} + E_{3} u_{t+k+1} \\
    E_{2} \delta_{t+k} + E_{3} z_{t+k+1} &\leq E_{1} x_{t+k} + E_{2} x_{t+k+1} + E_{3} u_{t+k+1} 
\end{align*}
\]

At current time \( t \), let \( x(t) \) be the current state. Consider the following min-max problem,

\[
\begin{align*}
    \min_{u} \max_{W} \ J(U, W, \Delta, Z, x(t)) \quad \text{s.t.} \quad & (3), (14), (19) \\
    \text{subject to} \quad & U := \{ u_{t}, \ldots, u_{t+k-1} \} \\
    \text{end set constraint} \quad & (3), (8) \\
J(U, W, \Delta, Z, x(t)) := & \| Px_{t+k} \| + \sum_{i=0}^{k-1} \| Q_{i} x_{t+i+1} \| + \| R u_{t+i} \| + \| Q_{2} \delta_{t+i+1} \| \quad \text{(17)}
\end{align*}
\]

where a notation \( \| v \| \) denotes \( \| v \| = \max \{ | v | \} \) for \( v = [v_{1}, v_{2}, \ldots, v_{n}]^{T} \). \( N \) is a predictive horizon and \( P, R \in \mathbb{R}^{n \times n} \), \( Q_{1} \in \mathbb{R}^{n \times m}, R_{t} \in \mathbb{R}^{m \times m}, Q_{2} \in \mathbb{R}^{n \times n}, Q_{3} \in \mathbb{R}^{n \times n} \), are nonsymmetric weighting matrices.

The formulation (15), (16) can be written as a mixed integer linear programming by using following approach. First we introduce a vector \( V \)

\[
V := [e_{0}, e_{1}, e_{2}, \ldots, e_{N-1}, e_{0}, e_{1}, e_{2}, \ldots, e_{N-1}]^{T}
\]

where \( e_{i} \) satisfies the following inequalities

\[
\begin{align*}
    -e_{0} 1_{n} &\leq -p_{1} x_{t+k+1} \\
    -e_{1} 1_{n} &\leq Q_{1} x_{t+k+1} \\
    -e_{2} 1_{n} &\leq Q_{2} \delta_{t+k+1} + Q_{3} u_{t+k+1} \\
    -e_{n-1} 1_{n} &\leq -p_{1} x_{t+k+1} \\
    -e_{N-1} 1_{n} &\leq -p_{1} x_{t+k+1} \\
\end{align*}
\]

Firstly the min-max problem (15), (16) can be denoted as

\[
\begin{align*}
    \min_{u} \max_{W} \ J(U, W, \Delta, Z, x(t)) \quad \text{s.t.} \quad & (3), (14), (19) \\
J(U, W, \Delta, Z, x(t)) := & \| Px_{t+k} \| + \sum_{i=0}^{k-1} \| Q_{i} x_{t+i+1} \| + \| R u_{t+i} \| + \| Q_{2} \delta_{t+i+1} \| \quad \text{(17)}
\end{align*}
\]

By plugging (14) into (21) and (22), and by defining the matrices \( O, S, F \) respectively the min-max problem (eqn.:) can be rewritten in the more simple form

\[
\begin{align*}
    \min_{u \in \mathcal{P}} \ J(p_{c}, p_{d}, q) \quad \text{s.t.} & (p_{c}, p_{d}, q) \in S \\
    \tilde{J}(p_{c}, p_{d}, q) := & f_{c}^{T} p_{c} + f_{d}^{T} p_{d} + g^{T} q \\
S := & \{ (p_{c}, p_{d}, q) : F_{c} p_{c} + F_{d} p_{d} + G_{q} \leq H x(t) + r \}
\end{align*}
\]

where \( p_{c} \) denotes the continuous components of \( (U, V, \Delta, Z) \) and \( p_{d} \) denotes discrete ones. And the vector \( q \) represents a component of \( W \).

By relaxing the conditions for the discrete component \( p_{d} \) as \( 0 \leq p_{d} \leq 1 \), the min-max problem (23) can be rewritten as

\[
\begin{align*}
    \min_{(p_{c}, p_{d}, q)} \ J(p_{c}, p_{d}, q) \quad \text{s.t.} & (p_{c}, p_{d}, q) \in S' \\
    \hat{J}(p_{c}, p_{d}, q) := & [ f_{c}^{T} f_{d}^{T} ] \begin{bmatrix} p_{c} & p_{d} \end{bmatrix} + g^{T} q \\
& f^{T} p + g^{T} q \\
S' := & \{ (p_{c}, p_{d}, q) : F_{c} p_{c} + F_{d} p_{d} + G_{q} \leq H x(t) + r \}
\end{align*}
\]

\[
\begin{align*}
    \min_{(p_{c}, p_{d}, q)} \ J(p_{c}, p_{d}, q) \quad \text{s.t.} & (p_{c}, p_{d}, q) \in S' \\
    \hat{J}(p_{c}, p_{d}, q) := & [ f_{c}^{T} f_{d}^{T} ] \begin{bmatrix} p_{c} & p_{d} \end{bmatrix} + g^{T} q \\
& f^{T} p + g^{T} q \\
S' := & \{ (p_{c}, p_{d}, q) : F_{c} p_{c} + F_{d} p_{d} + G_{q} \leq H x(t) + r \}
\end{align*}
\]

3.2 The control algorithm

A min-max model predictive control algorithm for the piecewise linear systems (1) with disturbance \( w(t) \in \mathcal{W} \) is presented as follows. Suppose \( u_{t}^{o} \) denotes the first element of optimal input sequence for the optimization problem (24).

**Algorithm 1:**

**Data:** \( x(t) \)

**Algorithm:** IF \( x(t) \in \mathcal{P} \) THEN (mode 1) \( u(t) = K x(t) \).

ELSE (mode 2) \( u(t) = u_{t}^{o} \).

**Theorem 1** Suppose that \( \varepsilon_{t}^{o} = 0 \ \forall x(t) \in \mathcal{P} \), \( u_{t} = K x(t) \). Then the control law given by Algorithm 1 satisfies the constraints (3) and drives the state \( x(t) \) to the union of constrained positively invariant sets \( U_{t = 0}^{\infty} \).

**Proof:** At time \( t \), state \( x(t) \), let \( V_{t}^{*} := \tilde{J}(U_{t}, \Delta_{t}, Z_{t}, V_{t}, W_{t}, x(t)) \) denotes the optimal cost which responds to the optimal input sequence \( U_{t}, \Delta_{t}, Z_{t}, V_{t}, W_{t} \) and the disturbance sequence \( W_{t}^{*} \). At time \( t \), the first element of the optimal sequence is applied, and disturbance takes a certain value \( w(t) \).

At time \( t + 1 \), consider an input sequence \( U_{t+1} = \{ u_{t+1}, u_{t+2}, \ldots, u_{t+k-1}, K x(t+k+1) \} \) in which the last element might not be optimal. If the input sequence \( U_{t+1} \) is used we obtain the following inequality.

\[
V_{t+1}^{*} \leq \tilde{J}(U_{t+1}, \Delta_{t+1}, Z_{t+1}, V_{t+1}, W_{t+1}, x(t+1))
\]

The right hand side of inequality (27) equals

\[
r.h.s. = V_{t}^{*} - (e_{0}^{o} + e_{0}^{o} + e_{0}^{o} + e_{0}^{o}) + e_{N+1}^{o}.
\]

At time step \( N + 1 \) we will obtain \( x \in \mathcal{P} \) because of the terminal constraint \( x_{t+N+1} \in \mathcal{P} \). Then the input must
be \( u(k) = K_x x_k + \eta_{k+1} \) from the Algorithm 1. We leads

\[
\varepsilon_{k+1} = \varepsilon_k - \left( \varepsilon_0^* + \varepsilon_0^* + \varepsilon_0^* + \varepsilon_0^* \right).
\]

Because \( \varepsilon \geq 0 \), the cost is monotonically nonincreasing. As it is bounded below by zero, it must consequently converge to a constant value, so that \( V_{k+1} = V_{k+1} \rightarrow 0 \) as \( t \rightarrow \infty \). Then we have

\[
\varepsilon_k^* + \varepsilon_k^* + \varepsilon_k^* + \varepsilon_k^* \leq V_{k+1}^* - V_{k+1}^*.
\]

This leads to the state converges to \( P \) which includes the origin. Further when the state in the set \( P \), the control law changes to \( u = K_x x \). Consequently the control algorithm satisfies the constraints and drives the state in \( U_{\infty} \). And in the control algorithm the constraints is satisfied since the two control modes guarantee the constraints satisfaction.

\begin{proof}
If the state of the system can be steered to the set \( U_{\infty} \), with no constraint violation in spite of existence of disturbance. The set \( U_{\infty} \) is made on the design of feedback gain \( K_x \), and \( \mu \) and we can design the gain \( K_x \). The control mode 1 is for keeping the state in the set \( P \) and mode 2 is for steering the state in \( \text{P} \). However, the computation of the algorithm is demanding since mode 2 solves the min-max optimization problem each time steps.
\end{proof}

3.3 Piecewise Affine Controller

Here we consider the off line computation of (23) to reduce the on line computation. We can obtain the min-max solution by calculating the vertices because the min-max problem (23) is linear [30]. The sequence is summarized algorithm 2.

\begin{algorithm}
1) Obtain vertices of the polyhedron \( S' \) [21].
2) By the vertices obtained 1), define the vertices \( (p_i, q_i) \), \( i = 1, 2, \ldots, k \) which satisfy

\[
p_i = \arg \min_p J(p, q), \quad \text{s.t.} \quad (p, q) \in S'.
\]

3) The vertex which maximize \( J \) is a min-max solution.

\begin{lemma}
Let the vector \( (p, q) \) be one of a vertex of the polyhedron \( S' \) for \( x(t) \). For the vertex \( (p, q) \) let \( F_A, G_A, H_A, r_A \) represent the matrices corresponding to active constraints (denotes inactive constraints)

\[
\begin{bmatrix}
F_A \\
G_A
\end{bmatrix}
\begin{bmatrix}
p \\
q
\end{bmatrix} = H_A x(t) + r_A
\]

\[
\begin{bmatrix}
F_N \\
G_N
\end{bmatrix}
\begin{bmatrix}
p \\
q
\end{bmatrix} \leq H_N x(t) + r_N.
\]

And matrices \( V_A \) and \( V_N \) are defined as \( V_A := \begin{bmatrix} F_A & G_A \end{bmatrix} \), \( V_N := \begin{bmatrix} F_N & G_N \end{bmatrix} \). Then in the region \( D(x(t)) \)

\[
D(x(t)) := \{ x(t) : V_N (V_A^T V_A)^{-1} V_A^T H_A - H_N \} \leq \begin{bmatrix}
V_N \end{bmatrix} (x(t) + r_A). \]

the state \( x(t) \) and the vertex \( (p, q) \) are satisfy

\[
\begin{bmatrix}
p \\
q
\end{bmatrix} = \psi(x(t))
\]

\end{lemma}

\begin{proof}
When \( (p, q) \) is a vertex of the polyhedron \( P \), the matrix \( V_A \) is full rank. Hence \( V_A^T V_A \) is nonsingular and we obtain the equation (35). By substituting the equation (35) into the equation (33) the region (34) is obtained.

By lemma 1 the vertices of \( S' \) are piecewise affine with respect to the state \( x(t) \) and the objective function \( \hat{J} \) for the vertex \( (p, q) \) can be denoted as

\[
\hat{J}(x(t)) := \left[ J^T g^T \right] \psi(x(t)).
\]

\end{proof}

Then the conditions for the vector

\[
\begin{bmatrix}
p_1 \\
q_1
\end{bmatrix} = \psi(x(t))
\]

being the min-max solution for problem (23) are

(necessary condition) \( x(t) \in D_i(x(t)) \)

(sufficient condition)

\[
x(t) \in \left( \bigcap_{i=1}^{k} D_i(x(t)) \right) \cap \{ x(t) : \hat{J}_1(x(t)) > \hat{J}_i(x(t)) \}
\]

\end{algorithm}

3.3 Piecewise Affine Controller

\begin{algorithm}
1) Obtain vertices of the polyhedron \( S' \) [21].
2) By the vertices obtained 1), define the vertices \( (p_i, q_i) \), \( i = 1, 2, \ldots, k \) which satisfy

\[
p_i = \arg \min_p J(p, q), \quad \text{s.t.} \quad (p, q) \in S'.
\]

3) The vertex which maximize \( J \) is a min-max solution.

\begin{proof}
When \( (p, q) \) is a vertex of the polyhedron \( P \), the matrix \( V_A \) is full rank. Hence \( V_A^T V_A \) is nonsingular and we obtain the equation (35). By substituting the equation (35) into the equation (33) the region (34) is obtained.

By lemma 1 the vertices of \( S' \) are piecewise affine with respect to the state \( x(t) \) and the objective function \( \hat{J} \) for the vertex \( (p, q) \) can be denoted as

\[
\hat{J}(x(t)) := \left[ J^T g^T \right] \psi(x(t)).
\]

\end{proof}

Then the conditions for the vector

\[
\begin{bmatrix}
p_1 \\
q_1
\end{bmatrix} = \psi(x(t))
\]

being the min-max solution for problem (23) are

(necessary condition) \( x(t) \in D_i(x(t)) \)

(sufficient condition)

\[
x(t) \in \left( \bigcap_{i=1}^{k} D_i(x(t)) \right) \cap \{ x(t) : \hat{J}_1(x(t)) > \hat{J}_i(x(t)) \}
\]

\end{algorithm}
THEN (mode 1) \( u(t) = K_1 x(t) \).
ELSE (mode 2) \( u(t) = [0 \cdots 0 \ I \ 0 \ \cdots \ 0] \psi(x(t)) \).

If once the piecewise affine function \( \psi \) is obtained for each region, when we implement the control algorithm 1' only the region which the state is to be needed. Then the proposed control procedure is summarized as follows.

1. Set the feedback gain \( K_i \) for each region \( i \).
2. Calculate the maximal constrained positively invariant set \( \mathcal{O}_{\infty,i} \) for each region \( i \).
3. Let \( \bigcap_{i=1}^{n} \mathcal{O}_{\infty,i} \) be the end set constraint \( \mathcal{P} \).
4. Set horizon length \( N \), and weight matrices.
5. Calculate the piecewise affine function (37) off line.
6. Algorithm 1'.

4 Illustrative Example

To illustrate the points, we consider the following simple system:

\[
x(t) = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w(t)
\]

\[
\alpha(t) = \begin{cases} \frac{\pi}{3}, & \text{if } \frac{1}{2} \leq x(t) < 0 \\
-\frac{\pi}{3}, & \text{if } x(t) \geq 0 
\end{cases}
\]

\[
x(t) \in [-10, 10] \times [-10, 10]
\]

\[
u(t) \in [-2, 2]
\]

\[
w(t) \in [-0.2, 0.2]
\]

It can be rewritten as follows

If \( x(t) \in X_1 \):
\[
x(t + 1) = A_1 x(t) + B_1 u(t)
\]

If \( x(t) \in X_2 \):
\[
x(t + 1) = A_2 x(t) + B_2 u(t)
\]

where

\[
X_1 = \{ x \in [-1, 0] \} \text{ if } x(t) \geq 0,
\]

\[
X_2 = \{ x \in [1, 0] \} \text{ if } x(t) < 0
\]

\[
A_1 = \begin{bmatrix} 0.4 & 0.4 \sqrt{3} \\ -0.4 \sqrt{3} & 0.4 \end{bmatrix},
\]

\[
A_2 = \begin{bmatrix} 0.4 & -0.4 \sqrt{3} \\ -0.4 \sqrt{3} & 0.4 \end{bmatrix}
\]

\[
B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},
\]

\[
B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\]

and the output to be constrained \( Y \) is defined as

\[
Y := \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} z(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} w(t) \quad (39)
\]

Consider the state feedback \( u(t) = K_1 x(t) \) if the state of the system is in \( X_1 \) and \( u(t) = K_2 x(t) \) if it is in \( X_2 \), and calculate the constrained positively invariant sets \( \mathcal{O}_{\infty,1}, \mathcal{O}_{\infty,2} \) for each closed loop system. The sets \( \mathcal{O}_{\infty,1}, \mathcal{O}_{\infty,2} \) obtained with

\[
K_1 = [-1.2 \ -0.6], \quad K_2 = [0.6 \ -1.2] \quad (40)
\]

are shown in Figure 1. Then we define the set which is used for the end set constraint of receding horizon control as \( \mathcal{P} = \mathcal{O}_{\infty,1} \cap \mathcal{O}_{\infty,2} \).

By defining \( \delta(t) \) and \( z(t) \) (39) can be rewritten as

\[
x(t + 1) = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \delta(t) + B_2 w(t) \quad (41)
\]

Figure 1: The sets \( \mathcal{O}_{\infty,1} \) and \( \mathcal{O}_{\infty,2} \)

Figure 2: The state trajectory

Figure 3: The state response (solid line: \( x_1 \), dashed line: \( x_2 \))

The control law given by Algorithm 2 is adapted along with the predictive horizon \( N = 3 \), a terminal constraint \( x(N(t)) \in \mathcal{P} \) and weights \( Q_1 = 10 I \), \( Q_2 = 1 \). In Figure 2-4, we show the resulting trajectories obtained with \( x(0) = [-8 \ 4] \) and \( w(t) = 0.2 t \), \( t \geq 1 \). Figure 3 shows the state, which has the constraint \( x(t) \in [-10, 10] \times [-10, 10] \) and Figure 4 shows the constrained control input \( u(t) \in [-2, 2] \). Finally, we find that in spite of the disturbance the control law given by Algorithm 2 drives the state to the set \( \mathcal{O}_{\infty,1} \cap \mathcal{O}_{\infty,2} \) allowing no constraint violations.
5 Conclusion

In this paper, we propose min-max model predictive control algorithm for piecewise linear systems affected by additive bounded disturbances. It has two control modes based on feedback min-max model predictive control, in the min-max model predictive control mode we employ the equivalence of piecewise linear systems and MLD systems and propose the end set constraint which consists of constrained positively invariant sets. Proposed control algorithm has guaranteed convergence to the set and no constraint violations. We propose min-max techniques for calculation of the optimal solution which moves the computation off line. From now on, some applications will be strongly needed e.g. automotive applications [12].

References


