Closed Form Solutions for Connectivity of Fixed Radius Random Graphs in One-Dimensional Space

Ai Noshiro and Masahito Kurihara

Abstract—We consider the connectivity of a class of random graph models called graphs defined by the Fixed Radius Model \( G = G(n, R) \). The model has a clear relationship with state-of-the-art wireless communication networks, often called ad-hoc networks. In this model, each random graph is defined by \( n \) nodes placed in a Euclidean plane randomly according to some distribution; each pair of nodes is connected by an edge if and only if the distance between the nodes is within the common radius \( R \). Hence the model can be naturally interpreted as a mathematical model of wireless communication networks in which every mobile node can communicate with other nodes within the distance \( R \).

In this paper, we present some analytical results concerning the probability that such random graphs are connected (i.e., there is a path between every pair of nodes), assuming that the fixed number of nodes are distributed in one-dimensional space according to the uniform distribution. Related results have been obtained in our previous paper only implicitly in the form of recursive equations. On the other hand, the results of this paper are significant in that they are closed form solutions of the recursive equations thus present the probability explicitly.

I. INTRODUCTION

Random graphs \( G = G(V, E) \) can be described as graphs that are generated by some random mechanisms. One of the random graph models most well-known and studied extensively is the Bernoulli Model \( G = G(n, p) \), which, given \( n \) and \( p \), generates an undirected graph \( G \) with \( n \) nodes such that there is an edge between each pair of nodes with probability \( p \). There are a number of theoretical results on this model [1]-[3].

Recently, the Fixed Radius Model \( G = G(n, R) \) has received much attention because of clear relationship with wireless communication networks, often called ad-hoc networks. In this model, each random graph is defined by \( n \) nodes placed in the Euclidean plane randomly according to some distribution; each pair of nodes is connected by an edge if and only if the distance between the nodes is within the common radius \( R \). Hence the model can be naturally interpreted as a mathematical model of wireless communication networks in which every mobile node can communicate with other nodes within the distance \( R \). There are various topics of research about this model. However, in connection with application to wireless networks, one of the most interesting topics would be the connectivity of the graphs, i.e., the probability that random graphs are connected graphs, because if the graph is connected, there is a communication path between every pair of mobile nodes.

In this paper, we study the Fixed Radius Model (FRM) concerning connectivity. As in [4] and [5], we confine ourselves to one-dimensional Euclidean space (i.e., straight line segment or interval) in which the nodes are uniformly distributed. Under these assumptions, our objective is to calculate the probability that random graphs generated by FRM are connected. The results are useful for some restricted areas such as long almost-straight roads and rivers which can be considered as one-dimensional space. Also, the result could give an insight into the analysis of two-dimensional models as a special case. Actually, we analyze two variations of FRM: Fixed-End FRM and Free-End FRM. The former assumes that there always exist a node on both end points of the interval, while the latter ignores this assumption.

There are some instructive analytical results about the Poisson Model \( G = G(\lambda, R) \) [6][7]. In this model, each pair of nodes is connected by an edge if and only if the distance between the nodes is within the common radius \( R \), just like FRM, but the nodes are placed randomly in the space according to Poisson point process of given intensity \( \lambda \). In this case, the number of nodes is the random variable following the Poisson distribution with parameter \( \lambda \). Especially in one-dimensional case, it is easy to analyze this model because the distances between adjacent nodes are given explicitly as a random variable which follows the exponential distribution.

The authors have already presented the probability of connectivity for both Fixed-End and Free-End FRMs [4][5]. However, those results were presented only implicitly in the form of recursive equations. On the other hand, the results of this paper are significant in that they are closed form solutions of the recursive equations thus present the probability explicitly.

The rest of the paper is organized as follows. In section II and III, we describe Fixed-End and Free-End FRMs and derive the integral equations which must be satisfied by the probability of connectivity. Furthermore, we derive the closed form solutions for the equations by using Laplace transforms. We conclude with a summary of this paper and future works in section IV.

II. FIXED-END FRM

A. Model

Let \( I = [0, t] = \{x|0 \leq x \leq t\} \) be an interval of length \( t \) \((t>0)\). In the Fixed-End FRM, exactly \( n \) nodes are uniformly distributed inside \( I \) \((0<x<t)\), plus two nodes are placed on both end points of the interval. Thus the total number of nodes is \( n+2 \). Since the location \( X \) of each node is a random
variable that follows the uniform distribution, we have
\[ P_R(X \leq x) = \frac{x}{t} \quad (0 \leq x \leq t). \]  \hfill (1)

We denote the locations of the nodes by \(X_0, X_1, \ldots, X_{n+1}\) from left to right, as in Fig. 1. We also abuse the notation by denoting by \(X_i\) the node at location \(X_i\) itself.

As described in the preceding section, we define a random graph by connecting by an edge every pair of two nodes within a fixed distance \(R\). Let \(W_i\) be the distance of between two adjacent nodes \(X_i\) and \(X_{i+1}\).

\[ W_i = X_{i+1} - X_i \quad (i = 0, 1, \ldots, n). \]  \hfill (2)

Let \(W\) be the greatest value among \(W_0, \ldots, W_n\).

\[ W = \max\{W_i|0 \leq i \leq n\}. \]  \hfill (3)

Then it is clear that the random graph is a connected graph if and only if \(W \leq R\), because it is a connected graph if and only if there is an edge between every pair of adjacent nodes. Thus the probability \(P_n(t)\) that node \(X_0\) is connected to node \(X_{n+1}\) is equal to the probability that \(W \leq R\).

\[ P_n(t) = P_R(W \leq R). \]  \hfill (4)

Note that the right-hand side of this equation has a form of distribution function of \(W\) if we regard \(R\) as a parameter. Actually, however, we consider \(R\) as a fixed constant throughout this paper, and make only \(n\) and \(t\) appear explicitly as parameters in the notation given in the left-hand side of the equation. This notation will turn out to be useful in the following analysis when we derive a formula which is recursive with respect to \(n\) and \(t\).

\[ I \]
\[ X_0 \quad X_1 \quad X_2 \quad X_{n-1} \quad X_n \quad X_{n+1} \]
\[ W_0 \quad W_1 \quad \ldots \quad W_{n-1} \quad W_n \]

Fig. 1. The Fixed-End Fixed Radius Model.

In the following analysis, we denote by \(r(t)\) the probability that two adjacent nodes with distance \(t\) is connected by an edge. This function is simply formulated as follows.

\[ r(t) = \begin{cases} 1 & (t \leq R) \\ 0 & (t > R). \end{cases} \]  \hfill (5)

B. Recursive Equation

In this subsection, we derive a recursive equation satisfied by \(P_n(t)\), the probability that node \(X_0\) at the left end point is connected to node \(X_{n+1}\) at the right end point through \(n\) nodes in the interval of length \(t\). We have the following two base cases for the recursion.

When \(n = 0\), as there are nodes only at the both ends of \(I\), it is clear that

\[ P_0(t) = r(t) = \begin{cases} 1 & (t \leq R) \\ 0 & (t > R). \end{cases} \]  \hfill (6)

When \(n \geq 1\) and \(t \leq R\), we have

\[ P_n(t) = 1 \quad (t \leq R). \]  \hfill (7)

Now let us study the inductive case. When \(n \geq 1\) and \(R < t\), since \(X_n \leq x\) implies that all the \(n\) nodes \(X_1, \ldots, X_n\) are located in the sub-interval \(Z = \{q|0 \leq q \leq x\}\), the distribution function of \(X_n\) is given as follows.

\[ F_n(x) = P_R\{X_n \leq x\} = \left(\frac{x}{t}\right)^n. \]  \hfill (8)

By differentiating (8), we get the probability density function \(f_n(x)\) of \(X_n\) as follows.

\[ f_n(x) = \frac{n x^{n-1}}{t^n}. \]  \hfill (9)

Under the condition \(X_n = x\), the probability that \(X_0\) is connected to \(X_n\) is \(P_{n-1}(x)\) (in a recursive way) and the probability that \(X_{n+1}\) is connected to \(X_n\) is \(P_0(t-x)\). As \(P_0(t-x)\) denotes the probability that there is an edge between \(X_n\) and \(X_{n+1}\), we see that \(P_0(t-x) = r(t-x)\). Consequently, we obtain the following recursive formula representing the probability that \(X_0\) is connected to \(X_{n+1}\).

\[ P_n(t) = \int_0^t P_{n-1}(x)r(t-x)\frac{n x^{n-1}}{(n-1)!}dx. \]  \hfill (10)

Multiplying both sides of (10) by \(\frac{t^n}{n!}\), we have

\[ \frac{t^n}{n!}P_n(t) = \int_0^t P_{n-1}(x)r(t-x)\frac{x^{n-1}}{(n-1)!}dx. \]  \hfill (11)

Here we introduce a new function

\[ \hat{p}_n(t) = \frac{t^n}{n!}P_n(t) \]  \hfill (12)

to rewrite (11) as follows.

\[ \hat{p}_n(t) = \int_0^t \hat{p}_{n-1}(x)r(t-x)dx. \]  \hfill (13)

Therefore, \(\hat{p}_n(t)\) is represented by the convolution of \(\hat{p}_{n-1}(t)\) and \(r(t)\). This equation together with (6) and (7) (now transformed to (14) and (15) below) forms a system of recursive equations satisfied by \(\hat{p}_n(t)\).

\[ \hat{p}_0(t) = P_0(t) = r(t). \]  \hfill (14)

\[ \hat{p}_n(t) = \frac{t^n}{n!} \quad (t \leq R). \]  \hfill (15)

C. Closed Form Solution

To obtain the closed form solutions, we rewrite (13) into an algebraic form by using the Laplace transform as follows.

\[ \mathcal{L}[\hat{p}_n(t)] = \mathcal{L}[\hat{p}_{n-1}(t)]\mathcal{L}[r(t)]. \]  \hfill (16)

If we write \(\mathcal{L}[\hat{p}_n(t)] = \hat{P}_n(s)\) and \(\mathcal{L}[r(t)] = R(s)\), this equation can be further reduced as follows.

\[ \hat{P}_n(s) = \hat{P}_{n-1}(s)R(s) = \hat{P}_{n-2}(s)R(s)^2 \]
\[ \vdots \]
\[ = \hat{P}_0(s)R(s)^n. \]  \hfill (17)
Since \( \hat{P}_0(s) = R(s) \) by (14), we have
\[
\hat{P}_n(s) = R(s)^{n+1}. 
\tag{18}
\]

By the inverse Laplace transform and (12), we obtain \( P_n(t) \) that we originally intended to find.
\[
P_n(t) = \frac{n!}{t^n} \mathcal{L}^{-1}\left[R(s)^{n+1}\right]. \tag{19}
\]

In order to have calculate \( R(s) \), we represent \( r(t) \) defined in (5) as a difference of step functions.
\[
r(t) = u_0(t) - u_R(t) \tag{20}
\]
where
\[
u_c(t) = \begin{cases} 
0 & (t < c) \\
1 & (t \geq c).
\end{cases} 
\tag{21}
\]

The Laplace transform of (20) is
\[
R(s) = \mathcal{L}[r(t)] = \frac{1 - e^{-Rs}}{s}. \tag{22}
\]

By (19), we must calculate the inverse Laplace transform of \( R(s)^{n+1} \). This term is represented by the binomial theorem as follows.
\[
R(s)^{n+1} = \left(\frac{1 - e^{-Rs}}{s}\right)^{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k} (-e^{-Rs})^k s^{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k} (-1)^k e^{-kRs} s^{n+1}. \tag{23}
\]

Therefore, we need only calculate the inverse Laplace transform of \( e^{-kRs} \). Recalling that the Laplace transform of \( t^n \) is \( n!/s^{n+1} \) and the multiplication by \( e^{-as} \) means shifting on the \( t \)-axis by \( a \), we have
\[
\mathcal{L}^{-1}\left[R(s)^{n+1}\right] = \sum_{k=0}^{n+1} \binom{n+1}{k} (-1)^k \left(\frac{t-kR}{s}\right)^n u_{kR}(t) \tag{24}
\]
Substituting the result into (19), we obtain
\[
P_n(t) = \frac{n!}{t^n} \sum_{k=0}^{n+1} (-1)^k \binom{n+1}{k} \frac{(t-kR)^n u_{kR}(t)}{n!} \tag{25}
\]

\[\boxed{D. \text{ Numerical Results}}\]

By using (25), we show the instances of \( P_n(t) \) for \( n = 0, 1 \), and 2.
\[
\begin{itemize}
  \item \( n = 0 \)
  \[ \hat{P}_0(t) = \sum_{k=0}^{1} (-1)^k \binom{1}{k} \left(1 - \frac{kR}{t}\right)^0 u_{kR}(t) = 1 - u_{R}(t). \]
  Divide into two cases by the step function of this formula.
  \begin{enumerate}
    \item \( t \leq R \)
      \[ P_0(t) = 1. \]
    \item \( R < t \)
      \[ P_0(t) = 0. \]
  \end{enumerate}
\end{itemize}

Consequently, we have
\[
P_0(t) = \begin{cases} 
1 & (t \leq R) \\
0 & (R < t). 
\end{cases} \tag{26}
\]

This means that \( P_0(t) = r(t) \), which can also be obtained trivially from (19).
\[
\begin{itemize}
  \item \( n = 1 \)
  \[ P_1(t) = \sum_{k=0}^{2} (-1)^k \binom{2}{k} \left(1 - \frac{kR}{t}\right)^1 u_{kR}(t) = u_0(t) - 2\left(1 - \frac{R}{t}\right) u_{R}(t) + \left(1 - \frac{2R}{t}\right) u_{2R}(t). \]
  It is convenient to divide the domain of \( t \) into the following three cases.
  \begin{enumerate}
    \item \( t \leq R \)
      \[ P_1(t) = 1. \]
    \item \( R < t \leq 2R \)
      \[ P_1(t) = \frac{2R}{t - 1}. \]
    \item \( 2R < t \)
      \[ P_1(t) = 0. \]
  \end{enumerate}
\end{itemize}

Consequently, we have
\[
P_1(t) = \begin{cases} 
1 & (t \leq R) \\
2z - 1 & (R < t \leq 2R) \\
0 & (2R < t) 
\end{cases}. \tag{27}
\]

where \( z = R/t \).
\[
\begin{itemize}
  \item \( n = 2 \)
  Similarly, we have
  \[ P_2(t) = \begin{cases} 
1 & (t \leq R) \\
-3z^2 + 6z - 2 & (R < t \leq 2R) \\
9z^2 - 6z + 1 & (2R < t \leq 3R) \\
0 & (3R < t) 
\end{cases}. \tag{28}
\]
\end{itemize}

where \( z = R/t \).
To check the relevance of the results experimentally, we have also obtained the numerical results for $P_n(t)$ by Monte Carlo simulation. As shown in Fig. 2, the numerical results (shown as dots) correspond well with the analytical results (shown as lines).

![Fig. 2. The numerical results for $P_n(t)$.](image)

III. FREE-END FRM

A. Model

The Free-End FRM is almost the same as the Fixed-End FRM except that, unlike the Fixed-End model, the Free-End model places no nodes on both end points of the interval $I$, thus the total number of the nodes is $n$.

We denote the locations of the nodes by $X_1, X_2, \ldots, X_n$ from left to right, as in Fig. 3. We once again abuse the notation by denoting by $X_i$ the node at location $X_i$ itself.

By using $W$ defined in (3), the probability $Q_n(t)$ that node $X_1$ is connected to node $X_n$ is given as follows.

$$Q_n(t) = P_R\{W \leq R\} (n \geq 2).$$

$$\text{(29)}$$

![Fig. 3. The Free-End Fixed Radius Model.](image)

B. Analysis

In this subsection, we calculate $Q_n(t)$ by using the results of the preceding section.

If we assume that the leftmost node $X_1$ and the rightmost node $X_n$ are located at $x$ and $y$, respectively, this Free-End model is equivalent to the Fixed-End model specified by the interval $[x, y]$ of length $y - x$ with $n - 2$ nodes inside.

Therefore, we can write $Q_n(t)$ by using $P_{n-2}(y - x)$ as follows.

$$Q_n(t) = \int_0^t \int_0^y P_{n-2}(y - x)f_{1,n}(x,y) dxdy. \quad \text{(30)}$$

where $f_{1,n}(x,y)$ denotes the joint probability density function of $X_1$ and $X_n$ which, according to the theory of the order statistics, can be represented as follows.

$$f_{1,n}(x,y) = n(n-1)f(x)f(y)(F(y) - F(x))^{n-2}. \quad \text{(31)}$$

Substituting (31) into (30) and recalling that $F(x) = x/t$, we have

$$Q_n(t) = \frac{n(n-1)}{t^n} \int_0^t \int_0^y (y-x)^{n-2} P_{n-2}(y-x) dxdy. \quad \text{(32)}$$

By using (12), i.e., $n!\hat{p}_n(t) = t^n P_n(t)$, we have

$$Q_n(t) = \frac{n!}{t^n} \int_0^t \int_0^y \hat{p}_{n-2}(y-x) dxdy. \quad \text{(33)}$$

Here, we introduce $w = y - x$ and use the variable $w$ instead of $x$.

$$Q_n(t) = \frac{n!}{t^n} \int_0^t \int_0^w \hat{p}_{n-2}(w)(-dw)dy$$

$$= \frac{n!}{t^n} \int_0^t \int_0^w \hat{p}_{n-2}(w)dw. \quad \text{(34)}$$

Furthermore, by reversing the order of the sum, we can simplify this double integral form to a single integral form as follows.

$$Q_n(t) = \frac{n!}{t^n} \int_0^t \int_0^w \hat{p}_{n-2}(w)dydw$$

$$= \frac{n!}{t^n} \int_0^t (t-w)\hat{p}_{n-2}(w)dw. \quad \text{(35)}$$

Therefore, the integral part of $Q_n(t)$ is represented by the convolution of $\hat{p}_{n-2}(t)$ and $t$.

Let

$$\hat{q}_n(t) = \frac{n!}{n!} Q_n(t). \quad \text{(36)}$$

Then, from (35) we have

$$\hat{q}_n(t) = \int_0^t (t-w)\hat{p}_{n-2}(w)dw. \quad \text{(37)}$$

To obtain closed form solutions, we rewrite (37) into an algebraic form by using the Laplace transform as follows.

$$\mathcal{L}[\hat{q}_n(t)] = \mathcal{L}[\hat{p}_{n-2}(t)]\mathcal{L}[t]. \quad \text{(38)}$$

If we write $\mathcal{L}[\hat{q}_n(t)] = \hat{Q}_n(s)$ and $\mathcal{L}[\hat{p}_{n-2}(t)] = \hat{P}_{n-2}(s)$, this equation can be further reduced by (18) as follows.

$$\hat{Q}_n(s) = \frac{1}{s^2} \hat{P}_{n-2}(s)$$

$$= \frac{1}{s^2} R(s)^{n-1}. \quad \text{(39)}$$

By the inverse Laplace transform of (39), we obtain

$$Q_n(t) = \frac{n!}{t^n} \mathcal{L}^{-1}\left[\frac{1}{s^2} R(s)^{n-1}\right]. \quad \text{(40)}$$
By using (22) and the binomial theorem, we see that
\[
\frac{1}{s^2} R(s)^{n-1} = \frac{(1 - e^{-R/s})^{n-1}}{s^{n+1}}
\]
\[
= \frac{1}{s^{n+1}} \sum_{k=0}^{n-1} \binom{n-1}{k} (-e^{-R/s})^k
\]
\[
= \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^k \frac{e^{-kR/s}}{s^{n+1}}. \tag{41}
\]

Therefore, we need only calculate the inverse Laplace transform of \(e^{-kR/s}/s^{n+1}\).
\[
\mathcal{L}^{-1} \left[ \frac{1}{s^2} R(s)^{n-1} \right] = \sum_{k=0}^{n-1} \binom{n-1}{k} \mathcal{L}^{-1} \left[ \frac{e^{-kR/s}}{s^{n+1}} \right]
\]
\[
= \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \frac{(t-kR)^n u_{kR}(t)}{n!}. \tag{42}
\]

Substituting the result into (40), we have
\[
Q_n(t) = \frac{n!}{t^n} \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \left( \frac{t-kR}{t} \right)^n u_{kR}(t). \tag{43}
\]

C. Numerical Results

We show instances of \(Q_n(t)\) for \(n = 2\) and \(3\).
\[
Q_2(t) = \begin{cases} 
1 & (t \leq R) \\
-z^2 + 2z & (R < t)
\end{cases} \tag{44}
\]
where \(z = R/t\).
\[
Q_3(t) = \begin{cases} 
1 & (t \leq R) \\
2z^3 - 6z^2 + 6z - 1 & (R < t \leq 2R) \\
6z^3 - 6z^2 & (2R < t)
\end{cases}
\]
where \(z = R/t\).

To check the relevance of the results experimentally, we have also obtained the numerical results for \(Q_n(t)\) by Monte Carlo simulation. As shown in Fig. 4, the numerical results (shown as dots) correspond well with the analytical results (shown as lines).

IV. CONCLUSION

In this paper, we obtained the closed form solutions of the probability that random graphs generated by Fixed-End and Free-End FRMs are connected graphs by using Laplace transform. The results are useful for designing of ad-hoc network systems in which objects with a RFID attachment are set up on long straight roads and rivers, belt conveyors and display racks. Using the results, we can calculate the appropriate transmitting range and the number of nodes in the interval which are needed to design ad-hoc network systems. Although we have assumed that the number of nodes, \(n\), is fixed, it is straightforward to extend our results to the models where that number is variable. To show this, let \(N\) be a random variable which denotes the number of nodes inside the interval and let \(P_r\{N = n\} = p_n\). Then the probability that the generated random graph is connected is simply given as \(\sum_{n=0}^{\infty} p_n Q_n(t)\), where we define that graphs with no nodes \((n = 0)\) are connected.

As future works, we would like to analyze the asymptotic behavior of the connectivity when the number of nodes is sufficiently large. In [8] and [9], the phenomena called the phase transition are discussed in terms of random graphs associated with wireless networks. We would like to work on the issue based on the results found in this paper.

REFERENCES