Regular maps from Cayley graphs, Part 1: Balanced Cayley maps

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Received July 1990  
Revised 22 April 1991  
Dedicated to Gert Sabidussi.

Abstract  

A Cayley map is a Cayley graph 2-cell embedded in some orientable surface so that the local rotations at every vertex are identical. Two types of Cayley maps are introduced: the balanced and antibalanced Cayley maps. In Part 1, conditions are given under which a balanced Cayley map is regular or reflexible, and automorphism groups of such maps are determined. In addition, a characterization of regular balanced Cayley maps is presented.

1. Introduction

The study of regular (or, symmetrical) maps has a long and rich history which leads back to the 19th century. Among the variety of problems considered, the most important seem to be that of classifying all regular maps on a given closed surface and that of deciding whether or not a given graph admits a regular map. As far as the first problem is concerned, the classification has been accomplished only for surfaces of small genus. On the other hand, the second problem has been solved only for a small number of classes of graphs, for example for complete graphs. In the investigations, both group-theoretical as well as combinatorial methods have been employed. (For general information on regular maps, the reader may consult [2–4, 7].)
In our two-part series we contribute to the second problem. We shall focus on maps built up on Cayley graphs, the Cayley maps. In particular, two classes of Cayley maps will be thoroughly investigated. While the first of these classes, the balanced Cayley maps, has already appeared in the literature [1-3, 7], the second class, the antibalanced maps, seems to be completely new. Our aim is to give a description of such maps in algebraic terms, which in turn enables to characterize these maps as well as their automorphism groups. The methods used are both combinatorial and group-theoretical.

Part 1 will be devoted to balanced Cayley maps. We present conditions under which such maps are regular or reflexible, and determine their automorphism groups. At the end we characterize regular balanced Cayley maps in terms of certain group homomorphisms. In the follow up to this paper, antibalanced Cayley maps will be investigated from similar points of view. We shall show that these maps lead to interesting group-theoretical concepts such as groups with sign structure and their anti-automorphisms. The necessary group-theoretical background will appear in a separate paper [5].

2. Preliminaries

A map is a cell decomposition of a closed surface. Equivalently, it is a 2-cell embedding of some graph. For our purposes, only orientable maps will be considered. Therefore a map can be identified with a pair \((K, P)\), \(K\) being a connected graph and \(P\) being a rotation of \(K\). Let \(D(K)\) be the set of all arcs of \(K\), where an arc is an edge endowed with any of the two possible orientations. A rotation of \(K\) is a permutation which, for every vertex \(v\) of \(K\), cyclically permutes the set of arcs emanating from \(v\). Let \(T\) be the arc-reversing involution of \(D(K)\) interchanging, for any edge \(e\) of \(K\), the two arcs associated with \(e\). Then the cycles of the permutation \(PT = P \circ T\) correspond to face-boundaries of the embedding of \(K\) defined by the rotation \(P\). We note that the arc \(T(x)\) is called the inverse of \(x\) (\(x \in D(K)\)) and is sometimes denoted by \(x^{-1}\).

A map-automorphism is a 1-1 mapping which preserves the cell structure of a map and, if the map is orientable, also the orientation of the supporting surface. Combinatorially, an automorphism of a map \(M = (K, P)\) is a permutation \(A\) of \(D(K)\) such that \(AP = PA\) and \(AT = TA\) (see [4]). These conditions ensure that every map-automorphism is necessarily an automorphism of the underlying graph. Moreover, it is easily shown [2] that for any two arcs \(x\) and \(y\) there is at most one map-automorphism taking \(x\) to \(y\). It follows that the order of the automorphism group of \(M\), \(\text{Aut} M\), is less than or equal to \(|D(K)| = 2|E(K)|\); in fact, \(|\text{Aut} M|\) divides the number \(2|E(K)|\) (\(E(K)\) being the edge set of \(K\)). Thus, the maximal symmetry is obtained when \(|\text{Aut} M| = |D(K)|\). In this case the map is called regular. In a regular map, any arc can be mapped to any other arc by a map-automorphism. In other words, the map-automorphism group acts transi-
tively on arcs. As a consequence, all vertices, all edges, and all faces of a regular
map are equivalent under map-automorphisms.

Along with map automorphisms also reflections of maps have often been
considered. Roughly speaking, a reflection of a map $M$ is an isomorphism of $M$
on its mirror image. Taking our combinatorial approach, a reflection of a map
$M = (K, P)$ is a permutation $B$ of $D(K)$ such that $BP = P^{-1}B$ and $BT = TB$. A
reflection need not exist for every map, but if it does the map is called reflexible.
It is clear that the set of all reflections together with the set of all automorphisms
of $M$ forms a group in which $\text{Aut} M$ has index two or one depending on whether
$M$ does or does not admit a reflection. This group is known as the extended
automorphism group of $M$ and will be denoted by $\text{EAut} M$. For a regular and
reflexible map $M = (K, P)$ we therefore have that $|\text{EAut} M| = 4 |E(K)|$.

In many constructions of regular maps, Cayley graphs play a prominent role. In
order to define a Cayley graph, let $G$ be a finite group and let $\Omega$ be a subset of $G$
satisfying the following conditions: (i) $\Omega$ generates $G$, (ii) $\Omega$ does not contain $1$,
the identity element of $G$, and (iii) $x^{-1}$ belongs to $\Omega$ whenever $x$ does. Now, the
Cayley graph $C(G, \Omega)$ has $G$ as its vertex set, $G \times \Omega$ as its arc-set and the
incidence relation is defined by letting an arc $(g, x) \in G \times \Omega$ have initial vertex $g$
and terminal vertex $gx$. The inverse of $(g, x)$ is then $T(g, x) = (gx, x^{-1})$. The
generator $x$ is said to be the colour of the arc $(g, x)$. Observe that for every colour
$x$ there exists exactly one $x$-coloured edge of $C(G, \Omega)$ incident with a given
vertex $g \in G$. In particular, the degree of a vertex has the same parity as the
number of involutions in $\Omega$.

Finally, we shall deal with maps $(K, P)$ where $K$ is a Cayley graph $C(G, \Omega)$.
Assume that the rotation $P$ of $C(G, \Omega)$ is such that there exists a cyclic
permutation $p$ of $\Omega$ for which $P(g, x) = (g, p(x))$ for an arbitrary arc $(g, x) \in D(C(G, \Omega))$. Then the pair $(C(G, \Omega), P)$ will be called a Cayley map and
denoted by $\text{CM}(G, \Omega, p)$. Note that Cayley maps are sometimes referred to as
symmetrical embeddings of Cayley graphs. It is well known that every Cayley map
is vertex-transitive [2]. In general, however, it need not be a regular map. In
order to be able to give some conditions under which a Cayley map becomes
regular, throughout our series we shall confine to two types of Cayley maps called
balanced and antibalanced, respectively.

A Cayley map $\text{CM}(G, \Omega, p)$ will be said to be balanced if

$$p(x^{-1}) = p(x)^{-1}$$

for every $x \in \Omega$, and antibalanced if

$$p(x^{-1}) = (p^{-1}(x))^{-1}.$$  

3. Basic facts about balanced Cayley maps

The material in this section extends the work of Biggs [1] and Biggs and White
[2, Chapter 5] where balanced Cayley maps have been considered, although not
explicitly defined. We shall focus on conditions under which a Cayley map is
regular, and if so, whether it is reflexible.

Balanced Cayley maps split into two different classes according to whether \( \Omega \)
does or does not contain involutions. Suppose \( x \in \Omega \) is an involution. By virtue of
(1), \( p(x) \) is an involution as well, and hence so is every element of \( \Omega \). In addition,
if \( \Omega \) consists of involutions then every cyclic permutation of \( \Omega \) defines a balanced
Cayley map. Now, let \( \Omega \) contain no involution; in this case the corresponding
Cayley map is called involution-free. Then \( \Omega \) has an even number of elements,
say, \( |\Omega| = 2d \), and from (1) it follows that for any \( x \in \Omega \) it holds that \( p^d(x) = x^{-1} \).
Clearly, this condition is also sufficient for such a Cayley map to be balanced.
Finally we notice that in either case there exists an integer \( d \) \((d = 0 \text{ if } \Omega \text{ contains}
involutions)\) such that \( p^d(x) = x^{-1} = p^{-d}(x) \) for every \( x \in \Omega \).

The condition (1) imposes certain restrictions on possible lengths of faces in a
Cayley map. To see this, recall that the face boundaries of any map correspond to
cycles of the permutation \( PT \). Applying this to the arc \((1, x)\) where \( 1 \) is the unit
element of \( G \) and \( x \in \Omega \) we successively obtain

\[
PT(1, x) = P(x, x^{-1}) = (x, p(x^{-1})),
\]
\[
PT(x, p(x^{-1})) = P(xp(x^{-1}), p(x^{-1}^{-1})) = P(xp(x^{-1}), p(x))
\]
\[
= (xp(x^{-1}), p^2(x))
\]

and so on. Thus, letting

\[
R_j(x) = \prod_{i=0}^{j} p^{i}(x_{\varepsilon}) \quad \text{where } \varepsilon_i = (-1)^i \text{ and } j \geq 0,
\]

the cycle of \( PT \) containing \((1, x)\) can be written in the form

\[(1, x), (R_0(x), p(x^{-1})), (R_1(x), p^2(x)), \ldots, (R_{j-1}(x), p^j(x^v)), \ldots.\]

Let \( m \) be the smallest positive integer for which \( p^m(x^v) = x \). Then the length of
the face containing \((1, x)\) is the least common multiple of \( m \) and the order of
\( R_{m-1}(x) \). If \( \Omega \) is involution-free then (1) implies that \( m = 2d \) or \( m = d \) according
to whether \( d \) is even or odd, respectively. Note that if \( G \) is Abelian then in the
former case \( R_{2d-1}(x) = 1 \in G \). Thus, each face of \( M \) has length \( 2d \) in this case. If,
on the other hand, \( \Omega \) consists of involutions and \( |\Omega| = k \) then \( m = k \).

The above considerations are easily seen to be independent of the particular
choice of \( x \in \Omega \). Summing up, we have obtained the following result.

**Proposition 1.** Let \( M = CM(G, \Omega, p) \) be a \( k \)-valent balanced Cayley map. Then
the length of every face of \( M \) is a multiple of:

(a) \( k \), if \( \Omega \) is involution-free and \( k \equiv 0 \pmod{4} \);
(b) \( k/2 \), if \( \Omega \) is involution-free and \( k = 2 \pmod{4} \);
(c) \( k \), if \( \Omega \) consists of involutions.

In case (a), if \( G \) is Abelian then every face of \( M \) has length precisely \( k \).
Our next theorem contains a result of Biggs and White [2, Theorem 5.3.7] as well as its converse.

**Theorem 2.** Let \( M = CM(G, \Omega, p) \) be a Cayley map. If there exists an automorphism \( \rho \) of the group \( G \) whose restriction to \( \Omega \) is equal to \( p \) (\( p \mid \Omega = p \)) then \( M \) is regular and balanced. Conversely, if \( M \) is a regular balanced Cayley map then such an automorphism \( \rho \) exists.

**Proof.** The first statement of the theorem is proved in [2, p. 119]. For the sake of completeness we offer here a different proof in which the map automorphisms are explicitly defined. Let us define for every \( b \in G \) and every \( k, 0 \leq k \leq |\Omega| - 1 \), a mapping \( A_{b,k} \) on the set of arcs of the Cayley graph \( C(G, \Omega) \) by \( A_{b,k}(a, x) = (bp^k(a), \rho^k(x)) \) where \( a \in G \) and \( x \in \Omega \) are arbitrary and \( \rho \) is the automorphism of \( G \) for which \( \rho^1 \circ \rho^{-1} = \rho \). Clearly, each \( A_{b,k} \) is a bijection. It remains to prove that \( A_{b,k} \) always commutes with both \( P \) and \( T \) (see Section 2 for the definition), and this is verified as follows.

\[
A_{b,k}P(a, x) = A_{b,k}(a, \rho(x)) = A_{b,k}(a, \rho(x))
\]

\[
= (bp^k(a), \rho^k(x)) = (bp^k(a), \rho^k(x))
\]

\[
= P(bp^k(a), \rho^k(x)) = PA_{b,k}(a, x);
\]

\[
A_{b,k}T(a, x) = A_{b,k}(ax, x^{-1}) = (bp^k(ax), \rho^k(x^{-1}))
\]

\[
= (bp^k(a)\rho^k(x), (\rho^k(x))^{-1}) = T(bp^k(a), \rho^k(x))
\]

\[
= TA_{b,k}(a, x).
\]

Thus we have established that every \( A_{b,k} \) is a map automorphism. Moreover, for every arc \((a', x')\) of \( C(G, \Omega) \) there exist \( b \) and \( k \) such that \( A_{b,k}(a, x) = (a', x') \). Therefore \( M \) is a regular map, and since \( p \) is a restriction of a group automorphism, \( M \) is balanced.

For the converse, suppose \( M \) is a regular balanced Cayley map. Let \( A \) be the automorphism of \( M \) such that \( A(1, x) = (1, \rho(x)) \) for every \( x \in \Omega \). Further, let \( q(g, x) = g \) for every arc \((g, x)\) of \( C(G, \Omega) \). Now, define \( \rho(g) = qA(g, x) \) where \((g, x)\) is any arc emanating from \( g \in G \). Since \( A \) is a map automorphism and thereby an automorphism of \( C(G, \Omega) \), it maps adjacent arcs to adjacent arcs. Consequently, the mapping \( \rho \) is correctly defined. We first show that \( \rho \) on \( \Omega \) restricts to \( p \). Indeed, if \( x \in \Omega \) we have

\[
\rho(x) = qA(x, x^{-1}) - qAT(1, x) = qTA(1, x) = qT(1, p(x))
\]

\[
= q(p(x), p(x)^{-1}) = p(x).
\]

To finish the proof it is now sufficient to verify that \( \rho(xy) = \rho(x)\rho(y) \) for any \( x \) and \( y \) in \( \Omega \). Let us therefore determine \( A(xy, y^{-1}) \). Clearly, \( A(xy, y^{-1}) = AT(x, y) \). Now, there exists an integer \( k \geq 1 \) such that \( (x, y) = P^k(x, x^{-1}) = \)
(x, p^k(x^{-1})), whence p^k(x^{-1}) = y. So
\[
TA(x, y) = TAP^k(x, x^{-1}) = TAP^kT(1, x) = TP^kTA(1, x)
= TP^kT(1, p(x)) = TP^k(p(x), p(x)^{-1}).
\]

Since our map \(M\) is balanced we further obtain
\[
TP^k(p(x), p(x)^{-1}) = TP^k(p(x), p(x^{-1})) = T(p(x), p^k(p(x^{-1})))
= T(p(x), pp^k(x^{-1})) = T(p(x), p(y))
= (p(x)p(y), p(y^{-1})),
\]
and hence \(A(xy, y^{-1}) = (p(x)p(y), p(y^{-1}))\). At the end we have
\[
\rho(xy) = qA(xy, y^{-1}) = q(p(x)p(y), p(y^{-1}))
= p(x)p(y) = \rho(x)\rho(y),
\]
which means that \(\rho\) is the required automorphism of \(G\).  \(\square\)

As we have seen, the existence of a regular balanced Cayley map is equivalent to the existence of a certain group automorphism described in Theorem 2. This automorphism is obviously determined uniquely by the map in question, and will be referred to as the \textit{rotary automorphism} of the group \(G\).

4. Automorphisms and isomorphisms

Let \(\text{Aut}(G, \Omega)\) be the subgroup of the automorphism group of \(G\) comprising all the automorphisms of \(G\) which preserve \(\Omega\) setwise. Clearly, the rotary automorphism \(\rho\) belongs to \(\text{Aut}(G, \Omega)\). The way \(\text{Aut}(G, \Omega)\) affects the structure of Cayley maps is seen in our next result.

\textbf{Theorem 3.} Let \(M = \text{CM}(G, \Omega, \rho)\) be a balanced regular Cayley map and let \(\rho\) be the corresponding rotary automorphism of the group \(G\). Then, \(M\) is reflexible if and only if there exists an involution \(\tau \in \text{Aut}(G, \Omega)\) such that \(\tau\rho = \rho^{-1}\tau\).

\textbf{Proof.} Let \(M\) be reflexible and let \(B\) be the reflection of \(M\) for which \(B(1, x) = (1, x^{-1})\) for a fixed \(x \in \Omega\). Again, let \(q(a, y) = a\). Let us define a mapping \(\tau: G \rightarrow G\) by putting \(\tau(a) = qB(a, y)\) where \(y\) is an arbitrary element of \(\Omega\). Similarly as in the proof of Theorem 2 one can show that \(\tau\) is correctly defined.

We first prove that
\[
B(a, \rho^m(x)) = (\tau(a), \rho^{-m}(x^{-1}))
\] (3)
for every \(a \in G\) and every integer \(m\); as a by-product we obtain the equality
\[
\tau(a\rho^k(x)) = \tau(a)\rho^{-k}(x^{-1})
\] (4)
for every \( a \in G \) and every integer \( k \). So far we know that \( \tau(1) = 1 \) and \( B(1, x) = (1, x^{-1}) \), which readily implies \( B(1, \rho^k(x)) = (1, \rho^{-k}(x^{-1})) \). Employing induction on the length of the element \( a \in G \) considered as a word in symbols from \( \Omega \), the induction step for both (3) and (4) is accomplished by the following computation.

\[
B(a\rho^k(x), \rho^m(x)) = B(a\rho^k(x), \rho^{m+d}(x^{-1})) = BP^{m+d-k}(a\rho^k(x), \rho^k(x^{-1})) =
\]

\[
= P^{k-m-d}B(a\rho^k(x), \rho^k(x^{-1})) = P^{k-m-d}BT(a, \rho^k(x)) = T(\tau(a), \rho^{-k}(x^{-1}))
\]

\[
= P^{k-m-d}(\tau(a)\rho^{-k}(x^{-1}), \rho^{-k}(x)) = (\tau(a)\rho^{-k}(x^{-1}), \rho^{-m-d}(x)) = (\tau(a)\rho^{-k}(x^{-1}), \rho^{-m}(x^{-1})).
\]

(Note that in the above lines we made use of the fact that \( \rho \) is a group automorphism and \( x = \rho^t(x^{-1}) \); (*) indicates the place where the induction hypothesis has been applied.)

From (4) it follows that \( \tau \) is an automorphism of \( G \) which preserves \( \Omega \) setwise. It remains to prove that \( \tau \) is an involution such that \( \tau \_\rho = \rho^{-1} \tau \). Clearly, it is sufficient to compute \( \tau^2(y) \) and \( \tau \_\rho(y) \) for an arbitrary generator \( y = \rho^k(x) \in \Omega \).

By taking \( a = 1 \) in (4) we have

\[
\tau^2(y) = \tau(\tau(\rho^k(x))) = \tau(\rho^{-k}(x^{-1})) = \tau(\rho^{-k-d}(x)) = \rho^{k+d}(x^{-1}) = \rho^k(x) = y
\]

and

\[
\tau \_\rho(y) = \tau(\rho^{k+1}(x)) = \rho^{-k-1}(x^{-1}) = \rho^{-1}(\rho^{-k}(x^{-1})) = \rho^{-1}(\tau(\rho^k(x))) = \rho^{-1}(\tau(y)).
\]

Thus, the automorphism \( \tau \in \text{Aut}(G, \Omega) \) has all the required properties.

Conversely, let \( \tau \in \text{Aut}(G, \Omega) \) be an involution such that \( \tau \_\rho = \rho^{-1} \tau \). Then, \( \tau \_\rho^m = \rho^{-m} \tau \) for every integer \( m \). Let us fix an element \( x \in \Omega \) and define a mapping on the arcs of \( C(G, \Omega) \) by setting

\[
B(a, \rho^m(x)) = (\tau(a), \rho^{-m}(x)) = (\tau(a), \tau \_\rho^m(x)).
\]

It can be easily verified that \( B \) is an automorphism of the graph \( C(G, \Omega) \). To show that \( B \) is a reflection of \( \text{CM}(G, \Omega, \rho) \) it remains to verify that \( BP = P^{-1}B \) and \( BT = TB \); we leave it to the reader. \( \square \)

Thus, the reflexibility of a regular balanced Cayley map is equivalent to the existence of a special involution \( \tau \in \text{Aut}(G, \Omega) \). Any such involutory automorphism of \( G \) will be called reflective.
Our next aim is to describe automorphism groups of regular balanced Cayley maps. It is well known [2] that the automorphism group of an arbitrary Cayley map \( M = CM(G, \Omega, \rho) \) contains a subgroup isomorphic to \( G \). If \( M \) is a balanced regular Cayley map, much more can be said. In fact, we show that the automorphism group of \( M \) is then obtained by adjoining the rotary automorphism to \( G \). A similar description can be given for the extended automorphism group of a reflexible map.

Before stating the results we recall a definition. Let \( H \) be a group and let \( X \) be a set of automorphisms of \( H \). The extension \( H(X) \) of the group \( H \) by the set \( X \) is defined as follows. The elements of \( H(X) \) are pairs \( (h, \gamma) = h\gamma \), where \( h \in H \) and \( \gamma \) belongs to the group of automorphisms generated by \( X \); the group multiplication is given by \( (h, \gamma)(k, \delta) = (h\gamma(k), \gamma\delta) \). It is well known that the extension of a group by adjoining automorphisms is a split extension.

**Theorem 4.** Let \( M = CM(G, \Omega, \rho) \) be a balanced regular Cayley map. Then \( \text{Aut} M = G(\rho) \), where \( \rho \) is the rotary automorphism of \( G \). Moreover, if \( M \) is reflexible then the extended automorphism group is \( \text{EAut} M = G(\rho, \tau) \), where \( \tau \) is the reflective automorphism of \( G \).

**Proof.** Let \( A_{b,k} \) be the automorphism of \( M \) sending a fixed arc \((1, x)\) onto \((b, \rho^k(x))\). First of all we have
\[
A_{b,k}(a, \rho^m(x)) = (b\rho^k(a), \rho^{k+m}(x)) \tag{5}
\]
for an arbitrary arc \((a, \rho^m(x))\) of \( C(G, \Omega) \). This can be proved by induction similarly as (3) or (4), so we omit the details. Now, the identity (5) yields
\[
A_{b,k}A_{c,1}(1, x) = A_{b,k}(c, \rho^1(x)) = (b\rho^k(c), \rho^{k+1}(x)) = A_{b\rho^k(c), k+1}(1, x).
\]
Since the map automorphisms are uniquely determined by the image of a single arc [2], we obtain
\[
A_{b,k}A_{c,1} = A_{b\rho^k(c), k+1}.
\]
This means that the mapping \( \text{Aut} M \to G(\rho) \), \( A_{b,k} \mapsto (b, \rho^k) \) is a surjective homomorphism. In fact, it must be an isomorphism because \( |\text{Aut} M| = |D(C(G, \Omega))| = |G| |\Omega| = |G(\rho)| \). This proves the first part of the theorem.

To prove the second part, note that every element of \( \text{EAut} M \) can be written in the form \( A_{b,k}B^\varepsilon \) where \( B \) is the reflection of \( M \) defined by (3) and \( \varepsilon = 0 \) or 1. Using (3), (5) and the fact that \( \tau^\varepsilon(x) = \rho^{\varepsilon\delta}(x) \) it is easily verified that
\[
A_{b,k}B^\varepsilon(a, \rho^m(x)) = (b\rho^k\tau^\varepsilon(a), \rho^{k+(1-2\varepsilon)m}\tau^\varepsilon(x))
\]
for every \( a \in G \) and every integer \( m \). As a consequence we obtain
\[
(A_{b,k}B^\varepsilon)(A_{c,1}B^\alpha)(1, x) = A_{b\rho^k\tau^{\varepsilon}(c), k+(1-2\varepsilon)m}\tau^\varepsilon(1, x),
\]
which in turn implies that the mapping \( \text{EAut} M \to G(\rho, \tau) \), \( A_{b,k}B^\varepsilon \mapsto (b, \rho^k\tau^{\varepsilon}) \) is an isomorphism. This completes the proof. \( \square \)
Theorem 4 has an obvious corollary: the group \( G(\rho) \), being isomorphic to \( \text{Aut} \mathcal{M} \), can be generated by two generators (one involutory), no matter what the original group \( G \) is. Since \( \rho \) is the rotary automorphism of \( G \) which permutes a generating set of \( G \), the generators for \( G(\rho) \) can be chosen to be \( I = (x, \rho^d) = xp^d \) and \( J = (1, \rho) = \rho \) (recall our agreement about \( d \) before Proposition 1). Similarly, the group \( G(\rho, \tau) \), being isomorphic to \( \text{EAut} \mathcal{M} \), must contain a generating set consisting of three involutions. One possibility is, for instance, \( I = (x, \rho^d), Y = (1, \tau) \) and \( Z = (1, \rho \tau) \).

Finally, we shall be interested in the question when two different cyclic permutations \( r \) and \( s \) of \( \Omega \) lead to isomorphic maps. We shall say that two maps \( M_1 \) and \( M_2 \) of the same graph \( K \), with respective rotations \( R \) and \( S \), are isomorphic if there exists an automorphism \( F \) of \( K \) such that \( FR = SF \) and \( FT = TF \). For balanced regular Cayley maps we have the following result.

**Theorem 5.** Let \( M_1 = \text{CM}(G, \Omega, r) \) and \( M_2 = \text{CM}(G, \Omega, s) \) be balanced regular Cayley maps with the same underlying Cayley graph, and let \( \rho \) and \( \sigma \) be the corresponding rotary automorphisms of \( G \). Then, the following assertions are equivalent:

(a) \( M_1 \) and \( M_2 \) are isomorphic Cayley maps.

(b) \( \rho \) and \( \sigma \) are conjugate in the group \( \text{Aut}(G, \Omega) \).

(c) There is an isomorphism of the groups \( G(\rho) \) and \( G(\sigma) \) which sends \( \rho \) to \( \sigma \) and whose restriction to \( G \) belongs to \( \text{Aut}(G, \Omega) \).

**Proof.** (a) \( \Rightarrow \) (b): Let \( F : M_1 \rightarrow M_2 \) be a map isomorphism and let \( R \) and \( S \) be rotations of \( M_1 \) and \( M_2 \), respectively. By definition, \( FR = SF \) (and hence \( FR^k = S^kF \) for every \( k \)) and \( FT = TF \). By regularity of \( M_2 \) we may assume \( F(1, x) = (1, x) \) for a fixed \( x \in \Omega \). Let us define a mapping \( f : G \rightarrow G \) by setting \( f(a) = qF(a, y) \) where \( q \) is the projection onto the first coordinate. Since \( F \) is an isomorphism, \( f \) is well defined and is a bijection on the group \( G \). We want to show that \( f \in \text{Aut}(G, \Omega) \) and \( fp = sf \). As in the proof of Theorem 3 we obtain by induction that \( F(a, \rho^k(x)) = (f(a), \sigma^k(x)) \) and \( f(ap^k(x)) = f(a)\rho^k(x) \) for every \( a \in G \) and every integer \( k \). It follows that \( f \in \text{Aut}(G, \Omega) \). Further, note that \( f(x) = x \), whence \( f(\rho^k(x)) = \sigma^k(x) = \sigma^k(f(x)) \). This implies that \( fp(y) = sf(y) \) for every generator \( y \in \Omega \); therefore \( \rho \) and \( \sigma \) are conjugate in \( \text{Aut}(G, \Omega) \).

(b) \( \Rightarrow \) (c): Let \( f \in \text{Aut}(G, \Omega) \) such that \( fp = sf \). Define a mapping \( g : G(\rho) \rightarrow G(\sigma) \) by setting \( g(a, \rho^k) = (f(a), \sigma^k) \). Obviously, \( g \) is a bijection. Moreover,

\[
g((a, \rho^k)(b, \rho^m)) = g(ap^k(b), \rho^{k+m}) = (f(ap^k(b)), \sigma^{k+m}) = (f(a)f\rho^k(b), \sigma^{k+m}) = (f(a), \sigma^k)(f(b), \sigma^m) = g(a, \rho^k)g(b, \rho^m),
\]

implying that \( g \) is the required isomorphism.
(c) \Rightarrow (a): Let g : G(p) \rightarrow G(\sigma) be an isomorphism such that \( g(1, \rho) = (1, \sigma) \) and \( g \in \text{Aut}(G, \Omega) \). By this last assumption, the mapping \( h : G \rightarrow G \) given by the relation \( g(a, \text{id}) = (h(a), \text{id}) \) belongs to \( \text{Aut}(G, \Omega) \). Similarly as in the preceding cases, it is a matter of routine to show that \( g(a, \rho^k) = (h(a), \sigma^k) \) for every \( a \in G \) and every integer \( k \). Now, carrying out the same computation as done in the proof (b) \Rightarrow (c) we obtain \( h\rho^k = \sigma^kh \) for every \( k \).

Choose a fixed \( x \in \Omega \). Since \( h \) preserves \( \Omega \), there is an integer \( m \) such that \( h\rho^m(x) = x \). Put \( f = h\rho^m \). Clearly, \( f \in \text{Aut}(G, \Omega) \). Moreover, \( f\rho = h\rho^m\rho = h\rho\rho^m = \sigma f \rho^m = \sigma f \), whence \( f\rho^k = \sigma^kf \). Finally, define a mapping \( F \) on the edges of \( C(G, \Omega) \) by \( F(a, \rho^k(x)) = (f(a), \sigma^k(x)) \). Again, it is easily verified that \( FR = SF \) and \( FT = TF \) which implies that \( F \) is the desired map isomorphism.

5. A characterization

In the final section we prove a characterization result for regular balanced involution-free Cayley maps. This characterization stems from the fact that a Cayley graph without involutory generators covers a bouquet of circles. Therefore we first establish an auxiliary result concerning regular maps of bouquets of circles. In the rest of this section we only consider Cayley maps without involutory generators.

Recall that the bouquet \( B_d \) of \( d \) circles is the graph consisting of a single vertex and \( d \) loops incident with it. A map \((B_d, P)\) is said to be balanced if for any arc \( y \) of \( B_d \) it holds that \( P(y^{-1}) = (P(y))^{-1} \) (equivalently, \( P^d(y) = y^{-1} \)) where \( y^{-1} = T(y) \) is the inverse of an arc \( y \). For bouquets of circles this property is equivalent with regularity.

**Theorem 6.** A map \( M \) of the bouquet of \( d \) circles is regular if and only if it is balanced. In this case, \( \text{Aut} M \cong \mathbb{Z}_{2d} \).

**Proof.** It is easy to see that a balanced map of \( B_d \) is regular. For the converse, assume that \( M \) is a regular map of \( B_d \) with rotation \( P \). To prove that \( M \) is balanced take an arbitrary arc \( x \) of \( B_d \). Clearly, there exists an integer \( k \geq 1 \) such that \( P^k(x) = x^{-1} \). Choose \( k \) to be the smallest possible. We wish to show that \( P^k(x) = x \). Let \( A \) be an automorphism of \( M \) sending \( x \) to \( x^{-1} \), whose existence is guaranteed by regularity of \( M \). Then

\[ P^k(x^{-1}) = P^kA(x) = AP^k(x) = A(x^{-1}) = AT(x) = TA(x) = T(x^{-1}) = x, \]

and so \( P^k(x) = x^{-1} \). Since \( P^2k(x) = P^k(x^{-1}) = x = P^{2d}(x) \), by minimality of \( k \) we have \( k = d \). Thus, \( P^d(x) = x^{-1} \) and \( x \) is arbitrary. This proves that the map \( M \) is balanced.
It remains to show that if $M$ is regular then $\text{Aut} M \cong Z_{2d}$. Obviously, the assignment $P \mapsto 1 \in Z_{2d}$ and $T \mapsto d \in Z_{2d}$ defines an isomorphism of the permutation group $\langle P, T \rangle$ onto $Z_{2d}$. It is well known that for any regular map the group $\langle P, T \rangle$ is isomorphic with the map-automorphism group $[4]$. Consequently, $\text{Aut} M \cong Z_{2d}$ which completes the proof. □

From the theorem just proved it readily follows that there is exactly one regular map of $B_d$ for each $d$, up to isomorphism. This map is obviously reflexible and its extended automorphism group is isomorphic to the dihedral group $D_{2d}$.

As mentioned in the proof, for any regular map $M$ the group $\text{Aut} M$ is isomorphic to $[P, T]$. In what follows we shall need a more explicit description of this isomorphism. Let $R$ be a map-automorphism which cyclically permutes the arcs incident with a vertex $u$ of $M$ and let $S$ be an automorphism of $M$ which reverses the orientation of an arc at $u$. If a regular map is viewed as a permutation representation of the group $\langle P, T \rangle$ and map-automorphisms as equivariant mappings of this representation, the assignment $R \mapsto P$ and $S \mapsto T$ extends to an isomorphism $\text{Aut} M \rightarrow [P, T]$. This follows easily from well-known facts from the theory of permutation groups, see, e.g., [8].

Recall that if $M = \text{CM}(G, \Omega, p)$ is a 2-valent regular balanced Cayley map then, by Theorem 4, $\text{Aut} M \cong G(p)$ where $p$ is the rotary automorphism of the group $G$. Combining this with the preceding facts we obtain an isomorphism $G(p) \rightarrow [P, T]$ given by $p \mapsto P$ and $x p^d \mapsto T$ where $x$ is an arbitrary fixed element of $\Omega$.

We now prove the main result of this section.

**Theorem 7.** Let $M$ be a 2-valent regular map $M$ with rotation $P$ and arc-reversing involution $T$. Then $M$ is isomorphic to a balanced involution-free Cayley map $\text{CM}(G, \Omega, p)$ if and only if the assignment $P \mapsto 1$ and $T \mapsto d$ extends to a homomorphism $[P, T] \rightarrow Z_{2d}$.

**Proof.** First let $M$ be isomorphic to a Cayley map $\text{CM}(G, \Omega, p)$ satisfying the above assumption. It is well known [6] that $M$ is a (possibly branched) covering over a map $M'$ of $B_d$, the bouquet of $d$ circles. Let $P'$ and $T'$ be the permutations corresponding to $M'$. Since a covering preserves rotations, $M'$ is balanced and hence regular by Theorem 6. Moreover, as a consequence of the theory developed in [4] this covering induces a homomorphism $[P, T] \rightarrow [P', T'] \cong Z_{2d}$ such that $P \mapsto 1$ and $T \mapsto d$. This proves the necessity. Before proving the converse observe that, in addition, the isomorphism $G(p) \cong [P, T]$ gives rise to the homomorphism $f : G(p) \rightarrow Z_{2d}$ with $\text{Ker} f = G$.

Now, let $M$ be a 2-valent regular map with rotation $P$ and arc-reversing involution $T$. Suppose $f : [P, T] \rightarrow Z_{2d}$ is a homomorphism for which $f(P) = 1$ and $f(T) = d$. We have to find a group $G$, a generating set $\Omega$ and a cyclic permutation $p$ of $\Omega$ satisfying 1) such that $M$ is isomorphic to $\text{CM}(G, \Omega, p)$. In accordance
with the implication already proved let \( G = \text{Ker} f, \quad \Omega = \{P^iTP^{-i}; 0 \leq i \leq 2d - 1\}, \)
and define \( p : \Omega \rightarrow \Omega \) by setting \( p(x) = PxP^{-1} \) for \( x \) in \( \Omega \). Obviously, \( \Omega \) generates \( G \) and \( p \) is a cyclic permutation of \( \Omega \), so the triple \((G, \Omega, p)\) defines a Cayley map. Since \( G \) is a normal subgroup of \([P, T]\), the conjugation by the element \( P \)
induces an automorphism \( \rho \) of \( G \). However, \( p \) is clearly a restriction of \( \rho \) to \( \Omega \) so \( \text{CM}(G, \Omega, p) \) is a balanced regular map, by Theorem 2. It remains to verify that the maps \( M \) and \( \text{CM}(G, \Omega, p) \) are isomorphic. Let \( P' \) and \( T' \) be the permutations corresponding to the latter maps. Since both \( M \) and \( \text{CM}(G, \Omega, p) \) are regular, it suffices to show that the assignment \( P \mapsto P' \) and \( T \mapsto T' \) extends to a group isomorphism. By Theorem 4, there is an isomorphism \( \text{Ker} f(\rho) = G(\rho) \rightarrow [P', T'] \) sending \( \rho \) to \( P' \) and \( x\rho^d \) to \( T' \) for a fixed \( x \in \Omega \). Consequently, it is enough to prove that there is an isomorphism \( G(\rho) \rightarrow [P, T] \) such that \( \rho \mapsto P \) and \( x\rho^d \mapsto T \). Let \( H \) be the cyclic subgroup of \([P, T]\) generated by \( P \). Since \( G \triangleleft [P, T] \), \( G \cap H \) is the trivial group and \( GH = [P, T] \) we conclude that \([P, T]\) is a split extension of the group \( G \) by \( H \). This results in the isomorphism \( G(\rho) \rightarrow [P, T] \) such that \( \rho \mapsto P \) and \( x\rho^d \mapsto T \), as required. The proof is complete. \( \square \)

The problem of finding a similar characterization for regular balanced Cayley maps which are not involution-free remains open.

References