OPTIMAL ENERGY CONTROL IN FINITE TIME BY VARYING
THE LENGTH OF THE STRING∗

MARTIN GUGAT†

Abstract. We consider a finite string where, at both end points, a homogeneous Dirichlet
boundary condition holds. One boundary point is fixed, and the other is moving; hence the length
of the string is changing in time. The string is controlled through the movement of this boundary
point. We consider movements of the boundary that are Lipschitz continuous. Only movements
for which at the given finite terminal time the string has the same length as at the beginning are
admissible. Moreover, we impose an upper bound for the Lipschitz constant of the movement that
is smaller than the speed of wave propagation. We consider the optimal control problem to find an
admissible movement for which at the given terminal time the energy of the string is minimal. We
give a sufficient condition for the existence and uniqueness of an optimal movement and construct
an optimal control movement.

Key words. PDE-constrained optimization, optimal control of PDEs, optimal boundary control,
wave equation, optimal energy control, moving boundary, control constraint, optimal shape

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1. Introduction. We consider a string of finite length that is governed by the
wave equation with homogeneous Dirichlet boundary conditions. The left boundary
point is fixed, and the other boundary point of the string is moving. This system has
already been studied in [3].

The boundary control of the wave equation has been studied by many authors
(see, e.g., [16], [13], [11], [12], [1], [20], [8], and the references therein). In most studies
both boundary points of the string are fixed, and the string is controlled through
prescribed function values at the fixed boundary points.

In contrast to this approach, in this paper we control the system through the
movement of the boundary point, that is, through the length of the string as a function
of time. For the case of dimensions greater than one, this corresponds to the control of
the shape as a function of time. Thus our problem is a one-dimensional (1-d) case for
optimal shape control of a hyperbolic partial differential equation. The monograph
Controllability of the wave equation with point control where the point is moving in
the system’s fixed spatial domain is studied in [9]. Observability and stabilizability
of this system are considered in [10]. A problem with moving control of the heat
equation is studied in [4].

The well-posedness of the wave equation in a noncylindrical, time periodical do-
main in $R \times R^N$ has been studied in [17]. Distributed control of the wave equation in a
domain with a moving boundary has been studied in [2] for dimensions unequal to two.
For these dimensions, a contraction of the domain always leads to nondecreasing en-
ergy, and an expansion always leads to nonincreasing energy (see Theorem 2.1 in [2]).

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†Lehrstuhl für angewandte Mathematik AM2, Martensstr. 3, 91058 Erlangen, Germany (gugat@
am.uni-erlangen.de).

1705
The situation in dimension two is completely different. For certain expansions, the energy is nonincreasing, but also, if the interval is contracting sufficiently fast, a decay of the energy can be achieved. In [18], [19] the stabilization of the wave equation through the movement of the boundary is studied in dimension two. A movement is constructed that assures exponential decay of the energy.

In this paper, we study the 1-d-case. In [6] the stability of the string with varying length has been studied, and it has been shown that the energy cannot become arbitrarily small unless it is zero from the start. This follows from inequality (5.10), where a lower bound for the energy at time $t$ is given that depends on the initial state. A sufficient condition for exponential growth of the energy is given in [7]. Interestingly, this situation of exponential energy growth could have applications in photon creation from the vacuum; see [14].

We consider the following optimal boundary movement control problem: Let at time $t = 0$ the length $L$ of the string and an initial state be given. Let a terminal time $T$ and a positive real number $D$ that is strictly less than the speed of wave propagation be given. We admit movements of the right boundary point of the string that are given by a Lipschitz continuous function with a Lipschitz constant that is less than or equal to the number $D$. Moreover, we assume that at the terminal time $T$ the string has the same length as at the beginning. For the set of Lipschitz continuous functions on the time interval $[0, T]$, we use the notation

$$Lip = \{ \phi : [0, T] \to (0, \infty) \text{ such that } \phi \text{ is Lipschitz continuous} \}.$$ 

We consider the boundary movements in the admissible set

$$(1.1) \quad \Phi = \{ \phi \in Lip \text{ with Lipschitz constant } \leq D \text{ and } \phi(0) = L, \phi(T) = L \}.$$ 

Our problem is to find an admissible boundary movement for which the energy of the string at the terminal time $T$ is minimal. We present an explicit formula for optimal movements that solve this optimal control problem.

This paper has the following structure: In section 2, we define the problem of energy minimizing boundary movement control for a vibrating string with homogeneous Dirichlet boundary conditions at both ends. We assume that at the terminal time the string has the same length as at the beginning. As a control constraint, we prescribe an upper bound for the Lipschitz constants of the admissible boundary movements.

Section 3 contains our main results. In Theorem 3.2, we present an optimal boundary movement that depends on the initial state in a robust way. The energy decay that can be achieved depends on the initial state. For certain initial states, it is not possible to achieve an energy decay by boundary movement control. Theorem 3.1 contains sufficient conditions for the existence and uniqueness of optimal controls.

The proof of the main result uses the solution of the initial-boundary-value problem for a given boundary movement that is given in section 4. The construction is based upon the method of characteristics. It is necessary to make sure that the Lipschitz constants of the admissible boundary movements are strictly less than the speed of wave propagation, because, otherwise, the solution of the corresponding initial boundary value problem is, in general, not well-defined. The reason is that the information travels on the characteristic curves, and, if the length of the string increases too fast, it may happen that there exist points that are not reached by characteristic curves of both families.

In section 5, we introduce a set of functions that depend in a bijective way on the boundary movements (see Lemma 5.1). These functions are the unknowns in our
transformed optimization problem. To obtain the transformed problem, we write our objective function, that is, the energy of the string at time \( T \) in terms of the new unknowns. The solution of the transformed problem is given in Lemma 5.6.

2. The problem. Let the wave speed \( c > 0 \) be given. At the initial time \( t = 0 \), the string has the length \( L > 0 \). Define the control time \( T = 2L/c \). Let \( p \in [1, \infty) \) be given. Now we define the set \( B \) of admissible initial states. In the definition of the set \( B \) and in what follows, we use generalized derivatives. Let

\[
B = \{ (y_0, y_1) : y_0' \in L^p(0, L), \quad y_1 \in L^p(0, L), \quad y_0(0) = y_0(L) = 0 \}.
\]

Let a real number \( D \in (0, c) \) be given. Define the set \( \Phi \) of admissible boundary motions as in (1.1). Let \( (y_0, y_1) \in B \) be given. We consider the problem

\[
P: \quad \text{Find } \phi \in \Phi \text{ such that } W(T) = \int_0^L \left| v_x(x, T) + \frac{1}{c} v_t(x, T) \right|^p + \left| v_x(x, T) - \frac{1}{c} v_t(x, T) \right|^p \, dx
\]
is minimized, where \( v(x, t) \) is the solution of the initial boundary value problem

\[
\begin{align*}
&v(x, 0) = y_0(x), \quad v_t(x, 0) = y_1(x), \quad x \in (0, L), \quad (2.2) \\
&v(0, t) = 0, \quad v(\phi(t), t) = 0, \quad t \in (0, T), \quad (2.3) \\
&v_{tt}(x, t) = c^2 v_{xx}(x, t), \quad (x, t) \in \Omega = \{ (x, t) : t \in (0, T), \quad x \in (0, \phi(t)) \}. \quad (2.4)
\end{align*}
\]

For \( p = 2 \), we have

\[
W(T) = 2 \int_0^L \left( v_x(x, T)^2 + \frac{1}{c^2} v_t(x, T)^2 \right) \, dx;
\]
hence, \( W(T) \) is equivalent to the classical energy. In the general case, we further refer to \( W(T) \) as a generalized energy function. For any \( t \in (0, T) \), the definition of \( W(t) \) is given in (5.8), where the integration interval is \((0, \phi(t))\). If both boundary points are fixed (that is, \( \phi(t) \equiv L \)), the generalized energy defined in (5.8) is conserved; see Remark 5.1.


**Theorem 3.1** (existence and uniqueness of the solution of problem \( P \)). There exists a movement \( \phi \in \Phi \) that solves problem \( P \).

Let \( p \in (1, \infty) \) and \((y_0, y_1) \in B \) be given. Define the \( L^p \)-function \( A \) as

\[
A(x) = \begin{cases} 
  y_0'(-x) - (1/c)y_1(-x) & \text{if } x \in [-L, 0), \\
  y_0'(x) + (1/c)y_1(x) & \text{if } x \in [0, L]. 
\end{cases}
\]

Define the set \( M_x = \{ x \in [-L, L] : A(x) = 0 \} \). If the set \( M_x \) has measure zero, the solution of \( P \) is uniquely determined.

For a point \( z \) on the real axis define the projection on the interval \([c^{-D}/c+D, c+D]/c-D] \) in the usual way as

\[
\Pi_{[c^{-D}/c+D, c+D]/c-D]}(z) = \max\{(c-D)/(c+D), \min\{(c+D)/(c-D), z\} \}.
\]
Define $W(0)$ as follows:

$$W(0) = \int_0^L \left| y_0'(x) + \frac{1}{c} y_1(x) \right|^p + \left| y_0'(x) - \frac{1}{c} y_1(x) \right|^p \, dx.$$  

**THEOREM 3.2** (solution of problem $P$). For $p = 1$, for all $\phi \in \Phi$ we have $W(T) = W(0)$.

Let $p \in (1, \infty)$ and an initial state $(y_0, y_1) \in B$ be given. Define the $L^p$-function $A$ as in (3.1). If $\int_{-L}^L |A(y)| \, dy > 0$, there exists a real number $\lambda > 0$ such that the moment equation

$$(3.2) \quad \int_{-L}^L \Pi_{\frac{\lambda - D}{\lambda + D}} (\lambda |A(y)|) \, dy = 2L$$  

holds. With this choice of $\lambda$, define the function $h : [-L, L] \to [L, 3L]$ by

$$h(x) = L + \int_{-L}^x \Pi_{\frac{\lambda - D}{\lambda + D}} (\lambda |A(y)|) \, dy$$  

and the functions $H_1 : [-L, L] \to (0, \infty)$ and $H_2 : [-L, L] \to [0, 2L]$ by

$$H_1(x) = \frac{h(x) - x}{2}, \quad H_2(x) = \frac{h(x) + x}{2}.$$

Then an optimal control movement that is a solution of problem $P$ is given by the function $\phi \in \Phi$ defined by

$$(3.3) \quad \phi(t) = H_1(H_2^{-1}(ct)), \quad t \in (0, T),$$  

that yields the minimal value for

$$(3.4) \quad W(T) = \int_{-L}^L \frac{|A(s)|^p}{h'(s)^{p-1}} \, ds.$$  

If $\int_{-L}^L |A(y)| \, dy = 0$, we have $W(T) = 0$ for all $\phi \in \Phi$.

The proofs of Theorems 3.1 and 3.2 will be given in section 5.4. They are based upon an explicit representation of the solution of the initial-boundary-value problem (2.2), (2.3), (2.4) for a given boundary motion. This representation of the solution is given in section 4.2. It is used to transform the optimal control problem to a convex optimization problem in a function space that we can solve. The transformed set of admissible functions is defined in section 5.1 by positive pointwise bounds for the function values and a moment equation that prescribes the integral of the admissible functions. For the solution of the transformed problem, the convexity of the transformed objective function on the transformed set of admissible functions is essential.

**Remark 3.1.** The (non)movement $\phi(t) = L, \quad t \in [0, T]$ is admissible. For this movement, the energy is conserved for all $p$ (see Remark 5.1). Therefore the optimal control movement does not lead to energy increase.

**Remark 3.2.** Assume that there exists a real constant $r > 0$ such that $|A(x)| = r$ for all $x \in [-L, L]$. Then for the minimal value of $W(T)$ we have $W(T) = W(0)$. This follows from Theorem 3.2, since to satisfy the moment equation (3.2), the number $\lambda$ has to be chosen such that $\Pi_{\frac{\lambda - D}{\lambda + D}} (\lambda |A(y)|) = 1$. This yields $h(x) = x + 2L$, ...
OPTIMAL ENERGY CONTROL

\( H_1(x) = L, \phi(t) = L \); hence, in this case it is optimal not to move the boundary point, and thus the energy is conserved.

**Example 3.1.** Assume that \( c = 1 \) and \( D \in [1/2, 1) \). Then \((c + D)/(c - D) \geq 3\). Let \( \gamma > 0 \) be given. Assume that

\[
\gamma y_0(x) = \begin{cases} 
\frac{1}{3}x, & x \in [0, L/2], \\
(L/6) - (4/3)(x - (L/2)), & x \in (L/2, (5L/6)], \\
-(5L/18) + (5/3)(x - (5L/6)), & x \in (5L/6, L], 
\end{cases}
\]

\[
\gamma y_1(x) = \begin{cases} 
0, & x \in [0, L/2], \\
-5/3, & x \in (L/2, (5L/6]), \\
4/3, & x \in (5L/6, L]. 
\end{cases}
\]

Then we have

\[
\gamma A(x) = \begin{cases} 
\frac{1}{3}, & x \in [-L, L/2], \\
-3, & x \in (L/2, (5L/6)], \\
3, & x \in ((5L/6), L]. 
\end{cases}
\]

Equation (3.2) holds with \( \lambda = \gamma \). For the function \( h \) defined in Theorem 3.2 we have

\[
h(t) = \begin{cases} 
\frac{4}{3}L + \frac{1}{3}t, & t \in [-L, L/2], \\
3t, & t \in (L/2, L]. 
\end{cases}
\]

This yields the optimal movement

\[
\phi(t) = \frac{L}{2} + \frac{1}{2}|t - L|, \ t \in [0, 2L].
\]

For the energy we have

\[
W(0) = \frac{1}{\gamma^p} \left( \frac{1}{2} \left( 3^p + \frac{1}{3^{p-1}} \right) \right) L, \ W(T) = \frac{1}{\gamma^p} 2L.
\]

This yields the ratio

\[
\frac{W(T)}{W(0)} = \frac{4}{3^p + \frac{1}{3^{p-1}}};
\]

thus, for \( p = 2 \) we have \( W(T)/W(0) = 3/7 \). Hence the optimal movement absorbs more than half of the initial energy. Since the set \( M_z \) has measure zero, Theorem 3.1 implies that the solution \( \phi \) of \( P \) is uniquely determined.

**Example 3.2.** Let \( y_0(x) = |x - (L/2)| - (L/2), \ y_1(x) = 0, \ x \in [0, L] \). We have \(|A| = 1 \) almost everywhere; thus, the set \( M_z \) has measure zero. Theorem 3.1 implies the existence of a solution of \( P \). Since \( M_z \) has measure zero, Theorem 3.1 implies that the solution of \( P \) is unique. Theorem 3.2 implies that the unique optimal movement is the nonmovement \( \phi(t) = L, \ t \in [0, 2L] \) that corresponds to the function \( h(x) = 2L + x \). Remark 5.1 implies that \( W(T)/W(0) = 1 \), so by boundary movement an energy decrease cannot be achieved. Since the solution of \( P \) is unique, this implies that for every other admissible boundary movement an energy growth is produced at the terminal time.
Then \( M_z \) has measure zero; thus, Theorem 3.1 implies that \( P \) has a unique solution. For all \( x \in [-L, L] \): \( |A(x)| \in [(c - D)/(c + D), (c + D)/(c - D)] \), and (3.2) holds. We have \( h(x) = -x + \sqrt{c^2(1 - T)^2 + 4c(L + x)} - c(1 - T) \), and the optimal movement is given by \( \phi(x) = L + cx(T - x) \).

**Example 3.4.** Assume that \( L(c + D) \leq 2c \). Let \( \gamma = 2c/(L(c - D)) \). Then \( \gamma \geq (c + D)/(c - D) \). Let \( y_0(x) = 0 \) and

\[
y_0(x) = \begin{cases} 
\gamma x, & x \in [0, \frac{1}{2\gamma}], \\
1 - \gamma x, & x \in (\frac{1}{2\gamma}, 1], \\
0, & x > \frac{1}{\gamma},
\end{cases}
\]

With \( \lambda = 1 \), (3.2) holds. Theorem 3.2 implies that an optimal movement is given by the function \( \phi \) corresponding to

\[
h'(t) = \begin{cases} 
\frac{c + D}{c - D}, & t \in \left[-\frac{1}{\gamma}, \frac{1}{\gamma}\right], \\
\frac{c - D}{c + D}, & t \notin \left[-\frac{1}{\gamma}, \frac{1}{\gamma}\right].
\end{cases}
\]

For the energy, this yields

\[
\frac{W(T)}{W(0)} = \left(\frac{c - D}{c + D}\right)^{p-1}.
\]

For the corresponding optimal movement we have \( \phi'(H_2(x)/c) = cH_1(x)/H_2'(x) \), and thus \( \phi'(H_2(x)/c) = D \), if \( x \in [-1/\gamma, 1/\gamma] \) and \( \phi'(H_2(x)/c) = -D \) otherwise. Hence

\[
\phi'(t) = \begin{cases} 
D, & t \in \left[H_2(1/\gamma), \frac{H_2(1/\gamma)}{c}\right] = \left[\frac{L}{2c}, \frac{L}{c} \left(\frac{3}{2} + \frac{D}{c}\right)\right], \\
-D, & \text{otherwise}.
\end{cases}
\]

**Example 3.5.** Assume that \( \varepsilon \in (0, 2D/(c + D)) \) and \( |A(x)| = 1 + \varepsilon \sin(x) \). Then \( M_z \) has measure zero; thus, \( P \) has a unique solution. Equation (3.2) holds with \( \lambda = 1 \), \( H_1(x) = L + \varepsilon(\cos(L) - \cos(x))/2 \), \( H_2(x) = H_1(x) + x \). For the graph of \( \phi \), by (3.3) we obtain the curve \( G = \{(t, \phi(t)) : t \in [0, T]\} = \{(H_2(x)/c, H_1(x)) : x \in [-L, L]\} \).

**4. Transformation of the problem.** For a given boundary movement \( \phi \) from the set \( \Phi \) defined in (1.1), we want to find a representation of the solution of the initial-boundary-value problem (2.2), (2.3), (2.4) with one moving boundary point in terms of traveling waves; that is, we want to derive d’Alembert’s solution for our problem. In particular, we have to show that such a solution exists.

**4.1. Wave propagation auxiliary functions.** In this section we define some auxiliary functions that we need to derive d’Alembert’s solution for our problem. Let \( \phi \in \Phi \) be given. Since \( \phi \) is Lipschitz, \( \phi \) is absolutely continuous. For \( t \in [0, T] \), define

\[
\psi_1(t) = \phi(t) - ct, \quad \psi_2(t) = \phi(t) + ct.
\]

Then \( \psi_1(t) = \phi'(t) - c \). The definition of the set \( \Phi \) implies the inequality

\ [-D \leq \phi'(t) \leq D]
and thus \( \psi_1'(t) \leq D - c < 0 \); hence, \( \psi_1 \) is strictly decreasing on \([0, T]\) and thus invertible. We have \( \psi_1(0) = L \) and \( \psi_1(T) = -L \); therefore, \( \psi_1^{-1}(s) \) is defined for \( s \in [-L, L] \).

We have \( \psi_2'(t) = \phi'(t) + c \geq -D + c > 0 \); hence, \( \psi_2 \) is strictly increasing on \([0, T]\) and thus invertible. We have \( \psi_2(0) = L \) and \( \psi_2(T) = 3L \); therefore, \( \psi_2^{-1}(s) \) is defined for \( s \in [L, 3L] \).

Since \( \phi(t) \geq 0 \), for all \( t \in [0, T] \) we have

\[
\text{(4.2)} \quad -\psi_1(t) = ct - \phi(t) < ct + \phi(t) = \psi_2(t).
\]

For \( x \in [-L, L] \) define

\[
\text{(4.3)} \quad h(x) = \psi_2(\psi_1^{-1}(-x)).
\]

Then \( h \) is strictly increasing and

\[
\text{(4.4)} \quad h'(x) = -\frac{\psi_2'(\psi_1^{-1}(-x))}{\psi_1'(\psi_1^{-1}(-x))} > 0.
\]

On account of (4.2) for all \( x \in [-L, L] \) we have

\[
x = -\psi_1(\psi_1^{-1}(-x)) < \psi_2(\psi_1^{-1}(-x)) = h(x).
\]

For the inverse of \( h \) we have

\[
\text{(4.5)} \quad h^{-1}(x) = -\psi_1(\psi_2^{-1}(x)).
\]

We have

\[
\text{(4.6)} \quad h^{-1}(L) = -L, \quad h^{-1}(3L) = L.
\]

Note that

\[
\text{(4.7)} \quad L = \psi_2(0) < h(0) = \psi_2(\psi_1^{-1}(0)),
\]

since

\[
0 < \psi_1^{-1}(0),
\]

which is true on account of

\[
L = \psi_1(0) > 0.
\]

Let \( t_1 = \psi_1^{-1}(0) \). Then \( \phi(t_1) - ct_1 = 0 \), and thus \( \phi(t_1) = ct_1 \). Therefore

\[
\psi_2(t_1) = \phi(t_1) + ct_1 = 2ct_1.
\]

Hence

\[
h(0) = \psi_2(\psi_1^{-1}(0)) = \psi_2(t_1) = 2ct_1 = 2c\psi_1^{-1}(0).
\]

In fact, \( t_1 \) is the time that a characteristic curve starting at time zero at the left end point of the string needs to reach the moving end of the string.

Later in Lemma 5.1 we will show that there is a bijection between the maps \( h \) as defined in (4.3) and the corresponding maps \( \phi \), which allows one to transform our optimal control problem to an optimization problem in terms of the function \( h \).
4.2. Solution of the initial-boundary-value problem. In this section, we give a representation of the solution of the initial-boundary-value problem for a given fixed boundary movement \( \phi \in \Phi \).

**Theorem 4.1.** Let \( \phi \in \Phi \) and \( (y_0, y_1) \in B \) be given. With \( h^{-1} \) as in (4.5), define the functions

\[
\alpha(x) = \begin{cases} 
- y_0(-x) + (1/c) \int_0^{-x} y_1(s) \, ds, & x \in (-L, 0), \\
y_0(x) + (1/c) \int_0^x y_1(s) \, ds, & x \in [0, L), \\
y_0(-h^{-1}(x)) + (1/c) \int_0^{-h^{-1}(x)} y_1(s) \, ds, & x \in [L, h(0)), \\
y_0(h^{-1}(x)) + (1/c) \int_0^{h^{-1}(x)} y_1(s) \, ds, & x \in (h(0), 3L) 
\end{cases}
\]

and

\[
v(x, t) = [\alpha(x + ct) - \alpha(-x + ct)]/2, \quad (x, t) \in \Omega.
\]

We have \( \alpha' \in L^p(-L, 3L) \). The function \( v \) is continuous on \( \Omega \) and \( v_1, v_x \in L^1(\Omega) \).

Define the family of test functions \( T \) as

\[
T = \{ \phi \in C^2(\Omega) : \text{There exists a set } Q = [x_1, x_2] \times [t_1, t_2] \subset \Omega \text{ such that the support of } \phi \text{ is contained in the interior of } Q \}.
\]

The function \( v \) satisfies the wave equation (2.4) in the following weak sense:

\[
\int_{\Omega} v_t(x, t) \phi_t(x, t) \, d(x, t) = c^2 \int_{\Omega} v_x(x, t) \phi_x(x, t) \, d(x, t) \text{ for all } \phi \in T.
\]

The function \( v \) satisfies (2.2) and (2.3). In this sense, \( v \) is the solution of the initial-boundary-value problem (2.2), (2.3), (2.4).

**Proof.** Since \( y_0' \in L^p(0, L) \), the Sobolev imbedding theorem implies that \( y_0 \) is continuous. Moreover, \( y_1 \) is in \( L^p(0, L) \), and thus \( \alpha \) is well-defined. Now we discuss the regularity of \( \alpha \). On the intervals \((-L, 0), (0, L), [L, h(0)), \) and \([h(0), 3L)\) the function \( \alpha \) is continuous. Due to the definition of the set \( B \) we have

\[
\lim_{x \to 0^-} \alpha(x) = - y_0(0) = 0 = y_0(0) = \lim_{x \to 0^+} \alpha(x),
\]

\[
\lim_{x \to L^-} \alpha(x) = y_0(L) + \frac{1}{c} \int_0^L y_1(s) \, ds = y_0(L) + \frac{1}{c} \int_0^L y_1(s) \, ds = \lim_{x \to L^+} \alpha(x),
\]

\[
\lim_{x \to h(0)^-} \alpha(x) = - y_0(0) = y_0(0) = \lim_{x \to h(0)^+} \alpha(x),
\]

and hence \( \alpha \) is continuous on the interval \((-L, 3L)\). The derivative \( \alpha' \) in the sense of distributions exists on the intervals \((-L, 0), (0, L), (L, h(0)), \) and \([h(0), 3L)\) as a \( L^p \)-function. Since \( \alpha \) is continuous, this implies that \( \alpha \) is absolutely continuous on \((-L, 3L)\). Hence \( \alpha' \in L^1(-L, 3L) \), and the \( L^p \) regularity on the subintervals \((-L, 0), (0, L), (L, h(0)), \) and \([h(0), 3L)\) implies that \( \alpha' \in L^p(-L, 3L) \). The continuity of \( v \) follows from the continuity of \( \alpha \). For \( t = 0 \) and \( x \in (0, L) \) we have

\[
v(x, 0) = [\alpha(x) - \alpha(-x)]/2 = y_0(x).
\]

For \( (x, t) \in \Omega \) almost everywhere, we have

\[
v_t(x, t) = c[\alpha'(x + ct) - \alpha'(-x + ct)]/2.
\]
Thus the definition of \( \alpha \) implies the equation \( v_t(x,0) = y_1(x) \). Hence the initial conditions (2.2) are valid.

For \( (x,t) \in \Omega \) almost everywhere, we have

\[(4.12) \quad v_x(x,t) = [\alpha'(x + ct) + \alpha'(-x + ct)]/2.\]

By Tonelli’s theorem (see, e.g., [15]), (4.12) implies \( v_x \in L^1(\Omega) \), and (4.11) implies \( v_t \in L^1(\Omega) \).

For all \( \varphi \in T \), integration by parts, (4.12) and (4.11) yield

\[
\int_{\Omega} v_x(x,t) \varphi_x(x,t) \, d(x,t) = \int_{t_1}^{t_2} \int_{x_1}^{x_2} \varphi_x(x,t) [\alpha(x + ct) + \alpha(-x + ct)]/(2c) \, dt \, dx
\]

\[
= -\int_{x_1}^{x_2} \int_{t_1}^{t_2} \varphi_{tx}(x,t) [\alpha'(x + ct) + \alpha'(x - ct)]/(2c) \, dt \, dx
\]

\[
= -\int_{t_1}^{t_2} \int_{x_1}^{x_2} \varphi_t(x,t) [\alpha(x + ct) + \alpha(-x + ct)]/(2c) \, dx \, dt
\]

\[
= \int_{t_1}^{t_2} \int_{x_1}^{x_2} \varphi_t(x,t) [\alpha(x + ct) - \alpha(-x + ct)]/(2c) \, dx \, dt
\]

\[
= \int_{\Omega} \varphi_t(x,t) v_t(x,t)/c^2 \, d(x,t),
\]

and hence (4.10) holds.

For \( x = 0 \) we have \( v(0,t) = [\alpha(ct) - \alpha(-ct)]/2 = 0 \); hence, at \( x = 0 \) the boundary condition \( v(0,t) = 0 \) holds for all \( t \in (0,T) \).

We have \( v(\phi(t),t) = [\alpha(\phi(t) + ct) - \alpha(-\phi(t) + ct)]/2 = [\alpha(\psi_2(t)) - \alpha(-\psi_1(t))]/2 \).

The definition of the set \( \Phi \) implies that, on the interval \([0,T] \), \( h \) is strictly increasing and invertible (see section 4.1). By (4.6), for all \( s \in [L,3L] = [\psi_2(0),\psi_2(T)] \) we have \( h^{-1}(s) \in [-L,L] \). Hence the definition of \( \alpha \) implies the equation

\[(4.13) \quad \alpha(h^{-1}(s)) = \begin{cases} -y_0(-h^{-1}(s)) + (1/c) \int_0^{-h^{-1}(s)} y_1(t) \, dt, & h^{-1}(s) \in (-L,0), \\ y_0(h^{-1}(s)) + (1/c) \int_0^{h^{-1}(s)} y_1(t) \, dt, & h^{-1}(s) \in [0,L]. \end{cases}\]

On the other hand, for \( s \in (L,h(0)) \) the definition of \( \alpha \) implies that \( \alpha(s) = -y_0(-h^{-1}(s)) + (1/c) \int_0^{-h^{-1}(s)} y_1(t) \, dt \), and we have \( h^{-1}(s) \in (h^{-1}(L),0) = (-L,0) \).

Thus (4.13) implies that \( \alpha(h^{-1}(s)) = \alpha(s) \) for \( s \in (L,h(0)) \).

For \( s \in (h(0),3L) \), the definition of \( \alpha \) yields \( \alpha(s) = y_0(h^{-1}(s)) + (1/c) \int_0^{h^{-1}(s)} y_1(t) \, dt \), and we have \( h^{-1}(s) \in (0,h^{-1}(3L)) = (0,L) \). Thus also in this case (4.13) implies that \( \alpha(h^{-1}(s)) = \alpha(s) \).

Hence for all \( s \in (L,3L) \) the following equation holds:

\[(4.14) \quad \alpha(h^{-1}(s)) = \alpha(s).\]

Thus for all \( t \in (0,T) \) we have

\[\alpha(\psi_2(t)) = \alpha(h^{-1}(\psi_2(t))) = \alpha(-\psi_1(\psi_2^{-1}(\psi_2(t)))) = \alpha(-\psi_1(t)).\]

Therefore, the boundary condition \( v(\phi(t),t) = 0 \) holds for all \( t \in (0,T) \). \( \square \)
The following lemma contains regularity conditions for the initial state and the boundary movement \( \phi \) that guarantee the existence of a solution of the initial-boundary-value problem (2.2), (2.3), (2.4) that satisfies the wave equation pointwise almost everywhere.

**Lemma 4.2.** If \( y_0 \in C^2[0, L], \ y_1 \in C^1[0, L], \ y_0(0) = 0, \ y_0(L) = y'_0(L) = 0, \ y_1(0) = 0 = y_1(L), \) and \( \phi \in \Phi \cap C^2[0, T], \) then \( \alpha \) defined by (4.8) is in \( C^2(-L, 3L), \) \( \alpha' \) is absolutely continuous, \( \alpha'' \in L^\infty(-L, 3L), \) and \( v \) defined by (4.9) satisfies the wave equation (2.4) in the following weak sense:

\[
\int_\Omega v_{tt}(x,t)\varphi(x,t)\,d(x,t) = c^2 \int_\Omega v_{xx}(x,t)\varphi(x,t)\,d(x,t) \text{ for all } \varphi \in \mathcal{T}.
\]

Moreover, \( v \) satisfies (2.4) pointwise almost everywhere in \( \Omega. \)

**Proof.** Since \( \phi \in C^2[0, T], \) definition (4.1) implies that \( \psi_1 \) and \( \psi_2 \) are in \( C^2[0, T]. \) Moreover, \( \psi_1^{-1} \) and \( \psi_2^{-1} \) are in \( C^2[0, T]. \) Hence the definition (4.3) of \( h \) implies that \( h \) is in \( C^2[-L, L], \) and (4.5) implies that \( h^{-1} \) is also two times continuously differentiable. Due to the conditions \( y_0(0) = 0, \ y_0(L) = y'_0(L) = 0, \ y_1(0) = 0 = y_1(L) \) in the end point of these intervals, we have the one-sided derivatives

\[
\begin{align*}
\alpha'_-(0) &= y'_0(0) - (1/c)y_1(0) = y'_0(0) + (1/c)y_1(0) = \alpha'_+(0), \\
\alpha'_-(L) &= y'_0(L) + (1/c)y_1(L) = 0 = y'_0(L) (h^{-1})'(L) - (1/c)y_1(L) (h^{-1})'(L) \\
&= y'_0(-h^{-1}(L)) (h^{-1})'(L) - (1/c)y_1(-h^{-1}(L)) (h^{-1})'(L) \\
&= \alpha'_-(L), \\
\alpha'_-(h(0)) &= y'_0(0) (h^{-1})'(h(0)) - (1/c)y_1(0) (h^{-1})'(h(0)) = y'_0(0) (h^{-1})'(h(0)) \\
&= y'_0(0) (h^{-1})'(h(0)) + (1/c)y_1(0) (h^{-1})'(h(0)) = \alpha'_-(h(0)).
\end{align*}
\]

Since the one-sided derivatives are equal, \( \alpha' \) is continuous, and hence \( \alpha \in C^1(-L, 3L). \) Since \( \alpha'' \) exists on the intervals \((-L, 0), \ (0, L), \ (L, h(0)), \) and \((h(0), 3L)\) as a bounded continuous function, this implies that \( \alpha' \) is absolutely continuous on \((-L, 3L)\) and \( \alpha'' \in L^\infty(-L, 3L). \) For \((x, t)\) in \( \Omega \) almost everywhere we have

\[
\begin{align*}
v_{tt}(x,t) &= c^2[\alpha''(x+ct) - \alpha''(-x+ct)]/2, \\
v_{xx}(x,t) &= [\alpha''(x+ct) - \alpha''(-x+ct)]/2,
\end{align*}
\]

and hence (2.4) holds almost everywhere in \( \Omega. \) For all \( \varphi \in \mathcal{T}, \) integration by parts yields

\[
\begin{align*}
\int_\Omega \varphi_t(x,t) v_t(x,t)/c^2 \,d(x,t) &= - \int_\Omega \varphi(x,t) v_{tt}(x,t)/c^2 \,d(x,t) \\
\int_\Omega \varphi_x(x,t) v_x(x,t) \,d(x,t) &= - \int_\Omega \varphi(x,t) v_{xx}(x,t) \,d(x,t).
\end{align*}
\]

Hence (4.10) implies that (4.15) holds. \( \square \)

**Remark 4.1.** For \( x \) in the interval \((0, L)\) we have

\[
\alpha'(x)^2 + \alpha'(-x)^2 = \frac{[\alpha'(x) + \alpha'(-x)]^2}{2} + \frac{[\alpha'(x) - \alpha'(-x)]^2}{2} = 2 \ y_0(x)^2 + \frac{2}{c^2} y_1(x)^2.
\]
5. Solution of the optimal shape control problem.

5.1. The transformed set of admissible controls. Definition (4.3) states how \( h \) can be obtained from a given movement \( \phi \). The following lemma shows that, if \( h \) is known, the corresponding movement \( \phi \) is uniquely determined and can be computed.

Lemma 5.1. Let \( \phi \in \Phi \) be given. Define the function \( h \) by (4.3). For \( x \in [-L, L] \), define

\[
H_1(x) = \frac{h(x) - x}{2}, \quad H_2(x) = \frac{h(x) + x}{2}.
\]

Then for all \( t \in [0, T] \), we have

\[
\phi(t) = H_1(H_2^{-1}(ct)).
\]

Proof. In section 4.1, we have seen that \( h \) is strictly increasing, and hence \( H_2 \) is strictly increasing. Since \( h(-L) = L \), we have \( H_2(-L) = 0 \). Since \( h(L) = 3L \), \( H_2(L) = 2L \). Thus the assertion (5.2) is equivalent to the statement that for all \( x \in [-L, L] \) we have

\[
\phi \left( \frac{H_2(x)}{c} \right) = H_1(x).
\]

From (4.1), we have for all \( t \in [0, T] \) the equation \( \psi_1(t) + 2ct = \psi_2(t) \). For \( t = \psi_1^{-1}(x) \) with \( x \in [-L, L] \), this yields

\[
x + 2c\psi_1^{-1}(x) = \psi_2(\psi_1^{-1}(x)),
\]

and by (4.3) this implies

\[
h(x) = \psi_2(\psi_1^{-1}(-x)) = 2c\psi_1^{-1}(-x) - x.
\]

Therefore, the following equation holds: \( H_2(x) = [h(x) + x]/2 = c\psi_1^{-1}(-x) \). This implies the equation

\[
\psi_1 \left( \frac{H_2(x)}{c} \right) = \psi_1(\psi_1^{-1}(-x)) = -x = \phi \left( \frac{H_2(x)}{c} \right) - H_2(x),
\]

where we have again used the definition of \( \psi_1 \). Hence

\[
\phi \left( \frac{H_2(x)}{c} \right) = H_2(x) - x = H_1(x),
\]

where the last equation follows from definition (5.1). \( \qed \)

Define the set \( H \) of functions defined on the interval \([-L, L]\) as follows:

\[
H = \{ h : h(x) = \psi_2(\psi_1^{-1}(-x)), x \in [-L, L], \text{ with } \psi_1, \psi_2 \text{ as in (4.1) for some } \phi \in \Phi \}.
\]

Lemma 5.2. Define the map \( \theta : \Phi \to H \) by \( \theta(\phi) = h \), where \( h(x) = \psi_2(\psi_1^{-1}(-x)) \), \( x \in [-L, L] \) with \( \psi_1, \psi_2 \) as in (4.1). Then \( \theta \) is bijective.

Proof. First we show that \( \theta \) is injective. Let \( \phi_1, \phi_2 \in \Phi \) be given such that \( h_1 = \theta(\phi_1) = \theta(\phi_2) = h_2 \). Lemma 5.1 implies that, with

\[
H_i(x) = \frac{h_i(x) - x}{2}, \quad H'_i(x) = \frac{h_i(x) + x}{2}, \quad i \in \{1, 2\},
\]

A. Hence

\[
\phi_1(t) = H_1(H_2^{-1}(ct)) = \phi_2(t),
\]

then \( \phi_1 \) and \( \phi_2 \) have the same value at every point. Therefore, \( \phi_1 = \phi_2 \).

Define the map \( \theta : \Phi \to H \) by \( \theta(\phi) = h \), where \( h(x) = \psi_2(\psi_1^{-1}(-x)) \), \( x \in [-L, L] \) with \( \psi_1, \psi_2 \) as in (4.1). Then \( \theta \) is bijective.

Proof. First we show that \( \theta \) is injective. Let \( \phi_1, \phi_2 \in \Phi \) be given such that \( h_1 = \theta(\phi_1) = \theta(\phi_2) = h_2 \). Lemma 5.1 implies that, with

\[
H_i(x) = \frac{h_i(x) - x}{2}, \quad H'_i(x) = \frac{h_i(x) + x}{2}, \quad i \in \{1, 2\},
\]

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we have
\[ \phi_i(t) = H_i^1((H_2^1)^{-1}(ct)), \quad i \in \{1, 2\}. \]

Since \( h_1 = h_2 \), we have \( H_1^1 = H_2^1 \) and \( H_2^1 = H_2^2 \), and thus \( \phi_1(t) = \phi_2(t) \). Therefore \( \theta \) is injective. The definition of the set \( H \) implies that \( H = \theta(\Phi) \), and thus \( \theta \) is surjective. \( \square \)

Later we will need the following result in a proof.

**Lemma 5.3.** Let \( I = \left[ \frac{-D}{c+D}, \frac{c+D}{d} \right] \). The following equation holds:

\[
c \max_{z \in I} \left\{ \frac{z-1}{z+1}, \frac{1-z}{1+z} \right\} = D.
\]

**Proof.** For \( z \in I \), let \( \eta_1(z) = (z - 1)/(z + 1) \). Since \( \eta_1'(z) > 0 \), for all \( z \in I \) we have \( \eta_1(z) \leq \eta_1\left(\frac{c-D}{c+D}\right) = D/c \).

For \( z \in I \), let \( \eta_2(z) = (1 - z)/(1 + z) \). Since \( \eta_2'(z) < 0 \), for all \( z \in I \) we have \( \eta_2(z) \leq \eta_2\left(\frac{c-D}{c+D}\right) = D/c \). \( \square \)

**Lemma 5.4.** Define the set

\[
M = \left\{ h| h : [-L, L] \to [L, 3L] \text{ such that } h(x) = L + \int_{-L}^{x} f(s) \, ds \text{ with } f \in F \right\},
\]

where the set \( F \) is defined as

\[
F = \left\{ f| f : [-L, L] \to \left[ \frac{c-D}{c+D}, \frac{c+D}{c-D} \right] \text{ such that } f \text{ is Lebesgue integrable and } \int_{-L}^{L} f(x) \, dx = 2L \right\}.
\]

Then \( M = H \).

**Proof.** First we show that \( H \subseteq M \). Let \( h \in H = \theta(\Phi) \), \( h(x) = \psi_2(\psi_1^{-1}(-x)) \), \( x \in [-L, L] \). Since \( \psi_2 \) is Lipschitz continuous with Lipschitz constant \( c + D \), for all \( x_1, x_2 \in [-L, L] \) we have

\[
|h(x_1) - h(x_2)| = |\psi_2(\psi_1^{-1}(-x_1)) - \psi_2(\psi_1^{-1}(-x_2))| \leq (c + D)|\psi_1^{-1}(-x_1) - \psi_1^{-1}(-x_2)|.
\]

For \( \psi_1 \) and \( t_1, t_2 \in [0, T] \) we have the inequality

\[
|\psi_1(t_1) - \psi_1(t_2)| = |\phi(t_1) - \phi(t_2) - c(t_1 - t_2)| \geq c|t_1 - t_2| - |\phi(t_1) - \phi(t_2)| \geq (c-D)|t_1 - t_2|.
\]

With \( t_1 = \psi_1^{-1}(-x_1), t_2 = \psi_1^{-1}(-x_2) \), this yields

\[
|\psi_1^{-1}(-x_1) - \psi_1^{-1}(-x_2)| \leq \frac{1}{c-D}|x_1 - x_2|.
\]

This implies the inequality

\[
|h(x_1) - h(x_2)| \leq \frac{c+D}{c-D} |x_1 - x_2|,
\]

and thus \( h \) is Lipschitz continuous with Lipschitz constant \( \frac{c+D}{c-D} \). Since \( h \) is Lipschitz, \( h \) is absolutely continuous, and, for the derivative, we have the inequality \( |h'(x)| \leq (c+D)/(c-D) \) almost everywhere.
For all $t_1, t_2 \in [0, T]$, we have
\[
|\psi_2(t_1) - \psi_2(t_2)| \geq (c - D)|t_1 - t_2|, \quad |\psi_1(t_1) - \psi_1(t_2)| \leq (c + D)|t_1 - t_2|.
\]
With $t_1 = \psi_1^{-1}(-x_1)$, $t_2 = \psi_1^{-1}(-x_2)$, this yields
\[
|x_1 - x_2| \leq (c + D)|\psi_1^{-1}(-x_1) - \psi_1^{-1}(-x_2)|.
\]
Hence
\[
|h(x_1) - h(x_2)| \geq (c - D)|\psi_1^{-1}(-x_1) - \psi_1^{-1}(-x_2)| \geq \frac{c - D}{c + D}|x_1 - x_2|.
\]
This implies that $|h'(x)| \geq (c - D)/(c + D)$ almost everywhere. In section 4.1, we have shown that $h$ is strictly increasing. Hence we have the inequality
\[
(5.5) \quad \frac{c - D}{c + D} \leq h'(x) \leq \frac{c + D}{c - D}.
\]
We can write
\[
h(x) = h(-L) + \int_{-L}^x h'(s) \, ds = L + \int_{-L}^x h'(s) \, ds.
\]
Since $h(L) = L + \int_{-L}^L h'(s) \, ds = 3L$, we have $\int_{-L}^L h'(s) \, ds = 2L$. So we have shown $h' \in F$, which implies $h \in M$, and thus $H \subset M$.

Now we show that $M \subset H$. Let $m \in M$ be given, $m(x) = L + \int_{-L}^x f(s) \, ds$, with $f \in F$ and $x \in [-L, L]$. Then $m$ is strictly increasing. For $x \in [-L, L]$, define $H_1(x) = [m(x) - x]/2$, $H_2(x) = [m(x) + x]/2$. Then $H_2$ is strictly increasing, and $\{H_2(x) : x \in [-L, L]\} = [H_2(-L), H_2(L)] = [m(-L) - L, m(L) + L] = [0, 2L]$. For $t \in [0, T]$, we have $ct \in [0, 2L] = H_2[-L, L]$. Hence $H_2^{-1}(ct)$ is well-defined and in $[-L, L]$. So we can define
\[
(5.6) \quad \phi(t) = H_1(H_2^{-1}(ct)).
\]
Since $H_2(-L) = 0$ and $H_2(L) = 2L$ we have
\[
\phi(0) = H_1(H_2^{-1}(0)) = H_1(-L) = L,
\]
\[
\phi(T) = H_1(H_2^{-1}(2L)) = H_1(L) = L.
\]
Our assumptions imply that $H_1$ is Lipschitz continuous with Lipschitz constant $c/(c - D)$ and $H_2^{-1}$ is Lipschitz continuous with Lipschitz constant $(c + D)/c$. Hence $\phi$ is Lipschitz continuous and thus absolutely continuous. Since $\phi(H_2(x)/c) = H_1(x)$, the chain rule implies that for the derivative we have the equation $\phi'(H_2(x)/c) = cH_1'(x)/H_2'(x)$, which implies the inequality
\[
\left|\phi' \left( \frac{H_2(x)}{c} \right) \right| = c \frac{|H_1'(x)|}{H_2'(x)} = c \frac{|f(x) - 1|}{f(x) + 1} \leq c \max \left\{ \frac{f(x) - 1}{f(x) + 1}, \frac{1 - f(x)}{1 + f(x)} \right\} \leq D,
\]
where the last inequality follows from Lemma 5.3 since $f \in F$. Hence $\phi \in \Phi$. Therefore we can compute $\theta(\phi)$.

For $x \in [-L, L]$, we have $H_2(x)/c \in [0, 2L/c] = [0, T]$, and the definition of $\phi$ implies the equation $\phi(H_2(x)/c) = H_1(x) = [m(x) - x]/2$, and hence
\[
\psi_2 \left( \frac{H_2(x)}{c} \right) = \phi \left( \frac{H_2(x)}{c} \right) + H_2(x) = \frac{m(x) - x}{2} + \frac{m(x) + x}{2} = m(x).
\]
Moreover, we have
\[ -\psi_1 \left( \frac{H_2(x)}{c} \right) = H_2(x) - \phi \left( \frac{H_2(x)}{c} \right) = H_2(x) - H_1(x) = \frac{m(x) + x}{2} - \frac{m(x) - x}{2} = x. \]

This implies the equation \( \psi_1^{-1}(-x) = H_2(x)/c \). Hence \( \theta(\phi)(x) = \psi_2(\psi_1^{-1}(-x)) = \psi_2(H_2(x)/c) = m(x) \). So we have shown that \( \theta(\phi) = m \). Hence \( m \in \theta(\Phi) = H \). Since \( m \in M \) was arbitrary, this yields \( M \subset H \). Since we have already shown the inclusion \( H \subset M \), this yields the equation \( H = M \). Moreover, this implies the equation
\[
(5.7) \quad \phi = \theta^{-1}(m). \quad \square
\]

We see that the mapping \( \theta \) defined in Lemma 5.2 is a bijection between the admissible motions \( \phi \in \Phi \) and the functions \( h \in M \). Moreover, for all \( m \in M \), (5.7) implies that \( \phi = \theta^{-1}(m) \) is given by (5.6).

**5.2. The objective function: Computation of the energy.** Let \( t \in [0, T] \) be given. Define the integrals
\[
I_1(t) = \int_0^{\psi_1(t)} |v_x(x, t) + \frac{1}{c} v_t(x, t)|^p \, dx, \quad I_2(t) = \int_0^{\psi_1(t)} |v_x(x, t) - \frac{1}{c} v_t(x, t)|^p \, dx
\]
and the generalized energy by
\[
(5.8) \quad W(t) = I_1(t) + I_2(t).
\]

Equation (4.9) implies that
\[
I_1(t) = \int_{ct}^{\psi_2(t)} |\alpha'(x)|^p \, dx, \quad I_2(t) = \int_{-\psi_1(t)}^{ct} |\alpha'(x)|^p \, dx.
\]

Thus for all \( t \in [0, T] \), we have
\[
W(t) = \int_{-\psi_1(t)}^{ct} |\alpha'(x)|^p \, dx + \int_{ct}^{\psi_2(t)} |\alpha'(x)|^p \, dx = \int_{-\psi_1(t)}^{\psi_2(t)} |\alpha'(x)|^p \, dx.
\]

For our terminal time \( T \) this implies that
\[
W(T) = \int_L^{3L} |\alpha'(x)|^p \, dx = \int_{h^{-1}(3L)}^{h^{-1}(L)} |\alpha'(h(s))|^p h'(s) \, ds = \int_{-L}^{L} |\alpha'(h(s))|^p h'(s) \, ds.
\]

By (4.14), for all \( u \in [L, 3L] \) we have \( h^{-1}(u) \in [-L, L] \) and \( h^{-1}(u) = \alpha(u) \). Thus for all \( s \in [-L, L] \) we have \( \alpha(s) = \alpha(h(s)) \) and thus
\[
\alpha'(s) = \alpha'(h(s)) \, h'(s).
\]

This yields the equation
\[
(5.9) \quad W(T) = \int_{-L}^{L} \frac{\alpha'(x)}{h'(x)^{p-1}} \, dx.
\]

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We see that $W$ as a function of $t$ on the interval $[0, T]$ is absolutely continuous and the derivative is given by the $L^1$ function

$$W'(t) = |\alpha'(\psi_2(t))|^p \psi'_2(t) + |\alpha'(-\psi_1(t))|^p \psi'_1(t)$$

$$= |\alpha'(\psi_2(t))|^p \psi'_2(t) + |\alpha'(h(-\psi_1(t)))|^p |h'(-\psi_1(t))|^p \psi'_1(t)$$

$$= |\alpha'(\psi_2(t))|^p \psi'_2(t) + |\alpha'(\psi_2(t))|^p \left[\frac{\psi''_2(t)}{\psi'_2(t)}\right]^p \psi'_1(t)$$

$$= |\alpha'(\psi_2(t))|^p \psi'_2(t) \left|\psi'_1(t)\right|^{p-1} - |\psi'_2(t)|^{p-1}.$$ 

Here we have used (4.4) to evaluate $h'(-\psi_1(t))$. Since $|\phi'(t)| < c$, this implies that the sign of $W'(t)$ is equal to the sign of

$$|\psi'_1(t)|^{p-1} - |\psi'_2(t)|^{p-1} = (c - \phi'(t))^{p-1} - (c + \phi'(t))^{p-1}.$$ 

If, for almost all $t \in (0, t_1)$, $\phi'(t) > 0$, this implies that $W''(t) < 0$ on $(0, t_1)$, and thus $W(0) > W(t_1)$. This means that an expansion causes a decrease in energy.

On the other hand, if, for almost all $t \in (0, t_1)$, $\phi'(t) < 0$, we have $W''(t) > 0$ on $(0, t_1)$, and therefore $W(0) < W(t_1)$. Thus a contraction causes an increase in energy.

This means that the results given in Theorem 2.1 in [2] for the case $p = 2$ and dimension unequal to two are also valid for $p \neq 2$, $p \in (1, \infty)$ in the 1-d case.

**Remark 5.1** (conservation of the energy for $\phi(t) \equiv L$). Let $\phi(t) \equiv L$. Then for all $t \in (0, T)$, $\phi'(t) = 0$; hence, $W'(t) = 0$ and therefore $W(t) = W(0)$; that is, in the case of two fixed boundary points the generalized energy $W(t)$ is conserved.

**Remark 5.2** (conservation of the energy for $p = 1$). For $p = 1$ we have for all $\phi \in \Phi$ and for $t \in (0, T)$

$$W'(t) = |\alpha'(\psi_2(t))| \psi'_2(t) \left|\psi'_1(t)\right|^{1-1} - |\psi'_2(t)|^{1-1} = 0,$$

and thus the integral

$$W(t) = \int_{-L}^{L} |\alpha'(x)| \, dx = W(T)$$

is conserved for the limit case $p = 1$ regardless of $\phi$. In other words, for $p = 1$ the control problem $P$ is meaningless.

**Remark 5.3** (sharp lower bounds for the energy). Let $q$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Hölder’s inequality implies that

$$\int_{-L}^{L} |\alpha'(x)| \, dx \leq \left(\int_{-\psi_1(t)}^{\psi_2(t)} |\alpha'(x)|^p \, dx \right)^{1/p} \left(\int_{-\psi_1(t)}^{\psi_2(t)} 1^q \, dx \right)^{1/q}$$

$$= (2 \phi(t))^{1-1/p} W(t)^{1/p}.$$ 

Thus for all $t > 0$ we have the following lower bound for the energy:

$$W(t) \geq \frac{1}{(2 \phi(t))^{p-1}} \left(\int_{-L}^{L} |\alpha'(x)| \, dx \right)^p. \tag{5.10}$$

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Assume now that \( \alpha(x) \) is such that there exists a real number \( r \) such that for all \( x \in [-\psi_1(t), \psi_2(t)] \) we have \( |\alpha'(x)| = r \). Such a situation occurs in Example 3.2 with \( r = 1 \). Then \( \int_{-L}^{L} |\alpha'(x)| \, dx = \int_{-\psi_1(t)}^{\psi_2(t)} |\alpha'(x)| \, dx = 2\phi(t) \, r \) and

\[
W(t) = 2\phi(t) \, r^p = \frac{1}{(2 \phi(t))^{p-1}} \left( \int_{-L}^{L} |\alpha'(x)| \, dx \right)^p,
\]

which shows that the lower bound (5.10) is sharp.

Equation (5.9) yields a lower bound for the energy \( W(T) \) at the terminal time \( T \).

With (5.5), (5.9) implies the inequality

\[
W(T) \geq \left( \frac{c - D}{c + D} \right)^{p-1} \int_{-L}^{L} |\alpha'(x)|^p = \left( \frac{c - D}{c + D} \right)^{p-1} \, W(0).
\]

This lower bound is attained in Example 3.4.

5.3. The transformed optimization problem. Later in this section, we will need the following result in a proof.

**Lemma 5.5.** Assume that \( p \in (1, \infty) \). Let \( r \geq 0 \) be given. For \( z \in (0, [c + D]/[c - D]) \), define the function \( \tau(z) = r/z^{p-1} \), and let \( \kappa = \frac{1}{2} (p-1)(c-D)^{p+1}/(c+D)^{p+1} > 0 \). Then for all \( z_1, z_2 \in (0, [c + D]/[c - D]) \), the following inequality holds:

\[
\tau(z_2) - \tau(z_1) \geq r(1-p) \frac{1}{z_1^p}(z_2 - z_1) + r\kappa(z_2 - z_1)^2.
\]

**Proof.** We have \( \tau'(z) = r(1-p)/z^p \) and \( \tau''(z) = r(p-1)/z^{p+1} \). We consider the Taylor expansion for \( \tau \) which yields the existence of a point \( \mu \) between \( z_1 \) and \( z_2 \) such that

\[
\tau(z_2) = \tau(z_1) + \tau'(z_1)(z_2 - z_1) + \frac{\tau''(\mu)}{2}(z_2 - z_1)^2
\]

\[
\geq \tau(z_1) + \tau'(z_1)(z_2 - z_1) + \frac{1}{2}r(p-1)p(z_2 - z_1)^2 / \max\{z_1, z_2\}^{p+1}
\]

\[
\geq \tau(z_1) + \tau'(z_1)(z_2 - z_1) + r\kappa(z_2 - z_1)^2,
\]

and the assertion follows. \( \square \)

Now we come to the transformed optimization problem. Let the set \( F \) be defined as in Lemma 5.4. For \( f \in F \), define

\[
J(f) = \int_{-L}^{L} \frac{|\alpha'(x)|}{f(x)^{p-1}} \, dx.
\]

In Theorem 4.1, we have seen that \( \alpha' \in L^p(-L, L) \). Due to the bounds for \( f \) in the definition of the set \( F \), this implies that the number \( J(f) \) is well-defined. Lemma 5.4 states that \( M = H \). Due to (5.9), for all \( h \in M \), we have \( W(T) = J(h) \). Hence the definition (5.4) of the set \( M \) and the bijection \( \theta \) between the sets \( \Phi \) and \( M \) given in Lemma 5.2 imply that our problem \( P \) is equivalent to the problem

\[
Q : \quad \text{Find } f \in F \text{ such that } J(f) \text{ is minimized.}
\]

Problem \( Q \) is a convex optimization problem. The necessary optimality conditions lead us to the following solution of problem \( Q \).
Lemma 5.6. Let \( p \in (1, \infty) \), and let \( \alpha \) be defined as in (4.8). For \( \lambda > 0 \), define the function \( f \) on the interval \([-L, L]\) by
\[
f(x) = \begin{cases} \frac{c-D}{c+D} & \text{if } x \in [-L, L] \text{ and } \lambda|\alpha'(x)| \leq \frac{c-D}{c+D}, \\ \lambda|\alpha'(x)| & \text{if } x \in [-L, L] \text{ and } \lambda|\alpha'(x)| \in \left(\frac{c-D}{c+D}, \frac{c+D}{c-D}\right), \\ \frac{c+D}{c-D} & \text{if } x \in [-L, L] \text{ and } \lambda|\alpha'(x)| \geq \frac{c+D}{c-D}. \end{cases}
\]

If \( \int_L^L |\alpha'(y)| \, dy > 0 \), there exists a real number \( \lambda > 0 \) such that
\[
(5.12) \quad \int_{-L}^{L} f(x) \, dx = 2L,
\]
and, with this choice of \( \lambda \), we have \( f \in F \), and \( f \) is a solution of problem \( Q \).

Define the set \( M_\zeta = \{ x \in [-L, L] : \alpha'(x) = 0 \} \). If \( M_\zeta \) has measure zero, the solution of \( Q \) is uniquely determined.

If \( \int_L^L |\alpha'(y)| \, dy = 0 \), we have \( J(f) = 0 \) for all \( f \in F \).

Proof. A special case. For the special case that \( \int_L^L |\alpha'(y)| \, dy > 0 \) and for \( \lambda = \frac{2L}{\int_{-L}^{L} |\alpha'(x)| \, dx} \), we have \( \lambda|\alpha'(x)| \in \left[\frac{c-D}{c+D}, \frac{c+D}{c-D}\right] \) for all \( x \in [-L, L] \), and we give a proof for the optimality of \( f \) that is based upon Hölder’s inequality. We have
\[
J(f) = \frac{1}{\lambda^{p-1}} \int_{-L}^{L} \frac{|\alpha'(x)|^p}{|\alpha'(x)|^{p-1}} \, dx = \frac{\left(\int_{-L}^{L} |\alpha'(x)| \, dx\right)^p}{(2L)^{p-1}}.
\]

Let \( q \) be such that \( (1/p) + (1/q) = 1 \), and let \( g \in F \). Then we have
\[
\int_{-L}^{L} |\alpha'(x)| \, dx = \int_{-L}^{L} \frac{|\alpha'(x)|}{g(x)^{1/q}} \, g(x)^{1/q} \, dx \\
\leq \left(\int_{-L}^{L} \frac{|\alpha'(x)|^p}{g(x)^{p/q}} \, dx\right)^{1/p} \left(\int_{-L}^{L} g(x) \, dx\right)^{1/q} \\
\leq \left(\int_{-L}^{L} \frac{|\alpha'(x)|^p}{g(x)^{p/q}} \, dx\right)^{1/p} \, (2L)^{1/q} \\
= J(g)^{1/p} (2L)^{1/q}.
\]

This implies the desired inequality
\[
J(g) \geq \left(\frac{\int_{-L}^{L} |\alpha'(x)| \, dx}{(2L)^{p/q}}\right)^p = \frac{\left(\int_{-L}^{L} |\alpha'(x)| \, dx\right)^p}{(2L)^{p-1}} = J(f).
\]

The general case. For the general case where \( \int_L^L |\alpha'(y)| \, dy > 0 \), we give a proof that is based upon the convexity of the objective function. Assume that \( f \) is given as defined in Lemma 5.6 and that (5.12) holds. Then \( f \in F \). Let \( g \in F \) be given. Define
the difference function \( \Delta = g - f \). Then \( \int_{-L}^{L} \Delta(x) \, dx = 0 \). Define the sets

\[
M_{\leq} = \{ x \in [-L, L] : |\alpha'(x)| \leq \frac{c - D}{c + D} \}, \\
M_0 = \{ x \in [-L, L] : |\alpha'(x)| \in \left( \frac{c - D}{c + D} \frac{1}{\lambda}, \frac{c + D}{c - D} \frac{1}{\lambda} \right) \}, \\
M_{\geq} = \{ x \in [-L, L] : |\alpha'(x)| \geq \frac{c + D}{c - D} \frac{1}{\lambda} \}.
\]

For all \( x \in M_{\leq} \), we have \( \Delta(x) = g(x) - \frac{c - D}{c + D} \geq 0 \). Hence due to the definition of \( f \) the following inequality holds:

\[
\int_{M_{\leq}} |\alpha'(x)|^p \Delta(x) \, dx = \left( \frac{c + D}{c - D} \right)^p \int_{M_{\leq}} |\alpha'(x)|^p \Delta(x) \, dx \\
\leq \left( \frac{c + D}{c - D} \right)^p \left( \frac{c - D}{c + D} \right)^p \frac{1}{\lambda^p} \int_{M_{\leq}} \Delta(x) \, dx = \frac{1}{\lambda^p} \int_{M_{\leq}} \Delta(x) \, dx.
\]

For all \( x \in M_{\geq} \), we have \( \Delta(x) = g(x) - \frac{c - D}{c + D} \leq 0 \). Hence due to the definition of \( f \) the following inequality holds:

\[
\int_{M_{\geq}} |\alpha'(x)|^p \Delta(x) \, dx = \left( \frac{c - D}{c + D} \right)^p \int_{M_{\geq}} |\alpha'(x)|^p \Delta(x) \, dx \\
\leq \left( \frac{c - D}{c + D} \right)^p \left( \frac{c + D}{c - D} \right)^p \frac{1}{\lambda^p} \int_{M_{\geq}} \Delta(x) \, dx = \frac{1}{\lambda^p} \int_{M_{\geq}} \Delta(x) \, dx.
\]

Moreover, the definition of \( f \) implies the equation

\[
\int_{M_0} \frac{|\alpha'(x)|^p}{f(x)^p} \Delta(x) \, dx = \int_{M_0} \frac{|\alpha'(x)|^p}{\lambda^p |\alpha'(x)|^p} \Delta(x) \, dx = \frac{1}{\lambda^p} \int_{M_0} \Delta(x) \, dx.
\]

We have \( f(x), g(x) \in \left[ \frac{c - D}{c + D}, \frac{c + D}{c - D} \right] \) almost everywhere on the interval \([-L, L]\). Hence, for our objective function, the application of Lemma 5.5 pointwise for all \( x \in [-L, L] \) with \( r = |\alpha'(x)|^p \) and \( z_2 = g(x), z_1 = f(x) \) yields

\[
J(g) - J(f) = \int_{-L}^{L} \frac{|\alpha'(x)|^p}{g(x)^{p-1}} \frac{|\alpha'(x)|^p}{f(x)^{p-1}} \, dx \\
\geq \int_{-L}^{L} (1 - p) \frac{|\alpha'(x)|^p}{f(x)^p} \Delta(x) \, dx + \int_{-L}^{L} |\alpha'(x)|^p \kappa \Delta(x)^2 \, dx \\
= (1 - p) \int_{M_{\leq}} \frac{|\alpha'(x)|^p}{f(x)^p} \Delta(x) \, dx + (1 - p) \int_{M_0} \frac{|\alpha'(x)|^p}{f(x)^p} \Delta(x) \, dx \\
+ (1 - p) \int_{M_{\geq}} \frac{|\alpha'(x)|^p}{f(x)^p} \Delta(x) \, dx + \kappa \int_{-L}^{L} |\alpha'(x)|^p \Delta(x)^2 \, dx.
\]

Since \( (1 - p) < 0 \), with the inequalities for the integrals on \( M_{\leq}, M_{\geq}, \) and \( M_0 \) derived
above, this implies the inequality
\[
J(g) - J(f) \geq (1 - p) \int_{M_{\geq}} \frac{1}{\lambda^p} \Delta(x) \, dx + (1 - p) \int_{M_0} \frac{1}{\lambda^p} \Delta(x) \, dx \\
+ (1 - p) \int_{M_{\geq}} \frac{1}{\lambda^p} \Delta(x) \, dx + \kappa \int_{-L}^L |\alpha'(x)|^p \Delta(x)^2 \, dx
\]
\[
= \frac{1 - p}{\lambda^p} \int_{-L}^L \Delta(x) \, dx + \kappa \int_{-L}^L |\alpha'(x)|^p \Delta(x)^2 \, dx
\]
\[
= \kappa \int_{-L}^L |\alpha'(x)|^p \Delta(x)^2 \, dx \geq 0.
\]
Thus \( J(g) \geq J(f) \). Hence \( f \) is a solution of the optimization problem \( Q \).

Define the set \( M_1 = \{ x \in [-L, L] : \Delta(x) \neq 0 \} \). If \( M_2 \) has measure zero and \( M_1 \) has strictly positive measure, that is, \( g \neq f \), we have the inequality
\[
J(g) - J(f) \geq \kappa \int_{-L}^L |\alpha'(x)|^p \Delta(x)^2 \, dx > 0.
\]
Hence in this case, the solution of \( Q \) is uniquely determined.

Define the function \( U \) on the interval \((0, \infty)\) by the equation
\[
U(\lambda) = \int_{-L}^L \Pi_{\{\epsilon \leq \frac{D}{\epsilon^p}, \frac{D}{\epsilon^p} \leq \frac{D}{\epsilon^p}\}}(\lambda |\alpha'(y)|) \, dy, \ \lambda \in (0, \infty).
\]
For all \( z_1, z_2 \in (0, \infty) \), we have the inequality
\[
\left| \Pi_{\{\epsilon \leq \frac{D}{\epsilon^p}, \frac{D}{\epsilon^p} \leq \frac{D}{\epsilon^p}\}}(z_1) - \Pi_{\{\epsilon \leq \frac{D}{\epsilon^p}, \frac{D}{\epsilon^p} \leq \frac{D}{\epsilon^p}\}}(z_2) \right| \leq |z_1 - z_2|.
\]
Hence for all \( \lambda_1, \lambda_2 \in (0, \infty) \) we have
\[
|U(\lambda_1) - U(\lambda_2)| \leq \int_{-L}^L \left| \Pi_{\{\epsilon \leq \frac{D}{\epsilon^p}, \frac{D}{\epsilon^p} \leq \frac{D}{\epsilon^p}\}}(\lambda_1 |\alpha'(y)|) - \Pi_{\{\epsilon \leq \frac{D}{\epsilon^p}, \frac{D}{\epsilon^p} \leq \frac{D}{\epsilon^p}\}}(\lambda_2 |\alpha'(y)|) \right| \, dy
\]
\[
\leq \int_{-L}^L |\lambda_1 |\alpha'(y)| - \lambda_2 |\alpha'(y)| | \, dy
\]
\[
= |\lambda_1 - \lambda_2| \int_{-L}^L |\alpha'(y)| \, dy.
\]
Thus \( U \) is Lipschitz continuous. We have \( \lim_{\lambda \to 0} U(\lambda) = 2L \frac{\epsilon^p}{\lambda^p} < 2L \).

Since \( \int_{-L}^L |\alpha'(y)| \, dy > 0 \) we have \( \lim_{\lambda \to \infty} U(\lambda) = 2L \frac{\epsilon^p}{\lambda^p} > 2L \). Hence there exists a number \( \lambda > 0 \) such that \( U(\lambda) = 2L \), which means that (5.12) is valid.

The case where \( \int_{-L}^L |\alpha'(y)| \, dy = 0 \) is trivial. \( \Box \)

5.4. Proofs of the main results.

Proof of Theorem 3.1. For \( p = 1 \), \( W(0) = W(T) \) for all \( \phi \in \Phi \) (see Remark 5.2), so \( \phi(t) = L \in \Phi \) is a solution of \( P \).

Assume that \( p > 1 \). Due to (5.9) and the transformation of the set of admissible controls described in section 5.1, problem \( P \) is equivalent to the problem

\( P_1 : \) Find \( h \in H \) such that \( J(h') = \int_{-L}^L |\alpha'(x)|^p \frac{h'(x)}{h'(x)^{p-1}} \, dx \) is minimized.
For all \( h \in H \), we have \( h'(x) \in F \), and, for all \( f \in F \), we have \( h(x) = -L + \int_{-L}^x f(x) \, ds \in H \). Moreover, \( J(h') = J(f) \). Due to the representation (5.4) of the set \( H \), this implies that \( H_1 \) is equivalent to \( Q \).

Lemma 5.6 implies the existence of a solution of \( Q \), which yields in turn the existence of a solution of \( P \).

By Theorem 4.1, we have \( \alpha' \in L^p(-L, 3L) \), and, due to the compatibility conditions \( y_0(0) = y_0(L) = 0 \) in the definition of the set \( B \), the definition of the function \( \alpha' \) implies that, for all \( x \in [-L, L] \), we have \( \alpha'(x) = A(x) \).

Hence the definition of the set \( M_x \) in Theorem 3.1 is equivalent to the definition in Lemma 5.6. If the set \( M_x \) has measure zero, Lemma 5.6 implies the uniqueness of the solution of \( Q \). We have seen that each function \( f \in F \) corresponds to an admissible function \( \phi \in \Phi \) with \( J(f) = W(T) \). This implies the uniqueness of the solution of \( P \).

**Proof of Theorem 3.2.** If \( \int_L^L |A(y)| \, dy > 0 \), Lemma 5.6 implies the existence of a number \( \lambda > 0 \) such that (5.12) holds. If (5.12) holds for \( \lambda > 0 \), then (3.2) is valid with this value of \( \lambda \), and, for the function \( f \) defined in Lemma 5.6, we have

\[
 f(x) = \Pi_{\frac{x-D}{\lambda} \leq \frac{x+D}{\lambda}}(\lambda |A(x)|).
\]

For the function \( h \) defined in Theorem 3.2, we have \( h(x) = -L + \int_{-L}^x f(x) \, ds \). Since \( f \) solves \( Q \), \( h \) solves problem \( P_1 \) defined above. In Lemma 5.2, we have shown that the solution \( \Phi \) of \( P \) can then be obtained by \( \Phi = \theta^{-1}(h) \). Equations (5.7) and (5.6) show that \( \Phi \) is given by (3.3). Equation (5.9) yields the minimal value of \( W(T) \) and the result for the case \( \int_L^L |A(y)| \, dy = 0 \).

**6. Conclusion.** We study a system that is controlled through the movement of the boundary and where the boundary movements are described by Lipschitz continuous functions. To obtain a well-posed problem, we require that the Lipschitz constants for the admissible controls are less than or equal to a given number \( D \) that is strictly less than the speed of wave propagation. We give a representation of a boundary movement that generates a maximal decrease of the energy in the given finite time interval. In particular, we give sufficient conditions for the existence and uniqueness of an optimal movement. Due to the nature of our system it is impossible to drive the energy arbitrarily close to zero unless it is zero from the start. For some initial states, it is even impossible to achieve any energy decrease by boundary movement control. The optimal energy decrease depends on the initial state. Depending on the initial state a considerable reduction of the energy can be achieved.

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