A generalization of Cahn-Hilliard inpainting for grayvalue images

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The Cahn-Hilliard equation has its origin in material sciences and serves as a model for phase separation and phase coarsening in binary alloys. A new approach in the class of fourth order inpainting algorithms is inpainting of binary images using the Cahn-Hilliard equation. We will present a generalization of this fourth order approach for inpainting greyvalue images. This is realized by using subgradients of the total variation functional within the flow, which leads to structure inpainting with smooth curvature of level sets. We will present some numerical examples for this approach and analytic results concerning existence and convergence of solutions.

1 Introduction

Second order variational inpainting methods, like total variation inpainting [1], have drawbacks as in the connection of edges over large distances or the continuous propagation of level lines into the damaged domain. In an attempt to solve both the connectivity principle and the so-called staircasing effect resulting from second order image diffusions, a number of third and fourth order diffusions have been suggested for image inpainting.

One of the most important works in this direction is the algorithm of Chan, Kang and Shen [2] based on Euler’s elastica energy. Their approach leads to a continuous connection of level lines also over large distances. Another new approach in the class of fourth order inpainting algorithms is inpainting of binary images using the Cahn-Hilliard equation [3], [4]. Let f be a given image in a domain Ω with inpainting domain $D \subset \Omega$. The result of the Cahn-Hilliard inpainting approach $u$ evolves in time to become a fully inpainted version of $f$ under the following equation

$$\frac{\partial u}{\partial t} = \Delta ( -\epsilon \Delta u + \frac{1}{\epsilon} F(u)) + \lambda(f - u),$$

where $F$ is a so-called double-well potential, e.g. $F(u) = u^2(u - 1)^2$, and $\lambda(x) = \lambda_0$ in $\Omega \setminus D$, $\lambda(x) = 0$ in $D$.

In the following a generalization of this fourth-order inpainting approach for grayvalue images is shown. Starting with an iterative scheme to solve (1) we take the $\Gamma$-limit for $\epsilon \to 0$ to obtain a scheme with subgradients of the total variation functional. A similar form of this $\Gamma$-limit already appeared in the context of decomposition and restoration for grayvalue images, see for example [9] and [6]. Motivated by these works a fourth order TV-inpainting approach is proposed and first numerical examples are presented.

2 $TV - H^{-1}$ inpainting

We consider the following iterative scheme for (1)

$$u_{k+1} = \arg \min_u J_k^\epsilon(u)$$

with

$$J_k^\epsilon(u) = \int_\Omega \left( \frac{\epsilon}{2} |\nabla u|^2 + \frac{1}{\epsilon} F(u) \right) \ dx + \frac{1}{2r} \left\| \nabla \Delta^{-1} (u - u_k) \right\|^2 + \frac{\lambda_0}{2} \left\| \nabla \Delta^{-1} (u - \frac{\lambda}{\lambda_0} f - (1 - \frac{\lambda}{\lambda_0}) u_k) \right\|^2,$$

and $u_k = u(t_k), t_{k+1} - t_k = \tau$. Our first result is stated in the following theorem and shows that problem (2) attains a unique solution in $H^1(\Omega)$.

Theorem 2.1 The optimization problem (2) attains a unique minimum in $H^1(\Omega)$.

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Fig. 1 $TV - H^{-1}$ inpainting: $u(1500)$ with $\lambda = 10^3$

To build the connection to the inpainting of grayvalue images we will show (cf. [5]) that minimizers of (2) $\Gamma$-converge to solutions of an optimization problem regularized with the TV norm. In fact Modica and Mortola have shown in [7] and [8] that the sequence of Cahn-Hilliard functionals $CH(\epsilon) = \int_{\Omega} \frac{1}{2} |\nabla u|^2 + \frac{1}{\epsilon} F(u) \, dx$ $\Gamma$-converges in the topology $L^1(\Omega)$ to

$$TV(u) = \begin{cases} C_0 \int_\Omega |\nabla u| \, dx & \text{if } |u(x)| = 1 \text{ a.e. in } \Omega \\ +\infty & \text{otherwise} \end{cases}$$

as $\epsilon \to 0$, where $C_0 = 2 \int_{-1}^1 \sqrt{F(s)} \, ds$. Motivated by this $\Gamma$-limit we consider the minimization problem for the following sequence of functionals

$$J_k(u) := TV(u) + \frac{1}{2r} ||u - u_k||_{-1}^2 + \frac{\lambda_0}{2} ||u - \frac{\lambda}{\lambda_0} f - (1 - \frac{\lambda}{\lambda_0})u_k||_{-1}^2,$$

with

$$TV(u) = \begin{cases} ||u||_{BV} & \text{if } -1 \leq u(x) \leq 1 \text{ a.e. in } \Omega \\ +\infty & \text{otherwise,} \end{cases}$$

for $f \in BV(\Omega)$, $|f| \leq 1$ is the given grayvalue image destroyed inside the inpainting domain $D$. We obtain the following theorem,

**Theorem 2.2** The minimization problem for (3) attains a unique solution in $BV(\Omega)$.

Further we can prove that minimizers of $J_k(u)$ converge to a steady state.

**Theorem 2.3** Minimizers $u_{k+1}$ of (3) weakly converge in $H^{-1}$ for $k \to \infty$ to solutions $\hat{u}$ of the equation

$$\Delta^{-1}(\lambda(\hat{u} - f)) = \hat{\rho},$$

where $\hat{\rho} \in \partial TV(\hat{u})$ and $\hat{u}$ is a stationary solution of the evolution equation

$$-\Delta^{-1} u_t = \hat{\rho} - \Delta^{-1}(\lambda(u - f)).$$

A detailed description of the stated results can be found in [5].

**References**


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1 This work was partially supported by the WWTF (Wiener Wissenschafts-, Forschungs- und Technologiefonds) project nr.CI06 003 and by the FWF (Fonds zur Förderung der wissenschaftlichen Forschung) Wittgenstein award of Peter Markowich, project nr. Z-50 MAT