Discovering Point Sources in Unknown Environments

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Abstract: We consider the inverse problem of discovering the location of a source from very sparse point measurements in a bounded domain that contains impenetrable (and possibly unknown) obstacles. We present an adaptive algorithm for determining the measurement locations, and ultimately, the source locations. Specifically, we investigate source discovery for the Laplace operator, though the approach can be applied to more general linear partial differential operators. We propose a strategy for the case when the obstacles are unknown and the environment has to be mapped out using a range sensor concurrently with source discovery.

1 Introduction

This work is motivated by robotic applications in which a robot, sent into an unknown environment, is supposed to discover the location of a signal source and place it under its line-of-sight in an efficient manner. The unknown environment contains non-penetrable solid obstacles and should be avoided along the robot’s path. In this environment, the properties of the signal, such as the signal strength, are assumed to satisfy certain partial differential equations (PDEs) with appropriate boundary conditions. The robot can gather measurements from two different sensors: a range sensor that gives distance from the robot to the surrounding obstacles, and a sensor that measures the signal strength that is being emitted from the yet-to-be-located source. We will refer to the information from the range sensor as the visibility of the robot and to the information from the signal strength sensor as the signal. While measurements can be taken anywhere, we are interested in having the robot take very few measurements with its sensors. Our goal is to design an algorithm that determines how the robot should navigate through the environment and where along its path it should take measurements. Motion planning is a fundamental problem in robotics. It has been an active area of research since 1980s. A comprehensive presentation of motion planning techniques can be
found in the book by Latombe [16]. In [9, 10], the authors provide a broad review of problems specific to the research in robotics. In the most general form, motion planning consists of finding a robot’s path from a start position to a goal position, while avoiding obstacles and satisfying some constraints [16]. A large class of optimal path planning problems can be formulated as problems involving Hamilton-Jacobi equations [5, 1], and solved efficiently [28, 23].

This work is primarily concerned with visibility-based navigation towards a target whose position is unknown. As the observer moves in space, its visibility region changes, thus modifying the information available to the observer about the space or progress towards the goal. The visibility region is the set of points that can be joined with a line segment to the observers position, without intersecting the obstacle boundaries. Online motion planning strategies are considered in [2, 6, 11, 12, 17, 20, 22, 25] among many others. In these works, the robot is in an unknown or partially known planar environment and uses its on-board sensors to navigate toward a target whose position is either known a priori or is recognized upon arrival. In particular, the bug-family algorithms are considered, which have two reactive modes of motion: moving toward the target between the obstacles and following the obstacles’ boundaries. These two modes interact incrementally until the target is reached or is found to be out of reach. We remark here that this paper considers problems in which the source cannot be identified until suitable inverse problems are solved and problems in which one does not want to reach the source but to locate it.

Today, computational geometry and combinatorics are the primary tools to solve the visibility-based problems [29, 8]. These techniques are mainly concerned with defining visibility on polygons and more general planar environments with special structure. The combinatorial approach leads to fast and elegant solutions in simplified planar polygonal environments. However, this approach becomes increasingly complex in more realistic settings, especially in three dimensions.

This source problem is classified as an inverse problem, however, it differs greatly from many typical inverse problems, which assume simple domains and dense arrays of sensors at fixed locations. In [19], Ling et al explore such a situation in which they recover the exact locations of multiple sources in a Poisson equation, given an initial guess for the locations and Dirichlet data collected on the boundary of the domain. To accomplish this, they use the special form of the free space Green’s function for the Laplacian. For inverse problems related to the heat equations with sources see [3, 4] and for a more general discussion of partial differential equations see [5].

When the obstacles are unknown, the environment needs to be mapped out as the robot moves so that attempts to take measurements inside the obstacles are avoided and the robot’s path does not intersect obstacles. The previous work of [14] and [15] on mapping of obstacles in unknown domains using visibility is useful in this regard. In it the authors propose an algorithm to construct a high-order accurate representation of the portions of the solid surfaces that are visible from a vantage point and to generate the correspond-
ing occlusion volume. Also they propose an algorithm to construct a piecewise linear path so that any point on the solid surfaces is seen by at least one vertex of the path and an accurate representation of the solids is constructed from the point clouds that are collected at the vertices of the path. In [14] a novel visibility-based algorithm is introduced to navigate toward a known target in an unknown bounded planar environment with obstacles. Similarly to [25], the observer navigates toward one of the visible horizons, or edges on the visibility map. In particular, to reach the goal, the edge that is the nearest to the target is chosen. To proceed the observer must overshoot the horizon by the amount inversely proportional to the curvature of the obstacle near the horizon, thus no boundary-following motion is required.

The [14] algorithm was motivated by the work of LaValle, Tovar et al. [24, 18, 25]. In [25], a single robot (observer) must be able to navigate through an unknown simply or multiply connected piecewise-analytic planar environment. The robot is equipped with a sensor that detects discontinuities in depth information and their topological changes in time. As a result of exploration, the region is characterized by the number of gaps and their relative positions. No distance or angular information is accumulated. In contrast, the [14] algorithm maps the obstacles in Cartesian coordinates as the observer proceeds through the environment, and utilizes the recovered information for further path planning. At the termination of the path all the obstacle boundaries are reconstructed to give a complete map of the environment. A practical implementation of this algorithm on an economical cooperative control tank-based platform is described in [13].

From our experience, the discovery of signal sources may be very insensitive to the presence of (or parts of) obstacles in sub-regions of the domain, possibly due to the decay of the signal strength. This suggests that the visibility path algorithm in [14] and [15] should be modified adaptively according to previous measurements and estimations of the signal source location.

To illustrate the main ideas in this paper, consider the following problem

\[ \begin{align*}
\triangle u(x) &= \delta(x - y) \quad \text{in} \quad D \setminus \Omega \\
\quad u &= 0 \quad \text{on} \quad \Gamma ,
\end{align*} \tag{1} \]

where \( u \) denotes the signal strength, \( D \) denotes a bounded domain, \( \Omega \subseteq D \) denotes the solid obstacles in the domain, \( \Gamma = \partial D \), and \( y \) denotes the (unknown) source location. One can view this PDE as giving a description of the steady state of a diffusion problem. We have chosen Dirichlet boundary conditions in this example but our methods can be adapted to other common boundary conditions such as Neumann boundary conditions.

Let \( \psi(\cdot; z) : D \mapsto \mathbb{R} \) describe the visibility of the domain from an observing location \( z \in D_R \). We require that \( \psi(\cdot; z) \) be a signed distance function such that the set \( W_z := \{ x \in D : \psi(x; z) < 0 \} \) corresponds to the region, including the interior of the solid obstacles, that is occluded from the observing location \( z \). This means that the line segment connecting \( z \) and any point in \( W_z \) must
intersect with the obstacles $\Omega$. Such visibility functions can be computed efficiently using the algorithms described in [27, 26, 15, 14].

A first, rather simple approach, would be to use gradient descent to determine the sample locations via the ordinary differential equation,

$$\frac{dX}{ds} = -\nabla u, \text{ with } X(0) = z_0 .$$

However, there are two drawbacks to this method. First, it only works in cases where the Green’s function has a specific structure, such as in the case for the Laplace operator. For problems involving the Helmholtz operator, the solutions are typically oscillatory and this gradient approach does not apply at all. Second, even for the Laplace operator, one can come up with a pathological configuration for the obstacles where the gradient vanishes at points other than the source.

We continue with the method proposed in this paper. At an observing location $z_1$, we can measure the signal strength $I_1 = u(z_1)$. We look at the solution to the adjoint problem,

$$\begin{align*}
\nabla v_1 &= \delta(x - z_1) \text{ in } D_\Omega \\
v_1 &= 0 \text{ on } \Gamma .
\end{align*}$$

Now, for $y \neq z_1$ we have

$$\begin{align*}
v_1(y) &= \int_{D_\Omega} \delta(x - y)v_1 dx = \int_{D_\Omega} v_1 \nabla u dx \\
u(z_1) &= \int_{D_\Omega} \delta(x - z_1)u dx = \int_{D_\Omega} u \nabla v_1 dx ,
\end{align*}$$

and thus, by Green’s identity,

$$v_1(y) = u(z_1) = I_1 .$$

Therefore the source must lie on the $I_1$ level set of $v_1$,

$$y \in \{ x \in D_\Omega : v_1(x) = I_1 \} .$$

Next, based on the visibility information, we select the next observing location $z_2$ from the region that is not occluded to $z_1$, that is $z_2 \in D \setminus W_{z_1}$. Denote the signal strength at $z_2$ by $I_2 = u(z_2)$. The function $v_2$ can be computed and we can narrow down the possible locations of $y$,

$$y \in \{ v_1 = I_1 \} \cap \{ v_2 = I_2 \} .$$

We can repeat this procedure for more measurements.

Our proposed algorithm can handle obstacles of a rather large class, including very complicated non-convex obstacles, see Figures 5 and 6 for an example. The only constraint comes from the size of the underlying mesh.
used to obtain the solution of the PDE (3), since this grid has the resolve the features of the obstacles. The discretization of the domain results in a simple system of linear equations that has to be solved. We refer the reader to [7, 21] for a discussion and efficient methods for solving PDEs.

In the case when the obstacles $\Omega$ are unknown, the visibility functions $\psi(x; z_k)$ provide a convenient over approximation of $\Omega$, since $\Omega \subseteq W_{z_k}$. This can be used in conjunction with a maximum principle for Poisson’s equation to estimate the location of $y$. We will discuss this in greater detail in later sections.

2 Mathematical Formulation

In the most general setting that we will consider, the inverse source problem can be formulated as follows. Let $u(x)$ satisfy,

$$
Lu = \alpha \delta(x - y) \quad \text{in} \quad D_\Omega \equiv D \setminus \Omega \\
Bu = 0 \quad \text{on} \quad \Gamma,
$$

(8)

where $D$ is a bounded domain, $\Omega$ are the (possibly unknown) obstacles, $\Gamma = \partial D_\Omega$, $L$ is a linear partial differential operator, $B$ is an operator specifying the boundary conditions, and $\alpha > 0$. We will assume that at a given point $z \in D_\Omega$ we can sample $u$ and the domain. That is, at a given $z$ we can measure $u(z)$ and its derivatives and the visibility function $\psi(x; z)$. The inverse source location problem is to recover the source location $y$ and the source strength $\alpha$ from a sequence of sample locations $z_k$.

The main approach that we will use in this problem is to look at the adjoint operator, $L^*$, with the appropriate boundary conditions $B^*$:

$$
L^*v = F_z \quad \text{in} \quad D_\Omega \\
B^*v = 0 \quad \text{on} \quad \Gamma,
$$

(9)

for some distribution $F_z$ with support $\{z\}$. Now, using the properties of the adjoint and assuming that $z \neq y$,

$$(Lu, v) - (u, L^*v) = 0,$$

(10)

and hence,

$$\alpha v(y) = F_z[u].$$

(11)

For example, if we use $F = \delta$, we get

$$\alpha v(y) = u(z).$$

(12)

Similarly, for $F = -\partial_x \delta$, we get

$$\alpha v(y) = \partial_x u(z).$$

(13)

For unknown domains, we use the methods developed in [14] to find the visibility function and use it determine the sequence of sample locations $z_k$. 
Poisson’s Equation

In this section we consider the case when \( L = \triangle \) in 2 dimensions with Dirichlet boundary conditions:

\[
\triangle u = \alpha \delta(x - y) \quad \text{in} \quad D_\Omega \\
u = 0 \quad \text{on} \quad \Gamma .
\]

(14)

We note that this operator is self-adjoint and that the following maximum principle from PDE theory holds:

**Theorem 1.** Let \( D_\Omega \) be bounded and \( w \) satisfy

\[
\triangle w = 0 \quad \text{in} \quad D_\Omega \\
w \leq 0 \quad \text{on} \quad \Gamma ,
\]

(15)

then \( w \leq 0 \) in \( D_\Omega \).

Now, supposed that we have an over-estimate for the obstacles \( \Omega^+ \), so that \( \Omega \subseteq \Omega^+ \) and let \( v \) and \( v^+ \) satisfy (14) with obstacle sets \( \Omega \) and \( \Omega^+ \) respectively. Then, since the fundamental solution for the Laplacian for any domain is non-positive, \( v, v^+ \leq 0 \) on \( \Gamma^+ = \partial D_{\Omega^+} \). Let \( w = v - v^+ \), so that \( w \) satisfies the conditions of Theorem 1 for \( D_{\Omega^+} \). Thus, \( w = v - v^+ \leq 0 \) in \( D_{\Omega^+} \). Furthermore, if we extend \( v^+ \) to \( D_\Omega \) by 0, we have that \( v^+ \leq v \) in \( D_\Omega \). This fact will be used in the case of unknown obstacles.

3.1 Known Environment

At a sample location \( z_k \), equation (11) gives us

\[
\alpha v_k(y) = u(z_k) \\
\alpha w_{k1}(y) = \partial_{x_1} u(z_k) \\
\alpha w_{k2}(y) = \partial_{x_2} u(z_k)
\]

(16)

where \( v_k \), \( w_{k1} \) and \( w_{k2} \) satisfy (9) with \( F = \delta(x - z_k) \), \( -\partial_{x_1} \delta(x - z_k) \) and \( -\partial_{x_2} \delta(x - z_k) \), respectively. Since \( v_k \) is non-zero except at the boundary, we can also form,

\[
\frac{w_{k1}(y)}{v_k(y)} = \frac{\partial_{x_1} u(z_k)}{u(z_k)} \quad \text{and} \quad \frac{w_{k2}(y)}{v_k(y)} = \frac{\partial_{x_2} u(z_k)}{u(z_k)} .
\]

(17)

Thus, \( y \) is in the intersection of the \( u(z_k) \) level set of \( v \), the \( \partial_{x_1} u(z_k) \) level set of \( w_{k1} \) and so on. Note that in the last two equations, \( \alpha \) does not appear. As we take more and more measurements \( z_k \), the intersection of all of these sets will be smaller and smaller. Furthermore, for a pair of measurements \( j \) and \( k \), we can form,

\[
\frac{v_k(y)}{v_j(y)} = \frac{u(z_k)}{u(z_j)} ,
\]

(18)

and so on. Note that these are also independent of \( \alpha \).
3.2 Unknown Environment

At a sample location $z_k$, we have the visibility function $\psi(x; z_k)$. From this function we can obtain an over-estimate of the obstacles, $\Omega^+_k$, such that $\Omega \subseteq \Omega^+_k$. After $K$ measurements, we let

$$\Omega^+ = \bigcap_{k=1}^{K} \Omega^+_k,$$

and

$$\begin{align*}
\Delta v^+_k &= \delta(x - z_k) \quad \text{in } D_{\Omega^+} \equiv D \setminus \Omega^+ \\
v^+_k &= 0 \quad \text{on } \Gamma^+,
\end{align*}$$

where $\Gamma^+ = \partial D_{\Omega^+}$.

The results from section 3.1 apply and for the $v_k$ defined in that section,

$$\alpha v_k(y) = u(z_k),$$

but since we don’t know $\Omega$, we cannot find $v_k$. However, by Theorem 1, $v_k \leq 0$ on $\Gamma^+$, since $\Gamma^+ \subset D_{\Omega}$. If we consider the difference $w = v_k - v^+_k$, we see that $w$ satisfies (15), and Theorem 1 implies that $w \leq 0$. This maximum principle gives us that for $\alpha \geq 0$,

$$\alpha v^+_k(y) \geq \alpha v_k(y) = u(z_k).$$

Thus,

$$y \in \bigcap_{k=1}^{K} \{x|\alpha v^+_k(x) \geq u(z_k)\}.$$

In the case when $\alpha$ is unknown, we would like to find an $\alpha$ independent set which includes $y$. From section 3.1, for a pair of samples $k$ and $l$, we have

$$\frac{v_k(y)}{v_l(y)} = \frac{u(z_k)}{u(z_l)} \quad \text{and thus, } \frac{v^+_k(y)}{v^+_l(y)} \geq \frac{u(z_k)}{u(z_l)}.$$

Now, let $\Omega^-$ be an under-estimate of the obstacles, so that $\Omega^- \subseteq \Omega$. A simple choice for $\Omega^-$ is the empty set (no obstacles). Also, let $v^-$ satisfy (19) for $\Omega^-$. By the maximum principle, $v_l \geq v^-$. Thus,

$$\frac{v^+_k(y)}{v^-(y)} \geq \frac{u(z_k)}{u(z_l)},$$

which is independent of $\alpha$ as desired.
3.3 Numerical Experiments and the Proposed Algorithm

We model the $\delta$-source as a sharply rescaled Gaussian centered at the prescribed location. To determine the location of the source, we build a probability density as follows. For each measurement, we let the probability density, $p_k(x)$, be constant in the possible region (it may be a curve) and have Gaussian drop-off away from this region. After $k$ different measurements, we let the probability density be

$$p(x) = \frac{\left(\prod_{j=1}^{k} p_j(x)\right)^{\frac{1}{k}}}{\int_{\Omega} \left(\prod_{j=1}^{k} p_j(x)\right)^{\frac{1}{k}} \, dx}.$$

(20)

**Known Strength, Known Environment**

For this experiment, we assume that the source has strength $\alpha = 1$ and use Algorithm 1. We sample $u(x)$ at 3 locations, which results in 9 level sets if we use the level sets given by (16). The domain and results are given in Figure 1.

**Algorithm 1** Source detection in known environment.

1. $u(z)$: solution of equation (14) that can be measured for any $z$.
2. $k = 1$
3. $z_k$: vantage point outside the occluding objects
4. compute $v_k$: solution of Equation (14) and any of the $w_{k1}, \ldots$, that are available
5. compute $p$ as in Equation (20)
6. while $p$ is not localized do
7. \hspace{0.5cm} $k = k + 1$
8. \hspace{0.5cm} chose $z_k$ to be outside of the set \{$x : v_{k-1} > u(z_{k-1})$\}
9. \hspace{0.5cm} compute $v_k$: solution of Equation (14) and any of the $w_{k1}, \ldots$, that are available
10. \hspace{0.5cm} re-compute $p$ as in Equation (20)
11. end while

Alternatively, we can use only $v_k$ along with all pairs (18). The results for 3 measurements (total of 6 level sets after the 3rd) are shown in Figure 2.

**Unknown Strength, Known Environment**

For this case, we assume that the strength $\alpha$ is unknown and use Algorithm 1. We sample $u(x)$ at 3 locations. Since the strength is unknown, we use equations (18). After locating the source, its strength can be approximated using,

$$\alpha = \frac{u(z_k)}{v_k(y)}.$$

(21)
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Fig. 1. Location of a source with known strength in a known environment for Poisson’s equation. Location based on 3 measurements with \( v_k, w_{k1} \) and \( w_{k2} \). Figures: A) The known environment, source and sample (1, 2, 3) locations; B) \( p(x) \) after 1 measurement; C) \( p(x) \) after 2 measurements; D) \( p(x) \) after 3 measurements. Blue is zero probability; Red is high probability.

The results are shown in Figure 3. The actual source location is \((0.200, 0.400)\) and its strength is \( \alpha = 1.817547 \). The probability \( p \) after 3 measurements has a maximum at \((0.208, 0.400)\). The strength estimates (21) are 1.861106, 1.837540, 1.767651, and the average is 1.822099.

Fig. 2. Location of a source with known strength in a known environment for Poisson’s equation. Location based on 3 measurements with \( v_k \) and pairwise combinations. Figures: A) The known environment, source and sample (1, 2, 3) locations; B) \( p(x) \) after 1 measurement; C) \( p(x) \) after 2 measurements; D) \( p(x) \) after 3 measurements. Blue is zero probability; Red is high probability.

Fig. 3. Location of a source with unknown strength in a known environment for Poisson’s equation. Location based on 3 measurements with \( v_k \) pairwise combinations. Figures: A) The known environment, source and sample (1, 2, 3) locations; B) \( p(x) \) after 2 measurements; C) \( p(x) \) after 3 measurements. Blue is zero probability; Red is high probability. Actual parameters: source location \((0.200, 0.400)\) with strength 1.817547. Location results: \((0.208, 0.400)\). Strength estimates, 1) 1.861106, 2) 1.837540, 3) 1.767651. Averaged \( \alpha = 1.822099 \).
In order to detect a source of given strength in an unknown environment the observer utilizes visibility information to proceed through the environment and to narrow down the region of possible source locations. In particular, let $\psi(\cdot; z_k)$ be the visibility level set function corresponding to the vantage point $z_k$. Then, $\{\psi(\cdot; z_k) \geq 0\}$ is the visible portion of $D$ and $\{\psi(\cdot; z_k) < 0\}$ is invisible. Let $\Psi_k$ denote joint visibility along the path. In the level set framework, $\Psi_k = \max_{j=1,\ldots,k}\{\psi(\cdot, z_j)\}$. The remaining occluded set $\{x \in D : \Psi_k(x) < 0\}$ may be used as an over-approximation of the obstacles $\Omega_k^+$. Note, as the observer explores more of $D$, $\Omega_k^+$ becomes a better approximation of $\Omega$.

Thus, the $u(z_k)$ level set of $v_k^+$ would pass closer to the source location $y$.

Furthermore, let $\{q_j\}_{j=1}^M$ be the filtered out visible points on the boundaries of the obstacles, collected along the observer’s path $\{z_1, \ldots, z_k\}$, see [14, 15] for details. To construct an under-approximation of the obstacles $\Omega_k^-$, we take the union of all $\epsilon$-balls $B_\epsilon(q_j)$, touching the visible points, such that $B_\epsilon(q_j) \subseteq \{\Psi_k \leq 0\}$. Dirichlet boundary conditions are enforced at the union of the boundaries of $B_\epsilon(q_j)$. Then, using Theorem 1, we may “sandwich” the location of the source $y \in \{x \in D : v_k^-(x) \leq u(z_k) \leq v_k^+(x)\}$.

As the observer proceeds through the environment, the next step along the path is chosen in the currently visible region, so that the resulting path avoids obstacles and is continuous and consists of a finite number of steps as in [14]. We adopt the algorithm in [14] to navigate through the unknown environment, in which the observer approaches one of the visible horizons, or edges on the piecewise-smooth visibility map, defined in [14]. The next step $z_{k+1}$ is obtained by overshooting the horizon location by the amount inversely proportional to the curvature of the obstacle’s boundary near the horizon.

To optimize the search, we choose a direction so that $z_{k+1} \in \{W_k \geq 0\}$, if the continuity of the path can be preserved. Otherwise, we simply proceed towards the nearest horizon, as was proposed in [14]. The algorithm terminates when the entire set $\{W_k \geq 0\}$ is visible from the vertices along the path. Note, that in most cases the proposed location algorithm would terminate prior to full mapping of the environment. However, if the environment has been fully explored before the source was located, the algorithm for the known environment may be applied.

Finally, we would like to remark that according to [14], the environment is considered to be completely explored when all the horizons detected along the path have been cleared. The observer may return to an earlier vantage point along the path to see other horizons. Therefore, the resulting path may branch out. The complete search strategy is described in Algorithm 2 below.

Note that $v_k^+$ and $v_k^-$ are the level set functions. Then, for a given $k$, the set $\{x \in D : v_k^-(x) \leq u(z_k) \leq v_k^+(x)\}$, containing the source, is defined by another level set function $W_k$, positive in the interior of the set and negative outside. Numerically, $W_k$ is defined in step 10 of the above algorithm. As the observer proceeds through the environment, we take the intersection of all such
Algorithm 2 Source detection in unknown environment. Source strength is known.

1. $u(z)$: solution of equation (14) that can be measured for any $z$.
2. $k = 1$
3. $z_k$: vantage point outside the occluding objects
4. $\psi(.,z_k)$: visibility with respect to $z_k$
5. $\psi_k$: joint visibility along the path
6. construct $\Omega_k^+$: over-estimate of $\Omega$ with respect to $z_k$
7. construct $\Omega_k^-$: under-estimate of $\Omega$ with respect to $z_k$
8. compute $v_k^+$: solution of Equation (14) with obstacles $\Omega^+$
9. compute $v_k^-$: solution of Equation (14) with obstacles $\Omega^-$
10. set $W_k(x) := -(v_k^+(x) - u(z_k))(v_k^-(x) - u(z_k)), x \in D$
11. while $\{W_k \geq 0\} \not\subset \{\psi_k \geq 0\}$ do
12. $k = k + 1$
13. set $\psi_k = \max\{\psi(.,z_k), \psi_{k-1}\}$
14. construct $\Omega_k^+, \Omega_k^-$
15. compute $v_k^+, v_k^-$
16. set $W_k(x) := \min\{-(v_k^+(x) - u(z_k))(v_k^-(x) - u(z_k)), W_{k-1}(x)\}, x \in D$
17. if $\{W_k \geq 0\} \cap \{\psi(.,z_k) > 0\} \neq \emptyset$ then
18. choose $z_k \in \{W_k \geq 0\} \cap \{\psi(.,z_k) > 0\}$
19. else
20. choose $z_k \in \{\psi(.,z_k) > 0\}$ according to the exploration algorithm in [14]
21. end if
22. end while

sets corresponding to each observing location. In the level set framework, this translates to $\min_{j=1,\ldots,k}\{W_j\}$, computed in step 16. Similarly, joint visibility along the path $\psi_k$ is computed as $\max_{j=1,\ldots,k}\{\psi(.,z_j)\}$ in step 13.

Figures 4, 5, and 6 demonstrate the performance of Algorithm 2. In all these figures, the over-approximation of the obstacles $\Omega^+$, based on joint visibility, is depicted by the orange contour, and the under-approximation $\Omega^-$, based on $\epsilon$-balls around the visible boundary points, is depicted by the magenta contour. The $u(z_k)$ level set of $v_k^+$ is shown in green and the $u(z_k)$ level set of $v_k^-$ is shown in blue. The blue region is the set $\{W_k \geq 0\}$. The location of the source is marked by the red star and the path is shown in black, with circles indicating the discrete steps.

Figure 4 shows a simple environment with three disk-shaped obstacles. The source is located at $(0.75, 0.75)$. The observer may not see the source from its initial position at $(-0.82, -0.91)$. The blue region $\{W_1 \geq 0\}$ almost overlaps with the invisible set $\{\psi_1 < 0\}$. The next vantage point is chosen to be inside the blue region. One can see that after two steps the region $\{W_2 \geq 0\}$, containing the source, has shrunk significantly. Finally, after three steps, $u(z_k)$ level sets of $v_k^+$ and $v_k^-$ coincide, and the source is located somewhere on the curve $\{W_3 = 0\}$. Since this set is entirely visible from the observer’s position, the search is complete. Note that the environment has not been entirely explored up to this point.
Step 1

Step 2

Step 3

Fig. 4. Unknown environment, known source strength. The source is located at $(0.75, 0.75)$. Orange contour is boundary of $\Omega^+_k$ and the magenta contour is the boundary of $\Omega^-_k$. The blue region is $W_k \geq 0$, the green contour is the $u(z_k)$ level set of $u^+_k$ and the blue contour is the $u(z_k)$ level set of $v^-_k$.

Figure 5 depicts a much more complex example. Here, the region is constructed from a slice of Grand Canyon elevation data, which has a much more complex geometrical structure comparing to the example with three circles. We further increased the complexity of the Grand Canyon terrain by adding two disk-shaped holes to the interior of the region. The source is concealed in a small bay with coordinates $(0.25, -0.55)$. At step 1 the blue region $\{W_1 \geq 0\}$, containing the source overlaps with the invisible set $\{\psi_1 < 0\}$. Therefore the observer simply approaches the nearest edge to arrive at $z_2$. From now on there is a preferable direction to approach. The next observing position $z_3$ is chosen according to step 18 of the algorithm. Now there are two possible directions to investigate. The observer chooses the nearest one to arrive at $z_4$. Since there are no new horizons at $z_4$, the observer backtracks to $z_3$ and explores the second choice horizon. As the observer approaches the source, the blue region shrinks. At $z_5$ the observer chooses the nearest of three possible horizons. Finally, the entire set of possible source locations is visible from $z_6$ and, therefore, the algorithm terminates. We remark that the source has been found long before the entire environment has been explored.

Finally, Figure 6 depicts the most complicated example. The source is concealed in a small cave at $(0.112, 0.876)$. Steps 1 through 12 are chosen according to the original [14] exploration algorithm, since the sets $\{W_2 \geq 0\}$ and $\{\psi_k \geq 0\}$ coincide for $k = 1, \ldots, 12$. Finally, the observer backtracks to $z_2$ to clear previously unexplored horizons. At $z_{14}$ the set containing the source becomes visible. In this example, the observer must explore almost the entire region to finally locate the source.

4 Conclusion

In this paper, we have developed an algorithm that can locate a source of unknown strength for a generic partial differential operator in a bounded

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1 The terrain data were obtained from:
Fig. 5. Unknown environment, known source strength. The source is located at (0.25, −0.55). Orange contour is boundary of \( \Omega^+ \) and the magenta contour is the boundary of \( \Omega^- \). The blue region is \( W_k \geq 0 \), the green contour is the \( u(z_k) \) level set of \( v^+_k \) and the blue contour is the \( u(z_k) \) level set of \( v^-_k \).
Fig. 6. Unknown environment, known source strength. The source is located at $(0.112, 0.876)$. Orange contour is boundary of $\Omega^+_k$ and the magenta contour is the boundary of $\Omega^-_k$. The blue region is $W_k \geq 0$, the green contour is the $u(z_k)$ level set of $v^+_k$ and the blue contour is the $u(z_k)$ level set of $v^-_k$. Steps 2 through 11 are skipped since no information regarding the source location is available.
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domain with obstacles. The algorithm relies on the solution of the adjoint problem and the reciprocity that exists between the operator and its adjoint. We have shown examples for the case of Poisson’s equation.

In the case of unknown obstacles, we have proposed a method for locating the source which is based on previous unknown environment exploration methods and relies on the maximum principle to determine a set of possible source locations. This algorithm also works in the case of unknown source strength. Several examples for Poisson’s equation were shown.

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References