POLYNOMIAL PATH ORDERS: A MAXIMAL MODEL

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Abstract. This paper is concerned with the automated complexity analysis of term rewrite systems (TRSs for short) and the ramification of these in implicit computational complexity theory (ICC for short). We introduce a novel path order with multiset status, the polynomial path order $\succ_p^\ast$. Essentially relying on the principle of predicative recursion as proposed by Bellantoni and Cook, its distinct feature is the tight control of resources on compatible TRSs: The (innermost) runtime complexity of compatible TRSs is polynomially bounded. We have implemented the technique, as underpinned by our experimental evidence our approach to the automated runtime complexity analysis is not only feasible, but compared to existing methods incredibly fast.

As an application in the context of ICC we provide an order-theoretic characterisation of the polytime computable functions. To be precise, the polytime computable functions are exactly the functions computable by an orthogonal constructor TRS compatible with $\text{POP}^\ast$.

1. Introduction

As a special form of equational logic, term rewriting has found many applications in automated deduction and verification. Term rewriting is a conceptually simple but powerful abstract model of computation that underlies much of declarative programming, and the automated time complexity analysis of term rewrite systems (TRSs for short) is of particular interest. A natural way to measure the time complexity of a TRS $\mathcal{R}$ is to measure the length $\ell$ of derivations

$$f(v_1, \ldots, v_n) \rightarrow^\mathcal{R} s_1 \rightarrow^\mathcal{R} s_2 \cdots \rightarrow^\mathcal{R} s_\ell = w$$

in terms of the sizes of the initial arguments $v_1, \ldots, v_n$. Maybe surprisingly, this unitary cost model is polynomially invariant $[7, 18]$: the result $w$ of $f(v_1, \ldots, v_n)$ can be computed on a conventional model of computation in time polynomial in $\ell$. Runtime complexity analysis is an active research area in rewriting. See [32] for a broad overview in this research field. Since the feasible functions are often associated with the polytime computable functions, estimating polynomial bounds is of particular interest. Virtually all methods developed in this field go back to termination techniques. Termination of rewrite systems has been studied extensively, and majored to a state where it has become practical to study the...
termination of *real world programs* by translations to rewrite systems. Source languages cover not only functional programs (see for instance [27] that studies Haskell), but also logic (c.f. [35] or [43] for Prolog programs) and imperative programs (for Java™ bytecode in [36] and recently [15]). This trend is also reflected in the annual termination competition (TERMCOMP) that features dedicated categories for all mentioned programming languages. Verifying that such translations are complexity preserving, rewriting can provide a *unified backend* for complexity analysis of programs, written in different languages and different paradigms.

It is clear that *reduction orders*, for instance *polynomial interpretations* and *recursive path orders* not only verify termination but also bind the length of reductions. For instance, the longest possible rewrite sequence in polynomial terminating TRSs is double-exponentially bounded in the size of the initial term, cf. [25]. Similar, *multiset path orders* (MPO for short) induce primitive recursive complexity [24], the induced bound for the Knuth-Bendix order is two-recursive [31] and for *lexicographic path orders* it is even multiply recursive [44]. In a modern termination prover, these orders play a fundamental role in their combination with transformation techniques like *semantic labeling* [47] and the *dependency pair method* [3]. Based on a careful analysis of the induced derivational complexity [33], Schnabl conjectures

> the derivational complexity of any rewrite system that can be proven terminating using a recent termination prover is bounded by a multiply recursive function.

With our tool TCT, the *Tyrolean complexity tool* [2], we have demonstrated that a termination prover, employing only suitable miniaturised termination techniques, can form a powerful complexity analyser. TCT puts special focus on proving polynomial bounds on the runtime (respectively derivational) complexity of TRSs. However, it is worth emphasising that the most powerful techniques for polynomial runtime complexity analysis currently available, basically employ semantic considerations on the rewrite systems, which are notoriously inefficient. We just mention very recently work on a miniaturisation of matrix interpretations due to Middeldorp et al. [30]. Recent breakthroughs in complexity analysis have also been achieved with the development of variations of dependency pairs [22, 23, 34] as well as modularity results [46].

1.1. **Motivation and Contributions.** To overcome the notorious inefficiency of semantic techniques in runtime complexity analysis we aim at a syntactic method to analyse polynomial runtime complexity of rewrite systems. A suitable starting point for such an analysis is given by the multiset path order MPO. MPO not only induces primitive recursive bounds on the length of derivations, it even characterises the primitive recursive functions [17]: any function computed by an MPO-terminating TRS is primitive recursive, vice versa, any primitive recursive function can be stated as an MPO-terminating TRS.

It is well known that the principles of *data tiering* introduced by Simmons [41] and Leivant [28] can be used to characterise small complexity classes like FP in a purely syntactic manner. In particular Bellantoni and Cook [13] embodies the principle of *predicative recursion*, a form of tiering, on the definition of the primitive recursive functions, resulting in a recursion theoretic characterisation of FP. The here proposed *polynomial path order*
(POP\(^*\) for short), embodies the principle of predicative recursion onto MPO, with the distinctive feature that POP\(^*\) induces polynomial bounds on the length of derivations. To motivate this order, let us first recapitulate central ideas of [13]. For each function \( f \), the arguments to \( f \) are separated into normal and safe ones. To highlight this separation, we write \( f(x; y) \) where arguments to the left of the semicolon are normal, the remaining ones are safe. Bellantoni and Cooks define a class \( B \), consisting of a small set of initial functions and that is closed under safe composition and safe recursion on notation (safe recursion for brevity). The crucial ingredient in \( B \) is that a new function \( f \) is defined via safe recursion by the equations

\[
\begin{align*}
  f(0, x; y) &= g(x; y) \\
  f(2z + i, x; y) &= h_i(z, x; y, f(z, x; y)) & i \in \{1, 2\}
\end{align*}
\]

(SRN)

for functions \( g, h_1 \) and \( h_2 \) already defined in \( B \). Unlike primitive recursive functions, the stepping functions \( h_i \) cannot perform recursion on the impredicative value \( f(z, x; y) \). This is a consequence of data tiering. Recursion is performed on normal, and recursively computed result are substituted into safe argument position. To maintain the separation, safe composition restricts the usual composition operator so that safe arguments are not substituted into normal argument position. Precisely, for functions \( h, r \) and \( s \) already defined in \( B \), a function \( f \) is defined by safe composition using the equation

\[
f(x; y) = h(r(x); s(x; y)) .
\]

(SC)

Crucially, the safe arguments \( y \) are absent in normal arguments to \( h \). The main result from [13] states that \( B = \text{FP} \).

Polynomial path orders enforce safe recursion on compatible TRSs. In order to employ the separation of normal and safe arguments, we fix for each defined symbol a partitioning of argument positions into normal and safe positions. For constructors we fix that all argument positions are safe. Moreover POP\(^*\) restricts recursion to normal argument. Dual only safe argument positions allow the substitution of recursive calls. Via the order constraints we can also guarantee that functions are composed in a safe manner. This syntactic account of predicative recursion delineates a class of rewrite systems: a rewrite system \( R \) is called predicative recursive if \( R \) is compatible with POP\(^*\). For motivation consider the TRS \( R_{\text{sat}} \) given in Example 1.1 the that encodes the function problem FSAT associated to the well-known satisfiability problem SAT. Notably FSAT is complete for the class of function problems over NP (FNP for short), compare [37].

**Example 1.1.** The TRS \( R_{\text{sat}} \) is defined as follows. A conjunctive normal form is encoded as a list of non-empty clauses, clauses being lists of literals, in the obvious way. Lists are constructed as usual from the constant \( [\] \) and the binary constructor \( (,) \). Literals are encoded as binary strings (build from the \( \varepsilon \), \( 0 \) and \( 1 \)) with the most significant bit reserved for its plurality. The TRS \( R_{\text{sat}} \) contains a conditional

\[
\text{if}(:,:,t,e) \rightarrow t \quad \text{if}(:,:,f,t,e) \rightarrow e
\]

and defines negation

\[
\text{neg}(:,:,1(x)) \rightarrow 0(x) \quad \text{neg}(:,:,0(x)) \rightarrow 1(x)
\]

as well as equality:

\[
\begin{align*}
  \text{eq}(0(x); 0(y)) &\rightarrow \text{eq}(x; y) \\
  \text{eq}(0(x); 1(y)) &\rightarrow \text{ff} \\
  \text{eq}(1(x); 1(y)) &\rightarrow \text{eq}(x; y) \\
  \text{eq}(1(x); 0(y)) &\rightarrow \text{ff} \\
  \text{eq}(\varepsilon;\varepsilon) &\rightarrow \text{tt}
\end{align*}
\]
A list of literals is \textit{consistent} if an atom does not occur positively and negatively.

\[
\text{consistent}([]); \rightarrow \text{tt} \quad \text{consistent}(l: ls; ) \rightarrow \text{if} (\text{member}(\text{neg}(l), ls; ), \text{ff}, \text{consistent}(ls; )) \\
\text{member}(x, [ ]; ) \rightarrow \text{ff} \quad \text{member}(x, y: ys; ) \rightarrow \text{if} (\text{eq}(x, y), \text{tt}, \text{member}(x, ys; ))
\]

The computed assignment will be a consistent list of literals. Note that a satisfying assignment and verify whether it is consistent.

Here \text{choice} selects nondeterministically a literal from a clause. This concludes the definition of $\mathcal{R}_{\text{sat}}$.

It can be verified that $\mathcal{R}_{\text{sat}}$ is compatible with the multiset path order $\succ_{\text{mpo}}$ with underlying precedence $\succ$ satisfying

\[
\text{guess} > \text{choice} \quad \text{eq} > \text{tt, ff} \quad \text{sat} > \text{sat'}, \text{guess} \\
\text{member} > \text{tt, ff} \quad \text{consistent} > \text{if, member, neg, tt, ff} \quad \text{sat'} > \text{if, consistent, unsat} \quad \text{neg} > 0, 1
\]

Using the separation of argument positions as indicated in the rules, where in the spirit of $\mathcal{B}$ constructors admit only safe arguments, we can even prove compatibility with $\succ_{\text{pop}^*}$ based on the same precedence, i.e., $\mathcal{R}_{\text{sat}}$ is predicative recursive.

Note that $\mathcal{R}_{\text{sat}}$ does not rigidly follow safe recursion $\mathcal{SRN}$ and safe composition $\mathcal{SC}$. Notably values are formed from an arbitrary algebra and are not restricted to words. Also $\succ_{\text{pop}^*}$ allows in principle arbitrary deep right-hand sides. Still the main principle, namely prohibition of recursion on impredicative values, remains reflected. In total, we establish following results.

\textbf{Automated Runtime Complexity Analysis of TRSs:} We establish that for predicative recursive TRSs $\mathcal{R}$, the (innermost) runtime complexity function is polynomially bounded. To the best of our knowledge, the polynomial path order is the first purely syntactic approach that establishes feasible bound on the runtime complexity of TRSs. We have implemented the here proposed techniques in $\text{TCT}$. The experimental evidence obtained indicates the viability of the method.

For the predicative recursive TRS $\mathcal{R}_{\text{sat}}$ from Example \ref{example:pred-rec} this result implies that the number of rewrite steps starting from $\text{sat}(c; )$ is polynomially bounded in the size of the CNF $c$. This can even be automatically verified\footnote{To our best knowledge $\text{TCT}$ is currently the only complexity tool that can provide a complexity certificate for the TRS $\mathcal{R}_{\text{sat}}$, compare \url{http://termcomp.uibk.ac.at}}. Due to the polynomial invariance theorem \ref{thm:pol-invar} we can thus that FSAT belongs to $\text{FNP}$.

\textbf{Resource free characterisation of FP:} The class of predicative recursive rewrite systems entail new \textit{order-theoretic} characterisation of FP, the polytime computable functions. This bridges the gap to implicit computational complexity (ICC for short) theory.

POP* is \textit{sound} for FP, i.e., (confluent and) predicative recursive TRSs compute only polytime computable functions. Moreover we can also prove that predicative recursive
TRSs are complete for \( FP \), in the sense that every polytime computable function \( f \) is defined by a (orthogonal and) predicative recursive TRS \( \mathcal{R}_f \).

**Parameter Substitution:** We extend upon \( \text{POP}^\ast \) by proposing a generalisation \( \text{POP}^\ast_{\text{PS}} \), admitting the same properties as outlined above, but that allows to handle more general recursion schemes that make use of parameter substitution. As a corollary to this and the fact that the runtime complexity of a TRS forms an invariant cost model we conclude a non-trivial closure property of Bellantoni and Cooks definition of the feasible functions.

The present article collects our ongoing work on polynomial path orders. The order \( \text{POP}^\ast \) has been introduced first in \([4]\), extended to quasi-precedences in \([9]\) and the extension \( \text{POP}^\ast_{\text{PS}} \) appeared first in the Workshop on Termination of 2009 \([6]\). Apart from the usual corrections of technicalities, we make here the following new contributions:

- The presented definition of \( \text{POP}^\ast \) is more liberal and captures predicative recursion more precisely, compare \([4, \text{Definition 4}]\) and Definition 3.3 from Section 3.
- To show that \( \text{POP}^\ast \) is sound for \( FP \), we relied in \([4]\) on a certain typing of constructors that guaranteed that sizes of values are polynomial in their depth. In particular, the typing prohibited tree structures a priori. Our new soundness result (c.f Theorem 7.1 from Section 7) is more general and permits arbitrary values.
- The propositional encoding used in our automation of polynomial path orders (c.f. Section 9) has been considerably overhauled.

1.2. Related Work. There are several accounts of predicative analysis of recursion in the (ICC) literature. We mention only those related works which are directly comparable to our work. See \([11]\) for an overview on ICC. The mental predecessor of \( \text{POP}^\ast \) is the path order for \( FP \) as put forward in \([2]\). Our main motivation lies in the observation that this order is directly only applicable to a handful of simple TRSs. This order only gains power if addition transformations are performed. But unfortunately powerful transformations are difficult to find automatically.

Notable the clearest connection of our work is to Marion’s light multiset path order \((\text{LMPO} \text{ for short})\) \([29]\). This path order forms a strict extension of the here proposed order \( \text{POP}^\ast \). Similar to \( \text{POP}^\ast \) it characterises \( FP \). As exemplified below however, compatible TRSs do not admit polynomially bounded runtime complexity in general. This renders LMPO non-usable in our complexity analyser \( \text{TCT} \). The definition of \( \text{POP}^\ast \) has been calibrated with some effort to prevent such behaviour.

**Example 1.2.** The TRS \( \mathcal{R}_{\text{bin}} \) is given by the following rules:

\[
\text{bin}(x, 0 ;) \rightarrow s(0) \quad \text{bin}(0, s(y);) \rightarrow 0 \quad \text{bin}(s(x), s(y);) \rightarrow (+; \text{bin}(x, s(y);), \text{bin}(x, y;))
\]

For a precedence \( \succ \) that fulfils \( \text{bin} \succ s \) and \( \text{bin} \succ + \) we obtain that \( \mathcal{R}_{\text{bin}} \) is compatible with LMPO. However it is straightforward to verify that the family of terms \( \text{bin}(s^n(0), s^m(0)) \) admits (innermost) derivations whose length grows exponentially in \( n \). Still the underlying function can be proven polynomial, essentially relying on memoisation techniques, c.f. \([29]\).

The result of our main theorem can also be obtained if one considers polynomial interpretations, where the interpretations of constructor symbols is restricted. Such restricted polynomial interpretations are called additive in \([14]\). Note that additive polynomial interpretations also characterise the functions computable in polyme, cf. \([14]\). Although incomparable to our technique, unarguably such semantic techniques admit a better intensionality, but are difficult to implement efficiently in an automated setting. In our
complexity tool \( T_C \), we see POP\(^*\) as a fruitful and fast extension that handles systems in a fraction of a second.

We also want to mention recent approaches for the automated analysis of resource usage in programs. Notably, Hoffmann et al. [26] provide an automatic multivariate amortised cost analysis exploiting typing, which extends earlier results on amortised cost analysis. To indicate the applicability of our method we have employed a straightforward (and complexity preserving) transformation of the RAML programs considered in [26] into TRSs. Equipped with POP\(^*\) our complexity analyser \( T_C \) can handle all examples from [26]. Finally Albert et al. [1] present an automated complexity tool for Java™ Bytecode programs and Gulwani et al. [21] as well as Zuleger et al. [48] provide an automated complexity tool for C programs.

1.3. Outline. The remainder of this paper is organised as follows. In the next section we recall basic notions and starting points of this paper. In Section 3 we introduce polynomial path orders along with our main result. In the subsequent Sections 4–6 we show that the (innermost) runtime-complexity of predicative recursive TRSs is polynomially bounded: in Section 4 we set the stage by introducing a notion of predicative interpretation; in Section 5 we present an extended version of the aforementioned path order on sequences [2], and we show that our extension is still sound (c.f. Corollary 5.16); section 6 finally shows that predicative interpretations embed derivations into the order on sequences, establishing our central argument.

In Section 7 we then present our ramification of polynomial path orders in ICC. Parameter substitution is incorporated in Section 8. Our implementation is detailed in Section 9 and ample experimental evidence is provided in Section 10. Finally, we conclude and present future work in Section 11.

2. Preliminaries

We denote by \( \mathbb{N} \) the set of natural numbers \( \{0, 1, 2, \ldots \} \). Let \( R \) be a binary relation. The transitive closure of \( R \) is denoted by \( R^+ \) and its transitive and reflexive closure by \( R^* \). For \( n \in \mathbb{N} \) we denote by \( R^n \) the \( n \)-fold composition of \( R \). The binary relation \( R \) is well-founded if there exists no infinite chain \( a_0, a_1, \ldots \) with \( a_i R a_{i+1} \) for all \( i \in \mathbb{N} \). Moreover, we say that \( R \) is well-founded on a set \( A \) if there exists no such infinite chain with \( a_0 \in A \). The relation \( R \) is finitely branching if for all elements \( a \), the set \( \{ b \mid a R b \} \) is finite.

A proper order is an irreflexive and transitive binary relation. A preorder is a reflexive and transitive binary relation. An equivalence relation is reflexive, symmetric and transitive. For a preorder \( \geq \), we denote the induced equivalence relation by \( \sim \) and induced proper order by \( > \).

A multiset is a collection in which elements are allowed to occur more than once. We denote by \( \mathcal{M}(A) \) the set of multisets over \( A \) and write \( \{a_1, \ldots, a_n\} \) to denote multisets with elements \( a_1, \ldots, a_n \). We use \( m_1 \uplus m_2 \) for the summation and \( m_1 \setminus m_2 \) for difference on multisets \( m_1 \) and \( m_2 \). The multiset extension \( R^\text{mul} \) of a relation \( R \) on \( A \) is the relation on \( \mathcal{M}(A) \) such that \( M_1 R^\text{mul} M_2 \) if there exists multisets \( X, Y \in \mathcal{M}(A) \) satisfying

1. \( M_2 = (M_1 \setminus X) \uplus Y \),

2. \( \emptyset \notin X \subseteq M_1 \) and

3. for all \( y \in Y \) there exists an element \( x \in X \) such that \( x R y \).
In order to cleanly extend this definition to preorders and equivalences, we follow [20]. Let \( \sim \) denote an equivalence relation over the set \( A \) and let \( \succeq = \succeq \cup \sim \) be a binary relation over \( A \) so that \( \succeq \) and \( \sim \) are compatible in the following sense: \( \sim \cdot \succeq \cdot \sim \subseteq \succeq \). Let \([a]_\sim\) denotes the equivalence class of \( a \in A \) with respect to \( \sim \). By the compatibility requirement, the extension \( \succ \) of \( \succeq \) to equivalence classes such that \([a]_\sim \succeq [b]_\sim\) if and only if \( a \succ b \), is well defined. We define the strict multiset extension \( \succ^\text{mul} \) of \( \succ \) as \( M_1 \succ^\text{mul} M_2 \) if and only if \([M_1]_\sim \succ^\text{mul} [M_2]_\sim\). Further, the weak multiset extension \( \succ^\text{mul} \) of \( \succ \) is given by \( M_1 \succ^\text{mul} M_2 \) if and only if \([M_1]_\sim \succ^\text{mul} [M_2]_\sim\) or \([M_1]_\sim = [M_2]_\sim\) holds. Note that if \( \succeq \) is a preorder (on \( A \)) then \( \succ^\text{mul} \) is a proper order and \( \succ^\text{mul} \) a preorder on \( M A \), cf. [20]. Also \( \succ^\text{mul} \) is well-founded if \( \succeq \) is well-founded.

2.1. Complexity Theory. We assume modest familiarity in complexity theory, notations are taken from [37]. The functional problem \( F_R \) associated with an binary relation \( R \) is: given \( x \) find some \( y \) such that \((x,y) \in R\) holds if \( y \) exists, otherwise return no. A binary relation \( R \) on words is called polynomial balanced if for all \((x,y) \in R\), the size of \( y \) is polynomially bounded in \( x \). The relation \( R \) is polytime decidable if \((x,y) \in R\) is decided by a deterministic Turing machine (TM for short) \( M \) operating in polynomial time. The class \( \text{NP} \) is the class of languages \( L \) admitting polynomially balanced, polytime decidable relations \( R_L \) [37, Chapter 9]: \( L = \{ x \mid (x,y) \in R_L \text{ for some } y \} \). The class \( \text{FNP} \) is the class of function problems associated with \( \text{NP} \) in the above way. The class of polynomial time computable functions \( \text{FP} \) (polytime computable for short) is the subclass resulting if we only consider function problems in \( \text{FNP} \) that can be solved in polynomial time [37, Chapter 10].

We say that a function problem \( F \) reduces to a function problem \( G \) if there exist functions \( s \) and \( r \), both computable in logarithmic space, such that for all \( x, y \) with \( F \) computing \( y \) on input \( x \), \( G \) computes on input \( s(x) \) the output \( z \) with \( r(z) = y \). Note that both \( \text{FNP} \) and \( \text{FP} \) are closed under reductions. We also note that nondeterministic Turing machines running in polynomial time compute function problems from \( \text{FNP} \).

**Proposition 2.1.** Let \( N \) be a nondeterministic Turing machine that computes the function problem \( F \) in polynomial time. Then \( F \in \text{FNP} \).

**Proof.** Define the following relation \( R \): \((x,y) \in R\) if and only if \( y \) is the encoding of an accepting computation of \( N \) on input \( x \). Since \( N \) operates in polynomial time, \( R \) is polynomially balanced, as it can be checked in linear time that \( y \) encodes an accepting run of \( N \) on input \( x \), \( R \) is polytime decidable. Hence the functional problem \( F_R \) that computes an accepting run \( y \) of \( N \) on input \( x \) is in \( \text{FNP} \). Finally notice that \( F \) reduces to \( F_R \). To see this, employ following reduction: the function \( r \) is simply the identity function; the logspace computable function \( s \) extracts the result of \( N \) on input \( x \) from the accepting run \( y \) computed by \( F_R \) on input \( x \). We conclude the lemma since \( \text{FNP} \) is closed under reductions. \( \square \)

2.2. Term Rewriting. We assume at least nodding acquaintance with the basics of term rewriting [10]. We fix the bare essential of notions and notation that are used in the paper.

Throughout the paper, we fix a countably infinite set of variables \( V \) and a finite signature \( \mathcal{F} \) of function symbols. The signature \( \mathcal{F} \) is partitioned into defined symbols \( D \) and constructors.
C. The set of values, basic terms and terms is defined according to the grammar

\[
\begin{align*}
\text{(Values)} & \quad T(\mathcal{C}, \mathcal{V}) \ni v := x | c(v_1, \ldots, v_n) \\
\text{(Basic Terms)} & \quad t_0(\mathcal{F}, \mathcal{V}) \ni s := x | f(v_1, \ldots, v_n) \\
\text{(Terms)} & \quad T(\mathcal{F}, \mathcal{V}) \ni t := x | c(t_1, \ldots, t_n) | f(t_1, \ldots, t_n)
\end{align*}
\]

where \( x \in \mathcal{V}, c \in \mathcal{C}, \) and \( f \in \mathcal{D}. \)

The arity of a function symbol \( f \in \mathcal{F} \) is denoted by \( \text{ar}(f) \). We write \( s \triangleright t \) if \( t \) is a subterm of \( s \), the strict part of \( \triangleright \) is denoted by \( \triangleright \). The size of a term \( t \) is denoted by \( |t| \) and refers to the number of variables and function symbols contained in \( t \). We denote by \( \text{dp}(t) \) the depth of \( t \) which is defined as \( \text{dp}(t) = 1 \) if \( t \in \mathcal{V} \) and \( \text{dp}(f(t_1, \ldots, t_n)) = 1 + \max\{\text{dp}(t_i) \mid i = 1, \ldots, n\} \). Here we employ the convention that the maximum of an empty set is equal to 0.

Let \( \triangleright \) be a preorder on the signature \( \mathcal{F} \), called quasi-precedence or simply precedence. We always write \( > \) for the induced proper order and \( \sim \) for the induced equivalence on \( \mathcal{F} \). We lift the equivalence \( \sim \) to terms modulo argument permutation: \( s \sim t \) if either \( s = t \) or \( s = f(s_1, \ldots, s_n) \) and \( t = g(t_1, \ldots, t_n) \) where \( f \sim g \) and for some permutation \( \pi, s_i \sim t_{\pi(i)} \) for all \( i \in \{1, \ldots, n\} \). Further we write \( s \triangleright \sim t \) if \( t \) is a subterm of \( s \) modulo \( \sim \), i.e., \( s \triangleright \sim_t \).

Let \( \triangleright \) denote by \( F^{<f} := \{g \mid f \triangleright_g \} \) the set of function symbols below \( f \) in the precedence \( \triangleright \). This notion is extended to sets \( F \subseteq \mathcal{F} \) by \( F^{<f} := \bigcup_{f \in F} F^{<f} \). The rank of a function symbol is inductively defined by \( \text{rk}(f) = \max\{1 + \text{rk}(g) \mid f \triangleright g\} \).

A rewrite rule is a pair \((l, r)\) of terms, in notation \( l \to r \), such that \( l \) is not a variable and all variables in \( r \) occur also in \( l \). Here \( l \) is called the left-hand, and \( r \) the right-hand side of \( l \to r \). A term rewrite system (TRS for short) \( \mathcal{R} \) over \( T(\mathcal{F}, \mathcal{V}) \) is a set of rewrite rules. In the following, \( \mathcal{R} \) always denotes a TRS. If not mentioned otherwise, we assume \( \mathcal{R} \) is finite. A relation on \( T(\mathcal{F}, \mathcal{V}) \) is a rewrite relation if it is closed under contexts and closed under substitutions. The smallest rewrite relation that contains \( \mathcal{R} \) is denoted by \( \to_{\mathcal{R}} \).

A term \( s \in T(\mathcal{F}, \mathcal{V}) \) is called a normal form if there is no \( t \in T(\mathcal{F}, \mathcal{V}) \) such that \( s \to_{\mathcal{R}} t \). With \( \text{NF}(\mathcal{R}) \) we denote the set of all normal forms of a TRS \( \mathcal{R} \). Whenever \( t \) is a normal form of \( \mathcal{R} \) we write \( s \to_{\mathcal{R}}^* t \) for \( s \to_{\mathcal{R}} \). The innermost rewrite relation, denoted as \( \to^*_\mathcal{R} \), is the restriction of \( \to_{\mathcal{R}} \) where arguments are normal forms. The TRS \( \mathcal{R} \) is terminating if no infinite rewrite sequence exists. The TRS \( \mathcal{R} \) has unique normal forms if for all \( s, t_1, t_2 \in T(\mathcal{F}, \mathcal{V}) \) with \( s \to_{\mathcal{R}}^* t_1 \) and \( s \to_{\mathcal{R}}^* t_2 \) we have \( t_1 = t_2 \). The TRS \( \mathcal{R} \) is called confluent if for all \( s, t_1, t_2 \in T(\mathcal{F}, \mathcal{V}) \) with \( s \to_{\mathcal{R}}^* t_1 \) and \( s \to_{\mathcal{R}}^* t_2 \) there exists a term \( u \) such that \( t_1 \to_{\mathcal{R}}^* u \) and \( t_2 \to_{\mathcal{R}}^* u \). An orthogonal TRS is a left-linear and non-overlapping TRS \([10]\). Note that every orthogonal TRS is confluent. The TRS \( \mathcal{R} \) is a constructor TRS if all left-hand sides are basic terms. A defined function symbol is completely defined (with respect to \( \mathcal{R} \)) if it does not occur in any term in normal form, i.e., functions are reducible on all terms. The TRS \( \mathcal{R} \) is completely defined if each defined symbol is completely defined.

2.3. Rewriting as Computational Model. We fix call-by-value semantics and only consider constructor TRSs \( \mathcal{R} \). Input and output are taken from the set of values \( T(\mathcal{C}, \mathcal{V}) \), and defined symbols \( f \in \mathcal{D} \) denote computed functions. More precise, a (finite) computation of \( f \in \mathcal{D} \) on input \( v_1, \ldots, v_n \in T(\mathcal{C}, \mathcal{V}) \) is given by innermost reductions

\[
f(v_1, \ldots, v_n) = t_0 \rightarrow_{\mathcal{R}} t_1 \rightarrow_{\mathcal{R}} \cdots \rightarrow_{\mathcal{R}} t_\ell = w.
\]

If the above computation ends in a value, i.e., \( w \in T(\mathcal{C}, \mathcal{V}) \), we also say that \( f \) computes on input \( v_1, \ldots, v_n \) in \( \ell \) steps the value \( w \). To also account for nondeterministic computation, we capture semantics of \( \mathcal{R} \) by assigning to each \( n \)-ary defined symbol \( f \in \mathcal{D} \) an \( n + 1 \)-ary
relation \([f]\) that maps input arguments \(v_1, \ldots, v_n\) computed values \(w\). A finite set \(N\) of non-accepting patterns is used to distinguish meaningful outputs \(w\) from outputs that should not be considered part of the computation. A value \(w\) is accepting with respect to \(N\) if there exists no \(p \in N\) and substitution \(\sigma\) such that \(p\sigma = w\) holds. A typical example of a meaningful value that should not be accepted is the constant \(\text{unsat}\) appearing in the TRS \(\mathcal{R}_{\text{sat}}\) from Example 1.1. Below functional problem are extended to \(n + 1\)-ary relations in the obvious way.

**Definition 2.2.** Let \(N\) be a set of non-accepting patterns. For each \(n\)-ary symbol, \(f \in D\) the TRS \(\mathcal{R}\) the relation \([f] \subseteq T(C, V)^{n+1}\) defined by \(f\) is given by

\[
(v_1, \ldots, v_n, w) \in [f] \iff f(v_1, \ldots, v_n) \xrightarrow{1} \mathcal{R} w \text{ and } w \text{ is accepting}.
\]

We say that \(\mathcal{R}\) computes the functional problems associated with \([f]\).

Note that if \(\mathcal{R}\) is confluent, then \([f]\) is in fact a (partial) function. Following [7, 22] we adopt an unitary cost model for rewriting, where each reduction step accounts for one time unit. Reductions are of course measured in the size of the input.

**Definition 2.3.** The (innermost) runtime complexity function \(rc_{\mathcal{R}} : N \rightarrow N\) relates sizes of basic terms \(f(v_1, \ldots, v_n) \in T_0(F, V)\) to the maximal length of computation. Formally

\[
rc_{\mathcal{R}}(n) := \max\{\ell \mid \exists s \in T_0(F, V), |s| \leq n \text{ and } f(v_1, \ldots, v_n) = t_0 \mathcal{R} t_1 \mathcal{R} \cdots \mathcal{R} t_\ell\}.
\]

The restriction \(s \in T_0(F, V)\) accounts for the fact that computations start only from basic terms. We sometimes use \(dh(s) := \max\{\ell \mid \exists t. s \mathcal{R} t\}\) to refer to the derivation height of a single term \(s\). Note that the runtime complexity function is well-defined if \(\mathcal{R}\) is terminating, i.e., \(\mathcal{R}\) is well-founded. If \(rc_{\mathcal{R}}\) is asymptotically bounded from above by a linear, quadratic, ..., polynomial function, we simply say that the runtime of \(\mathcal{R}\) is linear, quadratic, ..., or respectively polynomial. By folklore it is known that rewriting can be implemented with only polynomial overhead if terms grow only polynomial during reductions.

In [7] we have shown that the unitary cost model is reasonable for full rewriting (the deterministic case was proven independently in [8, 18] using essentially the same approach). It is not difficult to see that the central Lemma [7, Lemma 5.9] that estimates the implementation cost of a single rewrite step can be specialised to innermost rewriting. We obtain following proposition by specialising [7, Theorem 6.2] to innermost rewriting.

**Proposition 2.4.** Let \(\mathcal{R}\) be a TRS whose is at least linear. There exists a polynomial \(p_{\mathcal{R}}\) such that for any \(f(v_1, \ldots, v_n) \in T_0(F, V)\) of size up to \(n\),

1. any normal form of \(f(v_1, \ldots, v_n)\) can be computed on a Turing machine in non-deterministic time \(p_{\mathcal{R}}(rc_{\mathcal{R}}(n))\); and
2. some normal form of \(f(v_1, \ldots, v_n)\) is computable on a Turing machine in deterministic time \(p_{\mathcal{R}}(rc_{\mathcal{R}}(n))\).

Hence there are no surprises here. By Proposition 2.4 and Proposition 2.5 we obtain:

**Proposition 2.5.** Let \(\mathcal{R}\) be a rewrite system with polynomial runtime. Then the functional problems associated with \([f]\) defined by \(\mathcal{R}\) are contained in \(\text{FNP}\). If \(\mathcal{R}\) is confluent, i.e. deterministic, then \([f]\) is a (partial) function contained in \(\text{FP}\).

Our choice of adopting call-by-value semantics is rested in the observation that the unitary cost model of unrestricted rewriting often overestimates the runtime complexity of computed functions. This has to do with the unnecessary duplication of redexes.
Example 2.6. Consider the constructor TRS $R_{btree}$ given by the following rules.

\[ 1 : btree(0; ) \rightarrow \text{leaf} \quad 2 : btree(s(n); ) \rightarrow \text{dup}(btree(n)) \quad 3 : \text{dup}(t) \rightarrow \text{node}(t, t). \]

Then for $n \in \mathbb{N}$, $btree(s^n(0))$ computes a binary tree of height $n$ in a linear number of steps. On the other hand, $R_{btree}$ gives also rise to a non-innermost reduction

\[ btree(s^n(0); ) \rightarrow_R \text{dup}(btree(s^{n-1}(0); )) \rightarrow_R \text{node}(btree(s^{n-1}(0); ), btree(s^{n-1}(0); )) \rightarrow_R \ldots \]

obtained by preferring $\text{dup}$ over $btree$. The length of the derivation is however exponential in $n$.

By Proposition 2.3 we obtain $[btree] \in \text{FP}$. As indicated later, our analysis can automatically classify the function $[btree]$ as feasible.

3. The Polynomial Path Order

We arrive at the formal definition of polynomial path order ($\text{POP}^*$ for short). Variants of the here presented definition have been presented in earlier conference publications, see [4, 5, 6].

The order $\text{POP}^*$ essentially embodies the predicative analysis of recursion set forth by Bellantoni and Cook [13]. In $\text{POP}^*$, the separation of argument positions is taken into account in the notion of safe mapping.

Definition 3.1. A safe mapping $\text{safe}$ is a function $\text{safe} : \mathbb{F} \rightarrow 2^\mathbb{N}$ that associates with every $n$-ary function symbol $f$ the set of safe argument positions $\{i_1, \ldots, i_m\} \subseteq \{1, \ldots, n\}$. Argument positions included in $\text{safe}(f)$ are called safe, those not included are called normal and collected in $\text{nm}(f)$. For $n$-ary constructors $c$ we require that all argument positions are safe, i.e., $\text{safe}(c) = \{1, \ldots, n\}$.

We refine term equivalence so that the safe mapping is taken into account.

Definition 3.2. Let $\preceq$ denote a precedence and $\text{safe}$ a safe mapping. We define safe equivalence $\equiv_s$ for terms $s, t \in \text{TERMS}$ inductively as follows: $s \equiv_s t$ if either $s = t$ or $s = f(s_1, \ldots, s_n)$, $t = g(t_1, \ldots, t_n)$, $f \sim g$ and there exists a permutation $\pi$ such that for all $i \in \{1, \ldots, n\}$, $s_i \equiv_s t_{\pi(i)}$ and $i \in \text{safe}(f)$ if and only if $\pi(i) \in \text{safe}(g)$.

To avoid notational overhead, we suppose that for each $k+l$ ary function symbol $f$, the first $k$ argument positions are normal, and the remaining argument positions are safe, i.e., $\text{safe}(f) = \{k + 1, \ldots, k + l\}$. This allows use to write $f(s_1, \ldots, s_k; s_{k+1}, \ldots, s_{k+l})$ where the separation of safe from normal arguments is directly indicated in terms.

Let $\succeq$ denote a quasi-precedence. We require that the precedence adheres the partitioning of $\mathbb{F}$ into defined symbols and constructors in the following sense. Then in particular $\equiv_s$ preserves values, i.e., if $s \in T(C, V)$ and $s \equiv_s t$ then also $t \in T(C, V)$.

Definition 3.3. A precedence $\succeq$ is admissible (for $\text{POP}^*$) if $f \sim g$ implies that either both $f$ and $g$ are defined symbols, or both are constructors.

The following definition introduces an auxiliary order $>_{\text{pop}}$, the full order $>_{\text{pop}*}$ is then presented in Definition 3.5.

Definition 3.4. Let $\succeq$ denote a precedence and $\text{safe}$ a safe mapping. Consider terms $s, t \in \mathbb{T}(F, V)$ such that $s = f(s_1, \ldots, s_k; s_{k+1}, \ldots, s_{k+l})$. Then $s >_{\text{pop}} t$ if one of the following alternatives holds:
(1) \( s_i >_{\text{pop}} t \) for some \( i \in \{1, \ldots, k+l\} \) and, if \( f \in D \) then \( i \) is a normal argument position \((i \in \{1, \ldots, k\})\);

(2) \( f \in D, t = g(t_1, \ldots, t_m; t_{m+1}, \ldots, t_{m+n}) \) where \( f > g \) and \( s >_{\text{pop}} t_i \) for all \( i = 1, \ldots, m+n \).

Here we set \( >_{\text{pop}} := >_{\text{pop}} \cup \not\sim \).

Consider a function \( f \) defined by safe composition from \( r \) and \( s \), compare scheme \((5C)\).

The purpose of this auxiliary order is to embody safe composition in the full order \( >_{\text{pop}*} \).

Note that the auxiliary order can orient \( f(\bar{x}; \bar{y}) >_{\text{pop}} r(\bar{x};) \) for defined symbol \( f \) with \( f > r \).

On the other hand, \( f(\bar{x}; \bar{y}) \) and safe arguments \( y_i \) are incomparable, and consequently the orientation of \( f(\bar{x}; \bar{y}) \) and \( s(\bar{x}; \bar{y}) \) fails.

**Definition 3.5.** Let \( \succeq \) denote a precedence and \( \text{safe} \) a safe mapping. Consider terms \( s, t \in T(F, V) \) such that \( s = f(s_1, \ldots, s_k; s_{k+1}, \ldots, s_{k+l}) \). Then \( s >_{\text{pop}*} t \) if one of the following alternatives holds:

(1) \( s_i >_{\text{pop}*} t \) for some \( i \in \{1, \ldots, k+l\} \), or

(2) \( f \in D, t = g(t_1, \ldots, t_m; t_{m+1}, \ldots, t_{m+n}) \) where \( f > g \) and the following conditions hold:

- \( s >_{\text{pop}} t_j \) for all normal argument positions \( j = 1, \ldots, m \);

- \( s >_{\text{pop}*} t_j \) for all safe argument positions \( j = m + 1, \ldots, m + n \);

- \( t_j \not\in T(F^\text{Fun}(s), V) \) for at most one safe argument position \( j \in \{m + 1, \ldots, m + n\} \);

(3) \( f \in D, t = g(t_1, \ldots, t_m; t_{m+1}, \ldots, t_{m+n}) \) where \( f \sim g \) and the following conditions hold:

- \( \llbracket s_1, \ldots, s_k \rrbracket >_{\text{pop}*} \llbracket t_1, \ldots, t_m \rrbracket \);

- \( \llbracket s_{k+1}, \ldots, s_{k+l} \rrbracket >_{\text{pop}*} \llbracket t_{m+1}, \ldots, t_{m+n} \rrbracket \).

Here \( >_{\text{pop}*} := >_{\text{pop}} \cup \not\sim \).

We say a constructor TRS \( R \) is *predicative recursive* if \( R \) is compatible with an instance \( >_{\text{pop}*} \) with underlying admissible precedence.

We use the notation \( >_{\text{pop}*}^{(i)} \) and respectively \( >_{\text{pop}}^{(i)} \) to refer to the \( i \)-th case in Definition 3.4 respectively Definition 3.5. We emphasise that \( >_{\text{pop}*} \) is *blind* on constructors, in particular \( >_{\text{pop}*} \) collapses to the subterm relation (modulo equivalence) on values.

**Lemma 3.6.** Suppose the precedence underlying \( >_{\text{pop}*} \) is admissible. If \( s >_{\text{pop}*} t \) and \( s \in T(C, V) \) then \( s >_{\text{pop}} t \), in particular \( t \in T(C, V) \).

The case \( >_{\text{pop}*}^{(3)} \) accounts for definitions by safe composition \((5C)\). The final restriction put onto \( >_{\text{pop}*}^{(4)} \) is used to prevent multiple recursive calls as indicated in Example 1.2. We remark that restrictions put onto \( >_{\text{pop}*}^{(4)} \) are weaker compared to the corresponding clause given in \([4]\). Definition 4. The case \( >_{\text{pop}*}^{(5)} \) restricts the corresponding case in MPO by taking the separation of normal and safe argument positions into account. Note that here normal arguments need to decrease. This reflects that as in \((5R)\) recursion is performed on normal argument positions. We arrive at the central theorem of this paper.

**Theorem 3.7.** Let \( R \) be predicative recursive TRS. Then the innermost derivation height of any basic term \( f(\bar{u}; \bar{v}) \) is bounded by a polynomial in the maximal depth of normal arguments \( \bar{u} \). The polynomial depends only on \( R \) and the signature \( F \). In particular, the runtime complexity of \( R \) is polynomial.

---

4The early definition from \([4]\), Definition 4, used the full order \( >_{\text{pop}*} \) only on one argument of the right-hand side (the one that possibly holds the recursive call), the remaining arguments were all oriented with the auxiliary order \( >_{\text{pop}} \). To retain completeness, in \([4]\) we allowed also the admittedly ad hoc use of a subterm comparison on safe arguments.
The proof of Theorem 3.7 is rather involved, and outlined at the end of this section. The formal proof is then presented in the subsequent Sections 4–6. We clarify first Definition 3.5 on several examples.

Example 3.8. Consider the TRS $R_{\text{mul}}$ expressing multiplication in Peano arithmetic.

1: $+(0; y) \rightarrow y$

2: $+(s(; x) ; y) \rightarrow s(+(x ; y))$

3: $\times(0 ; y) \rightarrow 0$

4: $\times(s(; x) , y) \rightarrow +(y , \times(x , y))$

The TRS $R_{\text{mul}}$ is predicative recursive, using the precedence $\times \triangleright + \triangleright s$ and the safe mapping as indicated in the rules: The rules 1 and 3 are oriented by $>_{\text{pop}}$. The rule 3 is oriented by $>_{\text{pop}}$ using $+$ and $+(s(; x) ; y) >_{\text{pop}} + (x ; y)$. Note that the latter enforces that the first argument to $+$ is normal. Similar, the final rule 4 is oriented by $>_{\text{pop}}$, employing $\times > +$ together with $\times(s(; x) , y) >_{\text{pop}} y$ and $\times(s(; x) , y) >_{\text{pop}} \times(x , y)$. Note that the latter two inequalities require that the both argument positions of $\times$ are normal, i.e., are used for recursion.

Example 3.9. Now consider the extension of $R_{\text{mul}}$ from Example 3.8 by the two rules

5: $\exp(0 , y) \rightarrow s(0)  

6: \exp(s(; x) , y) \rightarrow \times(y , \exp(x , y))$

that express exponentiation $y^x$ in an exponential number of steps. The definition of $\exp$ does not follow predicative recursion, in particular since $\times$ admits no safe argument positions it cannot serve as stepping function. Independent on the safe mapping for $\exp$, rule 6 cannot be oriented using polynomial path orders.

Example 3.10. Finally, for a negative example consider $R_{\text{mul}}$ from Example 3.8 where the rule 4 is replaced by the rule

4a: $\times(s(; x) , y) \rightarrow +(\times(x , y) ; y)$.

The resulting system admits polynomial runtime complexity, but does not follow the rigid scheme of predicative recursion. For this reason, it cannot be handled by POP*. Technically, terms $\times(s(; x) , y)$ and $\times(x , y)$ is incomparable with respect to $>_{\text{pop}}$ independent on the precedence, and consequently also orientation of left- and right-hand side with $>_{\text{pop}}$ fails.

Finally, we stress that the restriction to innermost reductions is essential for the correctness of Theorem 3.7. This has to do with unnecessary duplication of redexes as pointed out in Example 2.6.

Example 3.11. Reconsider the TRS $R_{\text{btree}}$ from Example 2.6. Then $R_{\text{btree}} \subseteq >_{\text{pop}}$ with any admissible precedence satisfying $\text{btree} > \text{dup}$. Theorem 3.7 thus implies that the (innermost) runtime complexity of $R_{\text{btree}}$ is polynomial. On the other hand, we already observed that $R_{\text{btree}}$ admits exponentially long outermost reductions.

Proof Outline. The proof of Theorem 3.7 requires a variety of ingredients. In Section 4, we define predicative interpretations $P_\vartriangleright$ that flatten terms to sequences of terms, essentially separating safe from normal arguments. This allows us to analyse terms independent from safe arguments. In Section 5 we introduce an order $\vartriangleright$ on sequences of terms, that is simpler compared to $>_{\text{pop}}$, and does not rely on the separation of argument positions. In Section 6 we show that predicative interpretations embeds innermost rewrite steps into $\vartriangleright$:
We will tacitly employ $S$ for short) of an element width $\text{wd}$ of $S$.

In Theorem 5.15 we show that the length of $\triangleright$ descending sequences starting from basic terms can be bounded appropriately.

4. Predicative Interpretations

Fix a safe mapping safe on the signature $F$. In this section, we define the predicative interpretation that guided by safe interpret terms as sequences. For this, define the normalised signature $F_n$ be given as

$$F_n := \{ f_n \mid f \in F, \text{nrm}(f) = \{ i_1, \ldots, i_k \} \text{ and } \text{ar}(f_n) = k \}$$

The predicative interpretation of a term $f(s_1, \ldots, s_k; s_{k+1}, \ldots, s_{k+l})$ results in a sequence $[f_n(a_1, \ldots, a_k)] \sim a_{k+1} \sim \cdots \sim a_{k+l}$, where $\sim$ denotes concatenation of sequences and the sequences $a_i$ are predicative interpretations of the corresponding arguments $s_i$ ($i = 1, \ldots, k + l$). To denote sequences, we use an auxiliary variadic function symbol $\circ$. Here variadic means that the arity of $\circ$ is finite but arbitrary. We always write $[t_1 \ldots t_n]$ for $\circ(t_1, \ldots, t_n)$, in particular if we write $f(t_1, \ldots, t_n)$ then $f \neq \circ$. Note that in the interpretations, terms have sequences as arguments. We reflect this in the next definition.

Definition 4.1. The set of terms with sequence arguments $S^*(F, V) \subseteq T(F_n \cup \{ \circ \}, V)$ and the set of sequences $S^*(F, V) \subseteq T(F_n \cup \{ \circ \}, V)$ is inductively defined as follows:

1. $V \in S(F, V)$, and
2. if $t_1, \ldots, t_n \in S(F, V)$ then $[t_1 \ldots t_n] \in S^*(F, V)$, and
3. if $a_1, \ldots, a_n \in S^*(F, V)$ and $f \in F_n$ then $f(a_1, \ldots, a_n) \in S(F, V)$.

We always write $a, b, \ldots$, possibly extended by subscripts, for elements from $S(F, V)$ and $S^*(F, V)$. The restriction of $S(F, V)$ and $S^*(F, V)$ to ground terms is denoted by $S(F)$ and $S^*(F)$ respectively. When no confusion can arise from this we call terms with sequence arguments simply terms. Further, we sometimes abuse set notation and write $b \in [a_1 \ldots a_n]$ if $b = a_i$ for some $i \in \{1, \ldots, n\}$. We denote by $a \sim b$ the concatenation of $a \in S(F, V) \cup S^*(F, V)$ and $b \in S(F, V) \cup S^*(F, V)$. To avoid notational overhead we identify terms with singleton sequences. Let lift$(a) := [a]$ if $a \in S(F, V)$ and lift$(a) := a$ if $a \in S^*(F, V)$. We set $a \sim b := [a_1 \ldots a_n b_1 \ldots b_m]$ where lift$(a) = [a_1 \ldots a_n]$ and lift$(b) = [b_1 \ldots b_m]$. We define the length over $S(F, V) \cup S^*(F, V)$ as $\text{len}(a) := n$ where lift$(a) = [a_1 \ldots a_n]$. The sequence width $\text{wd}$ (or width for short) of an element $a \in S(F, V) \cup S^*(F, V)$ is given recursively by

$$\text{wd}(a) := \begin{cases} 1 & \text{if } a \text{ is a variable}, \\ \max\{1, \text{wd}(a_1), \ldots, \text{wd}(a_n)\} & \text{if } a = f(a_1, \ldots, a_n), \text{ and} \\ \sum_{i=1}^{n} \text{wd}(a_i) & \text{if } a = [a_1 \ldots a_n]. \end{cases}$$

We will tacitly employ $\text{len}(a) \leq \text{wd}(a)$ and $\text{wd}(a \sim b) = \text{wd}(a) + \text{wd}(b)$ for all $a, b \in S(F, V) \cup S^*(F, V)$. We define the norm of $t \in T(F, V)$ in correspondence to the depth of $t$, but disregard normal argument positions.

$$\text{norm}(t) = \begin{cases} 1 & \text{if } t \text{ is a variable} \\ 1 + \max\{\text{norm}(t_{k+1}), \ldots, \text{norm}(t_{k+l})\} & \text{if } t = f(t_1, \ldots, t_k; t_{k+1}, \ldots, t_{k+l}). \end{cases}$$
Note that since all argument positions of constructors are safe, the norm \( \text{norm}(\cdot) \) and depth \( \text{dp}(\cdot) \) coincides on values. Predicative interpretations are given by two mappings \( P_s, P_n : T(F, V) \rightarrow S'(F \cup \{\bullet\}) \) defined as follows:

\[
P_s(t) := \begin{cases} \{ \} & \text{if } t \text{ is a value} \\ \left[ f_n(P_n(t_1), \ldots, P_n(t_k)) \right] \sim P_s(t_{k+1}) \sim \cdots \sim P_s(t_{k+l}) & \text{otherwise where } (*) \end{cases}
\]

\[
P_n(t) := P_s(t) \sim \text{norm}(t).
\]

Here (*) stands for \( t = f(t_1, \ldots, t_k; t_{k+1}, \ldots, t_{k+l}) \).

In the next section we introduce the order \( \bullet \) on sequences \( S(F) \cup S'(F) \). In the subsequent section we then embed innermost \( \mathcal{R} \)-steps into this order, and use \( \bullet \) to estimate the length of reductions accordingly. Since for basic terms \( s = f(u_1, \ldots, u_k; u_{k+1}, \ldots, u_{k+l}) \) in particular

\[
P_s(s) = \left[ f_n(P_n(u_1), \ldots, P_n(u_k)) \right] \sim P_s(u_{k+1}) \sim \cdots \sim P_s(u_{k+l}) = \left[ f_n(\text{dp}(u_1), \ldots, \text{dp}(u_k)) \right]
\]

the so obtained bound will depend on depths of normal arguments only. To get the reader prepared for the definition of \( \bullet \), we exemplify Definition 4.2 on a predicative recursive TRS.

**Example 4.3.** Consider following predicative recursive TRS \( \mathcal{R}_f \) where we suppose that besides \( f \), also \( g \) and \( h \) are defined symbols:

\[
1: \quad f(0; y) \rightarrow y \\
2: \quad f(s(x); y) \rightarrow g(h(x); f(x; y))
\]

Consider a substitution \( \sigma : \text{Var} \rightarrow T(C, V) \). Using that \( P_n(v) = \text{dp}(v) \) for all values \( v \), the embedding \( P_s(l\sigma) \bullet P_s(r\sigma) \) of root steps (\( l \rightarrow r \in \mathcal{R}_f \)) results in the following order constraints.

\[
\left[ f_n(1) \right] \bullet \left[ \right] \quad \text{from rule 1}
\]

\[
\left[ f_n(\text{dp}(x\sigma) + 1) \right] \bullet \left[ g_n(P_n(h(x\sigma);)) \right] \sim \text{norm}(h(x\sigma);)) \sim \left[ h_n(\text{dp}(x\sigma)) \right] \bullet.
\]

Closure under context follows using standard inductive reasoning. To deal with steps below normal argument positions, it is also necessary to orient images of \( P_n \). On the TRS \( \mathcal{R}_f \) this results additionally in following constraints:

\[
\left[ f_n(1) \right] \sim \text{dp}(y\sigma) + 1 \bullet \text{dp}(y\sigma) \quad \text{from rule 1}
\]

\[
\left[ f_n(\text{dp}(x\sigma) + 1) \right] \sim \text{dp}(y\sigma) + 1 \bullet \left[ g_n(P_n(h(x\sigma);)) \right] \sim \text{dp}(x\sigma)) \sim \left[ h_n(\text{dp}(x\sigma)) \right] \bullet \sim \text{dp}(y\sigma) + 1 \quad \text{from rule 2}.
\]

To get a polynomial bound on \( \bullet \) descending sequences, we need to control the length of right-hand sides appropriately. Precisely we will require that for a global constant \( k \in \mathbb{N} \), \( \text{len}(b) \leq \text{wd}(a) + k \) whenever \( a \bullet b \) holds. In particular \( k \) will be more than twice the maximal size of a right-hand side in the analysed TRS \( \mathcal{R} \). Note that due to the following lemma, if \( l\sigma \rightarrow_{\mathcal{R}} r\sigma \) with \( \sigma : V \rightarrow T(C, V) \) is a root step of a predicative TRS \( \mathcal{R} \), then \( \text{len}(P(r\sigma)) \leq \text{wd}(P(l\sigma)) + k \) for \( P \in \{P_s, P_n\} \).
Lemma 4.4. Let \( s = f(s_1, \ldots, s_k; s_{k+1}, \ldots, s_{k+l}) \in T_b \), \( t \in T \), \( \sigma : V \to T(C, V) \) and define \( k := 2 \cdot |t| \). Then

1. \( \text{len}(P_s(t)) \leq |t| \); and
2. if \( s >_\text{pop} t \) then \( \text{len}(P_s(t)) \leq \max\{\text{norm}(s_1\sigma), \ldots, \text{norm}(s_k\sigma)\} + k \); and
3. if \( s >_\text{pop} t \) then \( \text{len}(P_s(t)) \leq \max\{\text{norm}(s_1\sigma), \ldots, \text{norm}(s_k\sigma), \text{norm}(s\sigma)\} + k \).

Proof. The first property follows by induction on \( t \), employing that \( P_s(x\sigma) = [ ] \). A standard induction on \( >_\text{pop} \) (respectively \( >_\text{pop} \)) proves the second and third properties. For the cases \( s >_\text{pop} t \) (respectively \( s >_\text{pop} t \) and \( s >_\text{pop} t \)), we use Lemma 3.6; the remaining cases follow from induction hypothesis directly.

Proof.}

5. The Polynomial Path Order on Sequences

The polynomial path order on sequences (POP for short), denoted by \( \succ \), constitutes a generalisation of the path order for FP as put forward in [2]. Whereas we previously used the notion of safe mapping to dictate predicative recursion on compatible TRSs, the order on sequences relies on the explicit separation of safe arguments as given by predicative interpretations. Following Buchholz [16], we present finite approximations \( \succ_{k,l} \) of \( \succ \). The parameters \( k \in \mathbb{N} \) and \( l \in \mathbb{N} \) are used to control the width and depth of right-hand sides. Fix a precedence \( \succeq \) on the normalised signature \( P_n \). We extend term equivalence with respect to \( \succeq \) to sequences by disregarding the order on elements.

Definition 5.1. We define \( a \sim b \) if \( a = b \) or there exists a permutation \( \pi \) such that \( a_i \sim b_{\pi(i)} \) for all \( i = 1, \ldots, n \), where either (i) \( a = [a_1 \ldots a_n] \), \( b = [b_1 \ldots b_n] \), or (ii) \( a = f(a_1, \ldots, a_n) \), \( b = g(b_1, \ldots, b_n) \) and \( f \sim g \).

In correspondence to \( >_\text{pop} \), the order \( \succ_{k,l} \) is based on an auxiliary order \( \succ_{k,l} \) defined next. The full order is then introduced in Definition 5.4.

Definition 5.2. Let \( k, l \geq 1 \). We define \( \succ_{k,l} \) with respect to the precedence \( \succeq \) inductively as follows:

1. \( f(a_1, \ldots, a_n) \succ_{k,l} b \) if \( a_i \succ_{k,l} b \) for some \( i \in \{1, \ldots, n\} \);
2. \( f(a_1, \ldots, a_n) \succ_{k,l} g(b_1, \ldots, b_m) \) if \( f \geq g \) and the following conditions are satisfied:
   - \( f(a_1, \ldots, a_n) \succ_{k,l-1} b_j \) for all \( j = 1, \ldots, m \);
   - \( m \leq \text{wd}(f(a_1, \ldots, a_n)) + k \);
3. \( f(a_1, \ldots, a_n) \succ_{k,l} [b_1 \ldots b_m] \) if the following conditions are satisfied:
   - \( f(a_1, \ldots, a_n) \succ_{k,l-1} b_j \) for all \( j = 1, \ldots, m \);
   - \( m \leq \text{wd}(f(a_1, \ldots, a_n)) + k \);
4. \( [a_1 \ldots a_n] \succ_{k,l} [b_1 \ldots b_m] \) if the following conditions are satisfied:
   - \( [b_1 \ldots b_m] \succ_{k,l} c_1 \ldots c_n \);
   - \( a_i \succeq_{k,l} c_i \) for all \( i = 1, \ldots, n \);
   - \( a_{i_0} \succ_{k,l} c_{i_0} \) for at least one \( i_0 \in \{1, \ldots, n\} \);
   - \( m \leq \text{wd}([a_1 \ldots a_n]) + k \);

Here \( \succeq_{k,l} \) denotes \( \succeq_{k,l} \cup \sim \). We write \( \succ_k \) to abbreviate \( \succ_{k,k} \).

Recall that the auxiliary order \( >_\text{pop} \) underlying \( \text{POP}^* \) is used to orient normal arguments in right-hand sides. Similar, the auxiliary order \( \succ_{k,l} \) is to orient the predicative interpretations of this normal arguments. We exemplify the order \( \succ_{k,l} \) on the Example 4.3.
Example 5.3. Reconsider rule 2 from Example 4.3 where in particular \( f(s(x); y) \gg_{\text{pop}} h(x) \). Define the precedence \( f_n > h_n > \bullet \). First recall that by definition of the operator \( \sim \) we have
\[
\mathbf{n} = [\bullet \cdots \bullet] = \bullet \sim \cdots \sim \bullet \sim [\cdot] \sim \cdots \sim [\cdot]
\]
with \( n \) occurrences of \( \bullet \) and \( m \) occurrences of \([\cdot]\). Using \( n \) times \( \bullet \sim \bullet \) and \( m \) times \( \bullet \gg_{k,l} [\cdot] \) we can thus prove \( n + m >_{k,l} \mathbf{n} \) for all \( m \geq 1 \) whenever \( l \geq 2 \).

Let \( k \geq 12 \) be at least twice the size of the right-hand sides, and consider a substitution \( \sigma : \text{Var} \mapsto T(C, V) \). To show \( P(f(s(x\sigma); y\sigma)) >_k P(h(s(x\sigma);)) \) for \( P \in \{ P_5, P_6 \} \), we can even show the stronger property \( f_n(dp(x\sigma) + 1) >_{k,10} P(h(s(x\sigma);)) \) since
\[
1: \quad dp(x\sigma) + 1 >_{k,l} dp(x\sigma) \quad \text{as } dp(x\sigma) + 1 > dp(x\sigma)
\]
\[
2: \quad f_n(dp(x\sigma) + 1) >_{k,l} h_n(dp(x\sigma)) = P_5(h(x\sigma);) \quad \text{using } f_n > h_n \text{ and } 1
\]
\[
3: \quad f_n(dp(x\sigma) + 1) >_{k,l} [h_n(dp(x\sigma))] \bullet = P_6(h(x\sigma);) \quad \text{by 2 and } f_n(\ldots) >_{k,l} \bullet
\]

We arrive at the definition of the full order \( >_{k,l} \).

Definition 5.4. Let \( k, l \geq 1 \). We define \( >_{k,l} \) inductively as the least extension of \( >_{k,l} \) such that:

1. \( f(a_1, \ldots, a_n) >_{k,l} b \) if \( a_i >_{k,l} b \) for some \( i \in \{1, \ldots, n\} \);
2. \( f(a_1, \ldots, a_n) >_{k,l} g(b_1, \ldots, b_m) \) if \( f \sim g \) and following conditions are satisfied:
   - \( \{a_1, \ldots, a_n\} >_{k,l} \{b_1, \ldots, b_m\} \);
   - \( m \leq k \);
3. \( f(a_1, \ldots, a_n) >_{k,l} [b_1 \cdots b_m] \) and following conditions are satisfied:
   - \( f(a_1, \ldots, a_n) >_{k,l-1} b_{j_0} \) for at most one \( j_0 \in \{1, \ldots, m\} \);
   - \( f(a_1, \ldots, a_n) >_{k,l-1} b_j \) for all \( j \neq j_0 \);
   - \( m \leq \text{wd}(f(a_1, \ldots, a_n)) + k \);
4. \( [a_1 \cdots a_n] >_{k,l} [b_1 \cdots b_m] \) and following conditions are satisfied:
   - \( [b_1 \cdots b_m] > c_1 \sim \cdots \sim c_n \);
   - \( a_i >_{k,l} c_i \) for all \( i = 1, \ldots, n \);
   - \( a_{i_0} >_{k,l} c_{i_0} \) for at least one \( i_0 \in \{1, \ldots, n\} \);
   - \( m \leq \text{wd}([a_1 \cdots a_n]) + k \);

Here \( >_{k,l} \) denotes \( >_{k,l} \cup \sim \). We write \( >_{k} \) to abbreviate \( >_{k,k} \).

Example 5.5. Reconsider the rules from Example 4.3 and let \( k \geq 12 \). We consider only substitutions \( \sigma : V \mapsto T(C, V) \). First consider a rewrite step \( f(0; y\sigma) \mapsto_R y\sigma \) due to rule 1. Exploiting the shape of \( \sigma \), we have \( P_7(f(0; y\sigma)) = f_n(1) >_{k} [\cdot] = P_7(y\sigma) \) and similar
\[
P_n(f(0; y\sigma)) = [f_n(1)] \sim dp(y\sigma) + 1 >_{k} dp(y\sigma) = P_n(y\sigma).
\]
Finally consider a rewrite step \( f(s(x\sigma); y\sigma) \mapsto_R g(h(x\sigma); f(x\sigma; y\sigma)) \) caused by rule 2. This case is slightly more involved. Essentially we use \( >_{k,l} \) to orient the recursive call (proof
step 5), and \( \preceq_{k,j} \) for the remaining elements not containing \( f_n \) (proof step 6).

4: \( dp(x\sigma) + 1 \overset{\text{Lemma 5.6}}{=} k \cdot dp(x\sigma) \)

5: \( f_n(dp(x\sigma) + 1) \overset{\text{Lemma 5.6}}{=} k \cdot f_n(dp(x\sigma)) \quad \quad \text{by 4} \)

6: \( f_n(dp(x\sigma) + 1) \overset{\text{Lemma 5.6}}{=} k \cdot g_n(h(x\sigma)) \quad \quad \text{using } f_n \succ g_n \text{ and 3} \)

7: \( f_n(dp(x\sigma) + 1) \overset{\text{Lemma 5.6}}{=} k \cdot [g_n(h(x\sigma)) \cdot f_n(dp(x\sigma))] \quad \quad \text{using 5 and 6} \)

8: \( P_h(f(s(x\sigma); y\sigma)) = [f_n(dp(x\sigma) + 1)] \cdot dp(y\sigma) + 1 \)

The next lemma collects some frequently used properties.

Lemma 5.6. The following properties hold for all \( k \geq 1 \) and \( a, b, c_1, c_2 \in S(F, V) \cup S'(F, V) \).

1. \( \succ_i \subseteq \succ \subseteq \succ_k \) for all \( l \leq k \);
2. \( \sim \cdot \succ_k \cdot \sim \subseteq \succ_k \);
3. \( a \succ_k b \) implies \( c_1 \sim a \sim c_2 \succ_k c_1 \sim b \sim c_2 \).

Proof. The first two properties follow by standard reasoning. For the final property on proves \( a \sim c_2 \succ_k b \sim c_2 \) by case analysis on the assumption \( a \succ_k b \) Crucially, \( \text{len}(b \sim c_2) \) is bounded by \( \text{wd}(a \sim c_2) + k \) as required in \( \succ_k \). The general property is then an easy consequence from Property 2.

Following [2] we define \( G_k \) that measures the \( \succ_k \)-descending lengths on sequences. To simplify matters, we restrict the definition of \( G_k \) to ground sequences. As images of predicative interpretations are always ground, this suffices for our purposes.

Definition 5.7. We define \( G_k : S(F) \cup S'(F) \to \mathbb{N} \) as

\[
G_k(a) := 1 + \max\{G_k(b) \mid b \in S(F) \cup S'(F) \text{ and } a \succ_k b\}.
\]

Note that due to Lemma 5.6 [2], \( G_k(a) = G_k(b) \) whenever \( a \sim b \). Sequences are intended to act purely as a container, not contributing to \( G_k \) themselves. The next lemma confirms our intention, exploiting that conceptually clause \( \succ_k \) amounts to a product-wise extension of \( \succ_k \) to sequences.

Lemma 5.8. For \( [a_1 \cdots a_n] \in S'(F) \) it holds that \( G_k([a_1 \cdots a_n]) = \sum_{i=1}^{n} G_k(a_i) \).

Proof. Let \( a = [a_1 \cdots a_n] \in S'(F) \). We first show \( G_k(a) \geq \sum_{i=1}^{n} G_k(a_i) \). Let \( b, c \in S(F) \cup S'(F) \) and consider maximal sequences \( b \succ_k b_1 \succ_k b_2 \cdots b_o \) and \( c \succ_k c_1 \succ_k \cdots c_p \). Using Lemma 5.6 [3] repeatedly we get \( b \sim c \succ_k b_1 \sim c \succ_k b_2 \sim c \cdots \succ_k b_o \sim c \). Similar \( c \sim b_0 \succ_k c_1 \sim b_0 \succ_k \cdots \succ_k c_p \). Since \( b_o \sim c \sim b_o \) and employing Lemma 5.6 [3] we see \( G_k(b \sim c) \geq G_k(b) + G_k(c) \) for all \( b, c \in S(F) \cup S'(F) \). We conclude \( G_k(a) = G_k([a_1 \cdots a_n]) \geq \sum_{i=1}^{n} G_k(a_i) \) with a straightforward induction on \( n \).

It remains to verify \( G_k(a) \leq \sum_{i=1}^{n} G_k(a_i) \). For this we show that \( a \succ_k b \) implies \( G_k(b) < \sum_{i=1}^{n} G_k(c_i) \) by induction on \( G_k(a) \). Consider the base case \( G_k(a) = 0 \). Since \( a \) is ground it follows that \( a = \langle \rangle \), the claim is trivially satisfied. For the inductive case \( G_k(a) > 1 \), let \( a \succ_k b \). Since \( a \) is a sequence, \( a \succ_k b \). Hence \( b \sim b_1 \cdots \sim b_n \) where \( a_i \succ_k b_i \) and thus
\( G_k(b_i) \leq G_k(a_i) \) for all \( i = 1, \ldots, n \). Additionally \( a_{i_0} \succ k b_{i_0} \) and hence \( G_k(b_{i_0}) < G_k(a_{i_0}) \) for at least one \( i_0 \in \{1, \ldots, n\} \). As in the first half of the proof, one verifies \( G_k(b_i) \leq G_k(b) \) for all \( i = 1, \ldots, n \). Note \( G_k(b) < G_k(a) \) as \( a \succ k b \), hence induction hypothesis is applicable to \( b \) and all \( b_i \) (\( i = 1, \ldots, n \)). It follows that

\[
G_k(b) = \sum_{c \in b} G_k(c) = \sum_{i=1}^n \sum_{c \in b_i} c = \sum_{i=1}^n G_k(b_i) < \sum_{i=1}^n G_k(a_i)
\]

This concludes the second half of the proof.

The central theorem of this section states that \( G_k(f(a_1, \ldots, a_n)) \) is polynomial in \( \sum^n G_k(a_i) \), where the polynomial bound depends only on \( k \) and the rank \( p \) of \( f \). The proof of this is rather involved. To cope with the multiset comparison underlying \( \succ_k \), we introduce as a first step an order-preserving extension \( G_k^* \) of \( G_k \) to multisets of sequences, in the sense that \( G_k^* (a_1, \ldots, a_n) > G_k^* (b_1, \ldots, b_m) \) holds whenever \( \{a_1, \ldots, a_n\} \succ_k \{b_1, \ldots, b_m\} \) (provided \( k \geq m, n \), c.f. Lemma 5.12). As the next step toward our goal, we estimate \( G_k(f(a_1, \ldots, a_n)) \) in terms of \( G_k^* (a_1, \ldots, a_n) \) whenever \( n \leq k \) and \( rk(f) \leq k \). Technically we bind following functions by polynomials \( q_{k,p} \).

**Definition 5.9.** For all \( k, p \in \mathbb{N} \) with \( k \geq 1 \) we define \( F_{k,p} : \mathbb{N} \to \mathbb{N} \) as

\[
F_{k,p}(m) := \max\{G_k(f(a_1, \ldots, a_n)) \mid f(a_1, \ldots, a_n) \in S(F), \ rk(f) \leq p, \ n \leq k \ \text{and} \ \ G_k^* (a_1, \ldots, a_n) \leq m\}.
\]

Noting that also \( G_k^* (a_1, \ldots, a_n) \) is polynomial in \( \max_{i=1}^n G_k(a_i) \), say \( q_k \), which depends only on \( k \), we obtain \( G_k(f(a_1, \ldots, a_n)) \leq q_{k,p}(q_k(\max_{i=1}^n G_k(a_i))) \) whenever \( k \geq n \).

The definition of \( G_k^* \) is defined in terms of an order-preserving homomorphism from \( \mathcal{M}(\mathbb{N}) \) to \( \mathbb{N} \). To illustrate the construction carried out below, consider the following example.

**Example 5.10.** Let \( k \geq 1 \) and let \( c > m_1 \geq \cdots \geq m_k \) be natural numbers in descending order, dominated by \( c \in \mathbb{N} \). Consider multisets \( \mathcal{M}(\mathbb{N}) \) of size \( k \). If we conceive such multisets as base-\( c \) representations of numbers using \( k \) digits, then we can form a chain \( M_1 \succ_k M_2 \succ_k \cdots \) that can be understood as a decreasing counter that wrongly wraps from \( \{m_1, \ldots, m_i + 1, 0, \ldots, 0\} \) to \( \{m_1, \ldots, m_i, m_{i+1}, \ldots, m_k\} \). It is not difficult to prove that the maximal length of such a chain is bounded by

\[
\sum_{m_{k-1}=1}^{m_k} \cdots \sum_{m_2=1}^{m_k-1} \sum_{m_1=1}^{m_k-1} m_k \leq \sum_{m_{k-1}=1}^{m_k-2} \cdots \sum_{m_2=1}^{m_k-2} \sum_{m_1=1}^{m_k-1} \Omega(m_{k-1}^2) \in \Omega(m_1^k).
\]

We now show that this upper bound serves also as a lower bound for multisets \( \mathcal{M}(\mathbb{N}) \) of size \( n \leq k \). As in the example, the function \( h_{k,c}^n : \mathbb{N}^l \to \mathbb{N} \) (where \( n \leq k \)) defined below interprets multisets \( M \in \mathcal{M}(\mathbb{N}) \) as natural numbers encoded in base-\( c \) with \( k \) digits, where the \( i^{th} \) largest \( m_i \in M \) represents the \( i^{th} \) most significant digit. Formally, for \( k \geq n \in \mathbb{N} \) and \( c \in \mathbb{N} \) we define the family of functions \( h_{k,c}^n : \mathbb{N}^l \to \mathbb{N} \) such that

\[
h_{k,c}^n(m_1, \ldots, m_n) = \sum_{i=1}^{n} \text{sort}^n(m_1, \ldots, m_n, i) \cdot c^{(k-i)}.
\]

Here \( \text{sort}^n(m_1, \ldots, m_n, i) \) denote the \( i^{th} \) element of \( m_1, \ldots, m_n \) sorted in ascending order, i.e., \( \text{sort}^n(m_1, \ldots, m_n, i) := m_{\pi(i)} \) for \( i = 1, \ldots, n \) and some permutation \( \pi \) such that \( m_{\pi(i)} \geq m_{\pi(i+1)} \) for \( i \in \{1, \ldots, n-1\} \).

Lemma 5.11. Let $k,n,c \in \mathbb{N}$ such that $k \geq 1$ and $k \geq n$. Then for all $n_1, \ldots , n_l \in \mathbb{N}$ we have:

1. $c > \max\{n_1, \ldots , n_l\}$ implies $c^n > h_{n,c}(m_1, \ldots , m_n)$.
2. $\langle m_1, \ldots , m_n \rangle \succ_{\mu} \langle m'_1, \ldots , m'_{n'} \rangle$ implies $h_{k,c}(m_1, \ldots , m_n) > h_{k,c}(m'_1, \ldots , m'_{n'})$ for all $c > m_1, \ldots , m_n \geq 1$.

The mapping $G_k$ is obtained by extend $h_{k,c}$ to multisets over $S(F) \cup S'(F)$.

Definition 5.12. Let $k,n \in \mathbb{N}$ such that $k \geq n$. We define $G_k^n : S(F)^n \rightarrow \mathbb{N}$ as

$G_k^n(a_1, \ldots , a_n) := h_{k,c}(G_k(a_1), \ldots , G_k(a_n))$

where $c = 1 + \max\{G_k(a_i) \mid i \in \{1, \ldots , n\}\}$.

By Lemma 5.11, $G_k^n(a_1, \ldots , a_n)$ is polynomially bounded in $G_k(a_i)$ ($i = 1, \ldots , l$). By Lemma 5.12 we state order preserving as outlined above.

Lemma 5.13. Let $a_1, \ldots , a_n, b_1, \ldots , b_m \in S(F) \cup S'(F)$ and let $k \geq m, n$. Then

$\langle a_1, \ldots , a_n \rangle \succ_{\mu} \langle b_1, \ldots , b_n \rangle \implies G_k^n(a_1, \ldots , a_n) > G_k^n(b_1, \ldots , b_m)$.

In Theorem 5.15 below we prove $F_{k,p}(m) \leq c_{k,p} \cdot (m + 2)^{d_{k,p}}$, where the constants $c_{k,p}, d_{k,p} \in \mathbb{N}$ are defined as follows: $d_{k,0} := k + 1$ and $d_{k,p+1} := (d_{k,p})^k + 1$; further we set $c_{k,0} := k^k$ and $c_{k,p+1} := (c_{k,p} \cdot k)^e$ where $e = \sum_{i=0}^{k} (k \cdot d_{k,p})^i$. Inevitably the proof of Theorem 5.15 is technical, the reader is advised to skip the formal proof on the first read. Theorem 5.15 is proven by induction on $p$ and $m$. Consider term $f(a_1, \ldots , a_n)$ with $k \geq n$ and $G_k^n(a_1, \ldots , a_n) \leq m$. At the heart of the proof, we have to show that $c_{k,p} \cdot (m + 2)^{d_{k,p}} > G_k^n(b)$ for arbitrary $b$ with $f(a_1, \ldots , a_n) \succ_k b$. The most involved case is $f(a_1, \ldots , a_n) \succ_k b$ for $b = [b_1 \ldots b_o]$. Here it is fundamental to give precise bounds on the elements $b_j$ with $f(a_1, \ldots , a_n) \succ_{k,l} b_j$. Since $\succ_{k,l}$ constraints $b_j$ to only contain symbols ranked below $rk(f) = p$ in the precedence, conceptually $G_k(b_j)$ is bounded by iterated application of the induction hypothesis on $p$. Since $l$ essentially controls the depth of $b_j$ (compare Example 5.5), $l$ serves as a bound on the number of iterations. To properly account for all cases of $\succ_{k,l}$, matters get slightly more involved. To bind $G_k(b_j)$ sufficiently, we define for $l \in \mathbb{N}$ a family of auxiliary functions $g_{k,p} : \mathbb{N} \rightarrow \mathbb{N}$ such that

$g_{k,p}(m) = \begin{cases} k^l \cdot m^l & \text{if } p = 0 \text{ or } l = 0, \\ c_{k,p-1} \cdot (m \cdot g_{l-1,k,p}(m))^{k-d_{k,p}} & \text{otherwise.} \end{cases}$

Having as premise the induction hypothesis (on $p$) of the main proof, the next lemma verifies that $g_{k,rk(f)(m + 2)}$ sufficiently binds $\succ_{k,l}$-descendants of $f(a_1, \ldots , a_n)$.

Lemma 5.14. Let $f(a_1, \ldots , a_n) \in S(F)$. Let $k \geq n$ and $m \geq G_k^n(a_1, \ldots , a_n)$. Suppose $F_{k,p}(m') \leq c_{k,p}(m' + 2)^{d_{k,p}}$ for all $p < rk(f)$ and $m'$. Then $f(a_1, \ldots , a_n) \succ_{k,l} b$ implies $G_k(b) \leq g_{k,l,rk(f)(m + 2)}$ for all $b \in S(F) \cup S'(F)$.

Proof. We prove the claim by induction on $l$ and case analysis on $f(a_1, \ldots , a_n) \succ_{k,l} b$. First note that $f(a_1, \ldots , a_n) \succ_{k,l} b$ implies that $a_i \succ_{k,l} b$ for some $i \in \{1, \ldots , n\}$ and consequently $G_k(b) \leq G_k(a_i)$. As by definition $G_k(a_i) \leq m$ the lemma follows trivially.

As in the base case $l = 1$ either $b = [\ ]$ or $f(a_1, \ldots , a_n) \succ_{k,l} b$, it suffices to consider only the remaining cases of the inductive step. Assuming $f(a_1, \ldots , a_n) \succ_{k,l+1} b$ we show $G_k(b) \leq g_{k,l+1,rk(f)(m + 2)}$. 

Case $f(a_1, \ldots, a_n) \geq_{k, l+1} b$ where $b = g(b_1, \ldots, b_o)$: Then $f(a_1, \ldots, a_n) \succ_{k, l} b_j$ for all $j = 1, \ldots, o$, and $f > g$. Set $m' := G_k(b_1, \ldots, b_o)$. We have

$$m' < \max\{G_k(b_j) + 1 \mid j \in \{1, \ldots, o\}\}^k$$

by definition and Lemma 5.11. Applying induction hypothesis $o$ times.

As the assumption also gives $\text{rk}(g) < \text{rk}(f)$ we have

$$G_k(b) \leq F_{k, \text{rk}(g)}(m')$$

by definition of $F_{k, \text{rk}(g)}$

$$\leq c_{k, \text{rk}(g)} \cdot (m' + 2)^{d_{k, \text{rk}(g)}}$$

by assumption

$$\leq c_{k, \text{rk}(f)-1} \cdot (m' + 2)^{d_{k, \text{rk}(f)-1}}$$

as $\text{rk}(g) < \text{rk}(f)$

$$< c_{k, \text{rk}(f)-1} \cdot ((g, k, l, \text{rk}(f)(m + 2) + 1)^{k + 2} d_{k, \text{rk}(f)-1})$$

substituting bound for $m'$

$$\leq c_{k, \text{rk}(f)-1} \cdot (g, k, l, \text{rk}(f)(m + 2) + 3)^{k d_{k, \text{rk}(f)-1}}$$

using $1 \leq k$

$$\leq c_{k, \text{rk}(f)-1} \cdot ((m + 2) \cdot g, k, l, \text{rk}(f)(m + 2))^{k d_{k, \text{rk}(f)-1}}$$

using $2 \leq g, k, l, \text{rk}(f)(m + 2)$

$$= g, k, l, \text{rk}(f)(m + 2)$$

using $\text{rk}(f) > 0$.

Case $f(a_1, \ldots, a_n) \geq_{k, l+1} b$ where $b = \{b_1 \ldots b_o\}$: Ordering constraints give $o \leq \text{wd}(a) + k$ and $f(a_1, \ldots, a_n) \succ_{k, l} b_j$ ($j = 1, \ldots, o$). Exploiting that $a_i$ is ground, a standard induction shows that $\text{wd}(a_i) \leq G_k(a_i)$, and consequently $\text{wd}(a_i) \leq m$. Thus

$$o \leq \text{wd}(a) + k = \max\{1, \text{wd}(a_1), \ldots, \text{wd}(a_n)\} + k \leq m + k \leq k \cdot (m + 1).$$

(†)

If $\text{rk}(f) = 0$ then we see

$$G_k(b) = \sum_{j=1}^{o} G_k(b_i)$$

using Lemma 5.8

$$\leq \sum_{j=1}^{o} g_{k, l, 0}(m + 2)$$

applying induction hypothesis $o$ times

$$\leq k \cdot (m + 1) \cdot g_{k, l, 0}(m + 2)$$

using (†)

$$= k \cdot (m + 1) \cdot k^l \cdot (m + 2)^l$$

by assumption $\text{rk}(f) = 0$

$$< k^{l+1} \cdot (m + 2)^{l+1} = g_{k, l, 10}(m + 2).$$

Otherwise $\text{rk}(f) > 0$ and we conclude

$$G_k(b) \leq k \cdot (m + 1) \cdot g_{k, l, \text{rk}(f)}(m + 2)$$

as in the case $\text{rk}(f) = 0$

$$< c_{k, \text{rk}(f)-1} \cdot ((m + 2) \cdot g_{k, l, \text{rk}(f)}(m + 2))^{k d_{k, \text{rk}(f)-1}}$$

as $k \leq c_{k, \text{rk}(f)-1}$ and $1 < k \cdot d_{k, \text{rk}(f)-1}$

$$= g_{k, l, \text{rk}(f)}(m + 2)$$

by assumption $\text{rk}(f) > 0$.

\[ \square \]

Theorem 5.15. For all $k, p \in \mathbb{N}$ there exist constants $c, d \in \mathbb{N}$ (depending only on $k$ and $p$) such that for all $m$: $F_{k, p}(m) \leq c \cdot (m + 2)^d$.

Proof. Fix $a = f(a_1, \ldots, a_n) \in S(F)$ such that $\text{rk}(f) = p$, $k \geq n$ and $G_k(a_1, \ldots, a_n) \leq m$. To show the theorem, we prove that for all $b$ with $a \succ_k b$ we have $G_k(b) < c_{k, p} \cdot (m + 2)^{d_{k, p}}$ by induction on the rank $p$ and side induction on $m$. 


**Base Case** \( p = 0 \): The base case of the side induction is trivial, so consider the inductive step \( m > 0 \). We first prove \( G_k(b) < k^k \cdot (m + 1)^{k+1} + k^l \cdot (m + 2)^l \) by induction on \( \bullet_{k,l} \).

**Case** \( f(a_1, \ldots, a_n) \overset{1}{\sim}_{k,l} b \): Then \( a_\bullet \overset{1}{\sim}_{k,l} b \), and we conclude since \( G_k(b) \leq G_k(a_i) < m \) using the assumptions and Lemma 5.14.

**Case** \( f(a_1, \ldots, a_n) \overset{2}{\sim}_{k,l} b \) where \( g(b_1, \ldots, b_o) \): The ordering constraints give \( o < k, f \sim g \) and \( \langle a_1, \ldots, a_n \rangle \overset{\text{sid}}{\sim}_{k,l} \langle b_1, \ldots, b_o \rangle \). Set \( m' := G_k(b_1, \ldots, b_o). \) Hence \( m' < G_k^n(a_1, \ldots, a_n) \leq m \) by Lemma 5.13 and assumption \( n < k \). Thus side induction hypothesis gives \( F_{k,0}(m') \leq c_{k,0} \cdot (m' + 2)^{k+1} = k^k (m' + 2)^{k+1} \). As the ordering constraints imply \( rk(g) = rk(f) = 0 \) we conclude

\[
G_k(g(b_1, \ldots, b_o)) \leq F_{k,0}(m') = c_{k,0} \cdot (m' + 2)^{d_{k,0}} = k^k \cdot (m' + 2)^{k+1} \quad \text{by side induction hypothesis}
\]

\[
< k^k \cdot (m + 1)^{k+1} + k^l \cdot (m + 2)^l \quad \text{using } m' < m.
\]

**Case** \( f(a_1, \ldots, a_n) \overset{3}{\sim}_{k,l} b \) where \( \langle b_1 \ldots b_o \rangle \): The ordering constraints give (i) \( a \overset{1}{\sim}_{k,l-1} b_j \) for some \( j_0 \in \{1, \ldots, o\} \), (ii) \( a \nRightarrow b_j \) for all \( j \neq j_0 \), and (iii) \( o \leq wd(a) + k \). By induction hypothesis on (i) we get \( G_k(b_{j_0}) < k^k \cdot (m + 1)^{k+1} + k^l \cdot (m + 2)^l \), the preparatory step gives \( G_k(b_{j_0}) < k^k \cdot (m + 1)^{k+1} + k^l \cdot (m + 2)^l \)

\[
\text{using } o < k \cdot (m + 2) \text{ and Lemma 5.14 on (ii)}
\]

Since \( \bullet_{k} = \bullet_{k,k} \) this preparatory step gives

\[
G_k(b) < k^k \cdot (m + 1)^{k+1} + k^k \cdot (m + 2)^k \leq k^k \cdot (m + 2)^{k+1}
\]

and concludes the base case.

**Inductive Step**: By induction hypothesis on \( p \) we get \( F_{k,p}(m) \leq c_{k,p} \cdot (m + 2)^{d_{k,p}} \), side induction hypothesis gives \( F_{k,p+1}(m') \leq c_{k,p+1} \cdot (m + 2)^{d_{k,p+1}} \) for all \( m' < m \). A standard induction reveals \( g_{k,p+1}(n) \leq c_{k,p} \cdot (k-d_{k,p})^n \cdot n \sum_{i=0}^{l-1} (k-d_{k,p})^i \) for all \( n \in \mathbb{N} \). We continue with the proof of the lemma, and show that for all \( l \geq 1 \), if \( f(a_1, \ldots, a_n) \overset{4}{\sim}_{k,l} b \) then

\[
G_k(b) \leq c_{k,p+1} \cdot (m + 1)^{d_{k,p+1}} + c_{k,p+1} \cdot (m + 2)^{(k-d_{k,p})^l}
\]

by induction on \( l \). The only interesting case is \( a \overset{4}{\sim}_{k,l+1} b \). Then \( b = [b_1 \ldots b_o] \) with (i) \( a \overset{1}{\sim}_{k,l} b_{j_0} \) for some \( j_0 \in \{1, \ldots, o\} \), (ii) \( a \nRightarrow b_j \) for all \( j \neq j_0 \), and (iii) \( o \leq wd(a) + k \). By induction hypothesis on (i) we get \( G_k(b_{j_0}) \leq c_{k,p+1} \cdot (m + 1)^{d_{k,p+1}} + c_{k,p+1} \cdot (m + 2)^{(k-d_{k,p})^l} \), Lemma 5.14 on (ii) gives \( G_k(b_{j_0}) \leq g_{l,k,p+1}(m + 2) \) for \( j \neq j_0 \) and (iii) gives \( o \leq k \cdot (m + 1) \) as...
in Equation (\[1\]). Summing up we see
\[
G_k(b) = \sum_{j=1}^{o} G_k(b_j)
\]
\[
\leq c_{k,p+1} \cdot (m+1)^{d_{k,p+1}} + c_{k,p+1} \cdot (m+2)^{(k-d_{k,p})^j}
\]
\[
+ k \cdot (m+1) \cdot g{l,k,p+1}(m+2)
\]
\[
\leq c_{k,p+1} \cdot (m+1)^{d_{k,p+1}} + c_{k,p+1} \cdot (m+2)^{(k-d_{k,p})^j}
\]
\[
+ k \cdot (m+1) \cdot \sum_{i=0}^{\ell-1} (k-d_{k,p})^i \cdot (m+2) \sum_{i=0}^{\ell-1} (k-d_{k,p})^i
\]
\[
< c_{k,p+1} \cdot (m+1)^{d_{k,p+1}} + c_{k,p+1} \cdot (m+2)^{(k-d_{k,p})^j}
\]
\[
+ c_{k,p+1} \cdot (m+2)^{\sum_{i=0}^{\ell-1} (k-d_{k,p})^i}
\]
\[
\leq c_{k,p+1} \cdot (m+1)^{d_{k,p+1}} + c_{k,p+1} \cdot (m+2)^{(k-d_{k,p})^{j+1}}
\]
as desired. Using (\[1\]), \(\blacktriangledown_k = \blacktriangledown_k, k\) and \((k \cdot d_{k,p})^k < (k \cdot d_{k,p})^k + 1 < d_{k,p+1}\) we finally get
\[
G_k(b) \leq c_{k,p+1} \cdot (m+1)^{d_{k,p+1}} + c_{k,p+1} \cdot (m+2)^{(k-d_{k,p})^k}
\]
\[
= c_{k,p+1} \cdot ((m+1)^{d_{k,p+1}} + (m+2)^{(k-d_{k,p})^k})
\]
\[
< c_{k,p+1} \cdot (m+2)^{d_{k,p+1}}
\]
and conclude the inductive case. \(\square\)

As a consequence, the number of \(\blacktriangledown_k\)-descents on basic terms interpreted with predicative interpretation \(P_k\) is polynomial in sum of depths of normal arguments.

**Corollary 5.16.** Let \(f \in D\) with at most \(k\) normal arguments. There exists a constant \(d \in \mathbb{N}\) depending only on \(k\) such that:
\[
G_k(P_k(f(m_1, \ldots, m_n; \bar{v}))) \in O((\max_{i=1}^{m} dp(u_i))^d)
\]
for all \(u_1, \ldots, u_{m+n} \in T(C, V)\).

**Proof.** Let \(s = f(m_1, \ldots, m_n; \bar{v})\) be as given by the corollary. Recall that
\[
P_s(s) = [f_n(P_n(u_1), \ldots, P_n(t_{u_m}))] \sim P_s(u_{m+1}) \sim \cdots \sim P_s(u_{m+n})
\]
\[
= [f_n(dp(u_1), \ldots, dp(u_m))]
\]
As $G_k(\bullet)$ is constant, say $G_k(\bullet) = c$, by Lemma 5.8 we see that $G_k(dp(u_i)) = c \cdot dp(u_i)$. Putting things together is tedious but not difficult:

$$G_k(s) = G_k(f_n(dp(u_1), \ldots, dp(u_m)))$$

by Lemma 5.8

$$\leq F_{k, rk(f)}\left(G_k(dp(u_1), \ldots, dp(u_m))\right)$$

by Lemma 5.11

$$\leq F_{k, rk(f)}\left((1 + \max \limits_{i=1}^{m} G_k(norm(u_i)))^k\right)$$

using $G_k(norm(u_i)) \leq c \cdot dp(u_i)$

$$\in O\left((c \cdot (1 + \max \limits_{i=1}^{m} dp(u_i)))^{k+d'}\right)$$

by Theorem 5.15

Set $d := k + d'$ and note that $d$ depends only on $k$ and $rk(f)$ as desired.

6. Predicative Embedding

Fix a predicative recursive TRS $R$ and signature $F$, and let $\succ_{pop^*}$ be the polynomial path order underlying $R$ based on the (admissible) precedence $\succ$. We denote by $\succ$ also the homomorphic precedence on $T_0$ given by: $f_n \sim g_h$ if $f \sim g$ and $f_n \succ g_h$ if $f \succ g$. Further, we set $f \succ \bullet$ for all $f_n \in T_0$. We denote by $\succ_{\ell}$ (and respectively $\succ_{\ell}$) the approximation given in Definition 5.4 (respectively Definition 5.2) with underlying precedence $\succ$. We now establish the embedding of $\frac{\to}{R}$ into $\succ_{\ell}$ for some $\ell$ depending only on $R$. To simplify matters, we suppose for now that $R$ is completely defined. Since then normal forms and values coincide, $s \frac{\to}{R} t$ if $s = C[l\sigma]$ and $t = C[r\sigma]$ where $l \to r \in R$ and all arguments of $lr$ are values. In particular, this implies that the substitution $\sigma$ maps variables to values.

Lemma 6.2 below proves the embedding of root steps for the case $l \succ_{pop^*} r$. In Lemma 6.3, we then show that the embedding is closed under contexts. The next lemma, exploited in Lemma 6.2, connects the auxiliary orders $\succ_{pop}$ and $\succ_{k,l}$ (compare Example 3.3).

**Lemma 6.1.** Suppose $s = f(s_1, \ldots, s_k; s_{k+1}, \ldots, s_{k+\ell}) \in T_b(F, V)$, $t \in T(F, V)$ and $\sigma: V \to T(C, V)$. Then for predicative interpretations $P \in \{P_s, P_n\}$ we have

$$s \succ_{pop} t \implies f_n(P_n(s_1, \sigma), \ldots, P_n(s_k, \sigma)) \succ_{2|t|} P(t_\sigma).$$

**Proof.** Let $s = f(s_1, \ldots, s_k; s_{k+1}, \ldots, s_{k+\ell}) \in T_b(F, V)$, $t \in T(F, V)$. We continue by induction on the definition of $\succ_{pop}$.

**Case $s \succ_{pop} t$:** Then $s_i \succ_{pop} t$ for some normal argument position $i \in \{1, \ldots, k\}$.

Note that by assumption $s \in T_b(F, V)$, $s_i \in T(C, V)$ and so Lemma 3.6 (employing $\succ_{pop} \subseteq \succ_{pop^*}$) gives $s_i \not\in t$ and $t \in T(C, V)$, consequently $t\sigma \in T(C, V)$ and furthermore $norm(s_i, \sigma) \succ norm(t, \sigma)$. As $t\sigma \in T(C, V)$, we get $f_n(P_n(s_1, \sigma), \ldots, P_n(s_k, \sigma)) \succ_{2|t|} P(t_\sigma)$ which concludes the case $P = P_n$. For the case $P = P_s$, observe $P_n(s_i, \sigma) = norm(s, \sigma)$ and $P_n(t, \sigma) = norm(t, \sigma)$ since both $s_i, \sigma$ and $t, \sigma$ are values. If $norm(s_i, \sigma) = norm(t, \sigma)$ then obviously $P_n(s_i, \sigma) = P_n(t, \sigma)$. Otherwise $norm(s_i, \sigma) \succ norm(t, \sigma)$ and then $P_n(s_i, \sigma) \succ_{2|t|} P_n(t, \sigma)$, employing $\bullet \succ_{2|t|} \bullet$. Hence overall $P_n(s_1, \sigma) \succ_{2|t|} P_n(t, \sigma)$. Since the position
i is normal, \( P_n(s_i, \sigma) \) is a direct subterm of \( f_n(P_n(s_1, \sigma), \ldots, P_n(s_k, \sigma)) \) and we conclude \( f_n(P_n(s_1, \sigma), \ldots, P_n(s_k, \sigma)) \succ_{2|t|} P_n(t, \sigma) \) as desired.

**Case** \( s \succ_{pop} t \): By the assumption \( t = g(t_1, \ldots, t_m; t_{m+1}, \ldots, t_{m+n}) \) where \( f > g \) and \( s \succ_{pop} t \) for all \( j = 1, \ldots, m + n \).

We consider the more involved case \( t \notin T(C, V) \). Let \( P_\delta(t, \sigma) = [v_{i_1} \cdots v_{i_j}] \) for all safe argument positions \( i = m + 1, \ldots, m + n \) of \( g \), i.e.,

\[
P_\delta(t, \sigma) = [g_i(P_n(t_1), \ldots, P_n(t_m)), v_{m+1,1}, \ldots, v_{m+1,jm+1}, \ldots, v_{m+n,1}, \ldots, v_{m+n,jm+n}].
\]

By induction hypothesis on \( i = 1, \ldots, m \) we get \( f_n(P_n(s_1, \sigma), \ldots, P_n(s_k, \sigma)) \succ_{2|t|} P_n(t, \sigma) \).

Since \( |t_i| < |t| \), using Lemma 5.6(3) we have in particular \( f_n(P_n(s_1, \sigma), \ldots, P_n(s_k, \sigma)) \succ_{2|t| - 2} P_n(t, \sigma) \). Using this, \( \rho_i > g_i \), and \( m < |t| \) we conclude

\[
f_n(P_n(s_1, \sigma), \ldots, P_n(s_k, \sigma)) \succ_{2|t| - 1} g_i(P_n(t_1), \ldots, P_n(t_m)) \cdot (6.1)
\]

By induction hypothesis on safe argument positions of \( g \) we get

\[
f_n(P_n(s_1, \sigma), \ldots, P_n(s_k, \sigma)) \succ_{2|t_i|} [v_{i_1} \cdots v_{i_j}, \sigma] = P_\delta(t, \sigma)
\]

for all \( i = m + 1, \ldots, m + n \). Using a simple inductive argument one verifies

\[
f_n(P_n(s_1, \sigma), \ldots, P_n(s_k, \sigma)) \succ_{2|t_i| - 1} v_{i,j_i} \] for all \( i = m + 1, \ldots, m + n \) and \( j = 1, \ldots, j_i \) (6.2)

from this. Let \( P \in \{ P_n, P_\delta \} \). Observe

\[
\text{len}(P(t)) \leq 2 \cdot |t| + \max\{\text{norm}(s_1, \sigma), \ldots, \text{norm}(s_k, \sigma)\} \quad \text{by Lemma 4.4(2)}
\]

\[
\leq 2 \cdot |t| + \text{wd}(f_n(P_n(s_1, \sigma), \ldots, P_n(s_k, \sigma))).
\]

Using this, Equations (6.1), Equations (6.2) and possibly \( f_n(P_n(s_1, \sigma), \ldots, P_n(s_k, \sigma)) \succ_{2|t|} \bullet \) we have \( \tau \succ_{2|t|} P(t, \sigma) \) as desired.

We conclude this auxiliary lemma.

**Lemma 6.2.** Suppose \( s = f(s_1, \ldots, s_k; s_{k+1}, \ldots, s_{k+1}) \in T_b(F, V), t \in T(F, V) \) and \( \sigma : V \rightarrow T(C, V) \). Then for predicative interpretation \( P \in \{ P_n, P_\delta \} \) we have

\[
s \succ_{pop} t \quad \implies \quad P(s, \sigma) \succ_{2|t|} P(t, \sigma).
\]

Proof. Let \( s, t, \sigma \) be as given in the lemma. We prove the stronger assertions

1. \( f_n(P_n(s_1, \sigma), \ldots, P_n(s_k, \sigma)) \succ_{2|t|} P_n(t, \sigma) \),
2. \( f_n(P_n(s_1, \sigma), \ldots, P_n(s_k, \sigma)) \succ_{2|t_i|} P_n(t, \sigma) \) if \( t \in T(F \uparrow \text{Fun}(s), V) \), and
3. \( f_n(P_n(s_1, \sigma), \ldots, P_n(s_k, \sigma)) \sim \text{norm}(s, \sigma) \succ_{2|t|} P_n(t, \sigma) \).

We continue with the proof of the assertions by induction on \( \succ_{pop} \).

**Case** \( s \succ_{pop} t \): Exactly as in Lemma 6.1 we conclude \( s_1 \sim t_1, t_1 \in T(C, V) \). The latter implies \( P_\delta(t_1) = [\] and thus Assertion 1 and Assertion 2 are immediate. For Assertion 3 observe that \( \text{len} (\text{norm}(t, \sigma)) = \text{norm}(t) \leq \text{norm}(s) \leq \text{wd}(f_n(P_n(s_1, \sigma), \ldots, P_n(s_k, \sigma)) \sim \text{norm}(s)) \) where the latter inequality is obtained by case analysis on \( i \). From this and \( f_n(P_n(s_1, \sigma), \ldots, P_n(s_k, \sigma)) \succ_{2|t_i| - 1} \bullet \) we get \( f_n(P_n(s_1, \sigma), \ldots, P_n(s_k, \sigma)) \sim \text{norm}(s) \succ_{2|t|} \text{norm}(t, \sigma) = P_n(t, \sigma) \) as desired.
Case $s \gtrsim_{\text{pop}} t$:

The assumption gives $t = g(t_1, \ldots, t_m; t_{m+1}, \ldots, t_{m+n})$ where $f > g$ and further $s \gtrsim_{\text{pop}} t_i$ for all normal argument positions $i = 1, \ldots, m$ and $s \gtrsim_{\text{pop}} t_i$ for all safe argument positions $i = m + 1, \ldots, m + n$.

Additionally, $t_{i_0} \notin T(F^{<\text{Fun}}(s), V)$ for most argument positions $i_0$.

We first verify Assertion 1 and Assertion 2 for the non-trivial case $t \notin T(C, V)$. Set $v := g_n(P_n(t_1\sigma), \ldots, P_n(t_m\sigma))$ and let $P(t_i\sigma) = [v_{i,1} \ldots v_{i,j_i}]$ for all safe argument positions $i = m + 1, \ldots, m + n$, hence

$$P(t_\sigma) = [g_n(P_n(t_1\sigma), \ldots, P_n(t_m\sigma)) v_{m+1,1} \ldots v_{m+1,j_{m+1}} \ldots v_{m+n,1} \ldots v_{m+n,j_{m+n}}].$$

Applying Lemma 6.1 on all normal arguments of $t$ we see

$$f_n(P_n(s_1\sigma), \ldots, P_n(s_k\sigma)) \succ_{2|\ell-1} g_n(P_n(t_1\sigma), \ldots, P_n(t_m\sigma))$$

(6.3)

from the assumptions $f_n > g_n$ and $s \gtrsim_{\text{pop}} t_i$ for all $i = 1, \ldots, m$. Since $s \gtrsim_{\text{pop}} t_{i_0}$ by assumption, induction hypothesis on $i_0$ gives

$$f_n(P_n(s_1\sigma), \ldots, P_n(s_k\sigma)) \succ_{2|\ell_0-1} [v_{i_0,1} \ldots v_{i_0,j_{i_0}}] = P(t_{i_0}\sigma).$$

Employing $2|\ell_0| < 2|\ell| - 1$, it is not difficult to check that due to the above inequality we have

$$f_n(P_n(s_1\sigma), \ldots, P_n(s_k\sigma)) \succ_{2|\ell-1} v_{i_0,j_0} \quad \text{for some } j_0 \in \{1, \ldots, j_{i_0}\}$$

(6.4)

$$f_n(P_n(s_1\sigma), \ldots, P_n(s_k\sigma)) \succ_{2|\ell-1} v_{i,j} \quad \text{for all } j = 1, \ldots, j_{i_0}, j \neq j_0.$$  

(6.5)

Similar induction hypothesis on safe argument positions $i = m + 1, \ldots, m + n$ ($i \neq i_0$) of $g$, where in particular $t_i \in T(F^{<\text{Fun}}(s), V)$ by assumption, gives

$$f_n(P_n(s_1\sigma), \ldots, P_n(s_k\sigma)) \succ_{2|\ell-1} v_{i,j} \quad \text{for all } i = m + 1, \ldots, m + n, i \neq i_0 \text{ and } j = 1, \ldots, j_i.$$  

(6.6)

Observe $\text{len}(P(t_\sigma)) \leq |\ell|$ by Lemma 4.4(1). Summing up, Assertion 1 follows by $\Box$ using Equations (6.3), (6.4), (6.5) and (6.6). Likewise, Assertion 2 follows using additionally

$$f_n(P_n(s_1\sigma), \ldots, P_n(s_k\sigma)) \succ_{2|\ell-1} \ast$$

and

$$\text{len}(P(t_\sigma)) \leq 2|\ell| + \max\{\text{norm}(s_1\sigma), \ldots, \text{norm}(s_k\sigma), \text{norm}(s)\}$$

by Lemma 4.4(3)

$$\leq 2|\ell| + \text{wd}(f_n(P_n(s_1\sigma), \ldots, P_n(s_k\sigma)) \ast \text{norm}(s)).$$

Finally, for Assertion 2 we proceed exactly as above, but strengthen the inequality (6.4) to $f_n(P_n(s_1\sigma), \ldots, P_n(s_k\sigma)) \succ_{2|\ell-1} v_{i_0,j_0}$ which follows as $t_{i_0} \notin T(F^{<\text{Fun}}(s), V)$ by assumption, and thus induction hypothesis can be strengthened to $f_n(P_n(s_1\sigma), \ldots, P_n(s_k\sigma)) \succ_{2|\ell_0-1} P(t_{i_0}\sigma)$. This concludes the case $\Box$.

Case $s \gtrsim_{\text{pop}} t$:

Then $t = g(t_1, \ldots, t_m; t_{m+1}, \ldots, t_{m+n})$ where $f \sim g$. Further, the assumption gives $\{s_1, \ldots, s_k\} \gtrsim_{\text{pop}} \{t_1, \ldots, t_m\}$ and $\{s_{k+1}, \ldots, s_{k+l}\} \gtrsim_{\text{pop}} \{t_{m+1}, \ldots, t_{m+n}\}$. Hence $t \notin T(F^{<\text{Fun}}(s), V)$ and Property 2 is vacuously satisfied. We prove Property 1 and Property 3.

Using that $s_i \in T(C, V)$ for all normal argument positions $i = 1, \ldots, m$ and employing Lemma 5.6 we see exactly as in the case $\Box$ above that $\{s_1, \ldots, s_k\} \gtrsim_{\text{pop}} \{t_1, \ldots, t_m\}$ implies $\{P_n(s_1\sigma), \ldots, P_n(s_k\sigma)\} \succ_{2|\ell-1} \{P_n(t_1\sigma), \ldots, P_n(t_m\sigma)\}$. Hence

$$f_n(P_n(s_1\sigma), \ldots, P_n(s_k\sigma)) \succ_{2|\ell-1} g_n(P_n(t_1\sigma), \ldots, P_n(t_m\sigma))$$

(6.7)
follows as by assumption $f_n \sim g_n$ and clearly $m \leq |t| \leq 2 \cdot |t| - 1$. Note that the assumption 
$\{s_{k+1}, \ldots, s_{k+l}\} \cup \{t_{m+1}, \ldots, t_{m+k}\}$ together with $s_i \in T(C, V)$ for all $i = k+1, k+l$ gives 
$t_j \in T(C, V)$, and consequently $P_2(t_j) = [\cdot]$ for all $j = k+1, \ldots, k+l$. Hence 
\[ f_n(P_n(s_1), \ldots, P_n(s_k)) \cdot_{2|t|} [g_n(P_n(t_1), \ldots, P_n(t_m))] = P_2(t) \]
which concludes Assertion \[1\].

To prove Assertion \[3\] we additionally verify $\text{norm}(t) \leq \text{norm}(s)$ by case analysis on $\text{norm}(t)$. Thus $\text{norm}(s) \cdot \text{norm}(t)$ follows. Using this and Equation (6.7) we obtain 
\[ f_n(P_n(s_1), \ldots, P_n(s_k)) \sim \text{norm}(s) \cdot_{2|t|} g_n(P_n(t_1), \ldots, P_n(t_m)) \sim \text{norm}(t) \] 
by Lemma \[5.0(1)\] and Lemma \[5.0(3)\].

We conclude the lemma.

\[\]

**Lemma 6.3.** Let $\ell \geq \max\{\text{ar}(f_n) \mid f_n \in F_n\}$ and $s, t \in T(F, V)$. Then for $P \in \{P_0, P_3\}$,

\[ P(s) \cdot_\ell P(t) \implies P(C[s]) \cdot_\ell P(C[t]) . \]

**Proof.** It suffices to consider the inductive step. Consider terms $s = f(s_1, \ldots, s_i, \ldots, s_{k+l})$ and $t = f(t_1, \ldots, t_i, \ldots, s_{k+l})$. We show that for $P \in \{P_0, P_3\}$, under the assumption $P(s) \cdot_\ell P(t)$ also $P(f(s_1, \ldots, s_i, \ldots, s_{k+l})) \cdot_\ell P(f(s_1, \ldots, t_i, \ldots, s_{k+l}))$ holds.

**Case** $P = P_0$: Consider the non-trivial case $t \notin T(C, V)$. Without loss of generality, suppose the first $k$ argument positions of $f$ are normal, and the remaining $\ell$ positions are safe. Depending on the position $i$, we distinguish two cases. If $i \in \{k+1, \ldots, k+l\}$ is safe, then by definition 
\[ P_0(s) = [f_n(P_n(s_1), \ldots, P_n(s_k))] \sim P_0(s_{k+1}) \sim \cdots \sim P_0(s_i) \sim \cdots \sim P_0(s_{k+l}), \]

\[ P_0(t) = [f_n(P_n(s_1), \ldots, P_n(s_k))] \sim P_0(s_{k+1}) \sim \cdots \sim P_0(t_i) \sim \cdots \sim P_0(s_{k+l}) \]

If $i$ is a normal argument position, the assumption $P_0(s_i) \cdot_\ell P_0(t_i)$ and $\ell \geq k$ gives 
\[ f_n(P_n(s_1), \ldots, P_n(s_i), \ldots, P_n(s_k)) \cdot_\ell f_n(P_n(s_1), \ldots, P_n(t_i), \ldots, P_n(s_k)) \]

and the lemma follows again using Lemma \[5.0(3)\].

**Case** $P = P_3$: Recall that $P_3(s) = P_3(s) \sim \text{norm}(s)$ and $P_3(t) = P_3(t) \sim \text{norm}(t)$. If $\text{norm}(s) \geq \text{norm}(t)$ then $P_3(s) \cdot_\ell P_3(t)$ follows from $P_3(s) \cdot_\ell P_3(t)$ and Lemma \[5.0(3)\]. Hence suppose $\text{norm}(s) < \text{norm}(t)$. First, consider the more involved case $t \notin T(C, V)$. As $\text{norm}(s) < \text{norm}(t)$ implies that $i$ is a safe argument position of $f$, we thus have 
\[ P_3(s) = [f_n(P_n(s_1), \ldots, P_n(s_k))] \sim P_3(s_{k+1}) \sim \cdots \sim P_3(s_i) \sim \cdots \sim P_3(s_{k+l}) \sim \text{norm}(s), \]

\[ P_3(t) = [f_n(P_n(s_1), \ldots, P_n(s_k))] \sim P_3(s_{k+1}) \sim \cdots \sim P_3(t_i) \sim \cdots \sim P_3(s_{k+l}) \sim \text{norm}(t) \]

Using Lemma \[5.0(2)\] and Lemma \[5.0(3)\] we see that $P_3(s) \cdot_\ell P_3(t)$ follows from $P_3(s) \sim \text{norm}(s) \cdot_\ell P_3(t) \sim \text{norm}(t)$. Note $\text{norm}(s_i) < \text{norm}(s)$ and observe that the assumption $\text{norm}(s) < \text{norm}(t)$ gives $\text{norm}(t) = \text{norm}(t_i) + 1$ by the shape of $s$ and $t$. Thus using Lemma \[5.0(3)\] and the assumption $P_3(s_i) \cdot_\ell P_3(t_i)$ we can even prove the stronger property $P_3(s_i) \sim \text{norm}(s_i) \cdot_\ell P_3(t_i) \sim \text{norm}(t_i) \sim \text{norm}(t)$. By similar reasoning we can also prove $t \in T(C, V)$ where $P_3(t) = \text{norm}(t)$. This concludes the case analysis.

\[\]
We have established the embedding for completely defined TRSs. Putting things together we obtain: This allows us to estimate the derivation height in terms of \( G_\ell \).

**Lemma 6.4.** Let \( \mathcal{R} \) be a completely defined TRS compatible with \( \succ_{\text{pop}} \). Define \( \ell := \max\{\ar(f_n) \mid f_n \in F_n\} \cup \{2 \cdot |r| \mid l \rightarrow r \in \mathcal{R}\} \) and \( P \in \{P, \bar{P}\} \). Then \( dh(s, \downarrow_{\mathcal{R}}) \leq G_\ell(P(s)) \).

**Proof.** Suppose \( \mathcal{R} \) is completely defined TRS compatible with \( \succ_{\text{pop}} \), and let \( \ell \) be given by the Lemma. Consider a maximal \( \mathcal{R} \)-derivation

\[
s \xrightarrow{\mathcal{R}} s_1 \xrightarrow{\mathcal{R}} s_2 \xrightarrow{\mathcal{R}} \ldots \xrightarrow{\mathcal{R}} s_m ,
\]

starting from an arbitrary term \( s \), i.e., \( m = dh(s, \downarrow_{\mathcal{R}}) \). Using Lemma 6.2 together with Lemma 6.3 \( m \)-times we get

\[
P(s) \triangleright_\ell P(s_1) \triangleright_\ell P(s_2) \triangleright_\ell \ldots \triangleright_\ell P(s_m)
\]

and consequently \( m \leq G_\ell(P(s)) \) by definition.

The final proof step is to lift the requirement that \( \mathcal{R} \) is completely defined. Call a normal form \( s \) garbage if its root symbol is defined. Let \( \bot \notin F \) be a fresh constructor symbol. For each garbage term \( s \) we extend \( \mathcal{R} \) by a rule that replaces \( s \) with \( \bot \). Although infinite, the resulting system is completely defined.

**Definition 6.5.** Let \( \bot \) be a fresh constructor symbol \( \bot \notin F \) and \( \mathcal{R} \) a TRS over \( F \). We define \( S_\mathcal{R} \) over the signature \( F \cup \{\bot\} \) by

\[
S_\mathcal{R} := \{ t \rightarrow \bot \mid t \in T(F \cup \{\bot\}, V) \text{ is a normal form of } \mathcal{R} \text{ with defined root symbol} \} .
\]

We set \( \mathcal{R}_\bot := \mathcal{R} \cup S_\mathcal{R} \).

We extend the precedence \( \triangleright \) on \( F \) to \( F \cup \{\bot\} \) so that \( \bot \) is minimal. As clearly \( s \triangleright_{\text{pop}} \bot \) for each garbage term \( s \), for predicative TRS \( \mathcal{R} \) the TRS \( S_\mathcal{R} \) is compatible with \( \succ_{\text{pop}} \). Note that \( S_\mathcal{R} \) is confluent and terminating, in particular every term \( s \) has a unique normal form with respect to \( S_\mathcal{R} \), in notation \( s \downarrow \). Clearly \( f(s_1, \ldots, s_n) \downarrow = f(s_1 \downarrow, \ldots, s_n \downarrow) \downarrow \). Exploiting that the additional rules do not interfere with pattern matching of \( \mathcal{R} \), the TRS \( \mathcal{R}_\bot \) is able to simulate \( \mathcal{R} \) in the following sense.

**Lemma 6.6.** Suppose \( \mathcal{R} \) is a constructor TRS. Then

\[
s \xrightarrow{\mathcal{R}} t \implies s \downarrow \xrightarrow{\mathcal{R}_\bot} t \downarrow
\]

**Proof.** Suppose \( s \xrightarrow{\mathcal{R}} t \), i.e., \( s = C[f(l_1\sigma_1, \ldots, l_n\sigma_1)] \) and \( t = C[r\sigma] \) for some context \( C \), rule \( f(l_1, \ldots, l_n) \rightarrow r \in \mathcal{R} \) and substitution \( \sigma \) where \( l_i\sigma \in \text{NF}(\mathcal{R}) \) for all \( i = 1, \ldots, n \). We continue by induction on \( C \). Let \( \sigma_j(x) := x\sigma \uparrow \) for all \( x \in \text{dom}(\sigma) \).

Consider the base case \( C = \emptyset \). Since \( \mathcal{R} \) is by assumption a constructor TRS, the direct arguments of the left-hand sides of \( \mathcal{R} \) do not contain defined symbols, consequently \( l_i\sigma = l_i\sigma_1 \downarrow = l_i\sigma_1 \) is a constructor term for all \( i = 1, \ldots, n \). We conclude the inductive step

\[
f(l_1\sigma_1, \ldots, l_n\sigma_1) \downarrow = f(l_1\sigma_1, \ldots, l_n\sigma_1) \rightarrow_{\mathcal{R}_\bot} r\sigma_1 \rightarrow_{\mathcal{R}_\bot} (r\sigma) \downarrow .
\]

Here in the first equality we employ that \( f(l_1\sigma_1, \ldots, l_n\sigma_1) \) is not a normal form of \( \mathcal{R} \).

For the inductive step, let \( s = f(s_1, \ldots, s_i, \ldots, s_n) \) and \( t = f(s_1, \ldots, t_i, \ldots, s_n) \) where \( s_i \rightarrow_{\mathcal{R}} t_i \). Induction hypothesis gives \( s_i \rightarrow_{\mathcal{R}} t_i \downarrow \). Then

\[
s \downarrow = f(s_1 \downarrow, \ldots, s_i \downarrow, \ldots, s_n) \rightarrow_{\mathcal{R}_\bot} f(s_1 \downarrow, \ldots, t_i, \downarrow, \ldots, s_n \downarrow) \rightarrow_{\mathcal{R}_\bot} t \downarrow .
\]

For the first equality we employ that \( s_i \notin \text{NF}(\mathcal{R}) \). This concludes the proof.
An immediate consequence is the following.

**Lemma 6.7.** Let $\mathcal{R}$ be a predicative recursive TRS. Then $\mathcal{R}_1$ is a completely defined TRS compatible with $\succ_{pop^*}$. Further $dh(s, \stackrel{1}{\rightarrow}_{\mathcal{R}}) \leq dh(s, \stackrel{1}{\rightarrow}_{\mathcal{R}_1})$ for all basic terms $s$.

**Proof.** We have already observed that $\mathcal{R}_1$ is compatible with $\succ_{pop^*}$. Moreover it is completely defined by definition. As $\mathcal{R}$ is predicative recursive, it is a constructor TRS. To prove the second half of the assertion, consider a maximal derivation

$$s \stackrel{1}{\rightarrow}_{\mathcal{R}} s_1 \stackrel{1}{\rightarrow}_{\mathcal{R}} s_2 \stackrel{1}{\rightarrow}_{\mathcal{R}} \cdots \stackrel{1}{\rightarrow}_{\mathcal{R}} s_m$$

starting from a basic term $s$, i.e., $m = dh(s, \stackrel{1}{\rightarrow}_{\mathcal{R}})$. If $m = 0$ the lemma is immediate. For the case $m > 0$, $m$-times application of Lemma 6.6 gives

$$s \downarrow \stackrel{1}{\rightarrow}_{\mathcal{R}} s_1 \downarrow \stackrel{1}{\rightarrow}_{\mathcal{R}} s_2 \downarrow \stackrel{1}{\rightarrow}_{\mathcal{R}} \cdots \stackrel{1}{\rightarrow}_{\mathcal{R}} s_m \downarrow .$$

Hence overall, $dh(s, \stackrel{1}{\rightarrow}_{\mathcal{R}}) \leq dh(s, \stackrel{1}{\rightarrow}_{\mathcal{R}_1})$. Since by assumption $s$ is a basic term not in normal form, we have $s \downarrow = s$ and the lemma follows.

We arrive at the proof of the main theorem:

**Proof of Theorem 7.1.** Let $\mathcal{R}$ be a predicative recursive TRS and fix an arbitrary basic term $s = f(u_1, \ldots, u_m; u_{m+1}, \ldots, u_{m+n})$. Set $\ell = \max \{\text{ar}(f_n) \mid f_n \in F_0\} \cup \{2 \cdot |r| \mid l \rightarrow r \in \mathcal{R}_1\}$ and note that $\ell$ is well defined since $F_0$ and $\mathcal{R}$ are finite. Putting things together we see

$$dh(s, \stackrel{1}{\rightarrow}_{\mathcal{R}}) \leq dh(s, \stackrel{1}{\rightarrow}_{\mathcal{R}_1}) \quad \text{using Lemma 6.7}$$

$$\leq G_\ell(P(s)) \quad \text{using Lemma 6.4}$$

$$\in O\left(\left(\max_{i=1}^{m} dp(u_i)\right)^d\right) \quad \text{using Corollary 5.16}$$

where $d$ depends only on $\ell$. 

We now present the application of polynomial path orders in the context of *implicit computational complexity (ICC)*. As by-product of Proposition 2.3 and Theorem 3.7 we immediately obtain that POP* is sound for FNP respectively FP.

**Theorem 7.1.** Let $\mathcal{R}$ be a predicative recursive TRS. For every relation $[f]$ defined by $\mathcal{R}$, the functional problem $F_f$ associated with $[f]$ is in FNP. Moreover, if $\mathcal{R}$ is confluent then $[f] \in FP$.

Although it is decidable whether a TRS $\mathcal{R}$ is predicative recursive (we present a sound and complete automation in Section 10), confluence is undecidable in general. To get a decidable result for FP, one can replace by an decidable criteria, for instance orthogonality.

We will now also establish that POP* is complete for FP, that is, every function $f \in FP$ is computed by some confluent (even orthogonal) predicative recursive TRS. For this we use the term rewriting formulation of the predicative recursive functions from 12.

**Definition 7.2.** For each $k, l \in \mathbb{N}$ the set of function symbols $F_B^{k,l}$ with $k$ normal and $l$ safe argument positions is the least set of function symbols such that

1. $\epsilon \in F_B^{0,0}$, $S_1, S_2 \in F_B^{0,1}$, $P \in F_B^{0,1}$, $C \in F_B^{0,4}$ and $I_j^{k,l}, O^{k,l} \in F_B^{k,l}$, where $j = 1, \ldots, k + l$;
2. if $\bar{r} = r_1, \ldots, r_m \in F_B^{k,0}$, $\bar{s} = s_1, \ldots, s_n \in F_B^{k,l}$ and $h \in F_B^{m,n}$ then $SC[h, \bar{r}, \bar{s}] \in F_B^{k,l}$;
(3) if \( g \in \mathcal{F}_B^{k,l} \) and \( h_1, h_2 \in \mathcal{F}_B^{k+1,l+1} \) then \( \text{SRN}[g, h_1, h_2] \in \mathcal{F}_B^{k+1,l} \);

The *predicative signature* is given by \( \mathcal{F}_B := \bigcup_{k,l \in \mathbb{N}} \mathcal{F}_B^{k,l} \). Only the constant \( \epsilon \) and dyadic successors \( S_1, S_2 \), which serve the purpose of encoding natural numbers in binary, are constructors.

The remaining symbols from \( \mathcal{F}_B \) are defined by the following (infinite) schema of rewrite rules \( R_B \). Here we \( k, l \) range over \( \mathbb{N} \) and we abbreviate \( \bar{x} = x_1, \ldots, x_k \) and \( \bar{y} = y_1, \ldots, y_l \) for \( k \) respectively \( l \) distinct variables.

**Initial Functions**

\[
P(\epsilon) \rightarrow \epsilon
\]

\[
P(S_i(x)) \rightarrow x
\]

\[
I_j^{k,l}(\bar{x}; \bar{y}) \rightarrow x_j
\] for all \( j = 1, \ldots, k \)

\[
I_j^{k,l}(\bar{x}; \bar{y}) \rightarrow y_{j-k}
\] for all \( j = k + 1, \ldots, l + k \)

\[
C(\epsilon, \bar{y}, z_1, z_2) \rightarrow y
\]

\[
C(S_i(x), y, z_1, z_2) \rightarrow z_i
\] for \( i = 1, 2 \)

\[
O(\bar{x}; \bar{y}) \rightarrow \epsilon
\]

**Safe Composition (SC)**

\[
\text{SC}[h, \bar{r}, \bar{s}](\bar{x}; \bar{y}) \rightarrow h(\bar{r}(\bar{x}); \bar{s}(\bar{x}; \bar{y}))
\]

**Safe Recursion on Notation (SRN)**

\[
\text{SRN}[g, h_1, h_2](\epsilon, \bar{x}; \bar{y}) \rightarrow g(\bar{x}; \bar{y})
\]

\[
\text{SRN}[g, h_1, h_2](S_i(z), \bar{x}; \bar{y}) \rightarrow h_i(z, \bar{x}; \bar{y}, \text{SRN}[g, h_1, h_2](z, \bar{x}; \bar{y}))
\] for \( i = 1, 2 \)

We emphasise that the above rules are all orthogonal. Also, we stress that the system \( R_B \) is dupped infeasible in [12]. Indeed \( R_B \) admits an exponential lower bound on the derivation height which has to do with effects caused by duplicating redexes as explained already in Example 2.6 on page 10. Therefore \( R_B \) is not (directly) suitable as a term-rewriting characterisation of the predicative recursive functions. However this exponential lower-bound is only correct if we consider unrestricted rewriting. The following proposition verifies that \( R_B \) generates only polytime computable functions.

**Proposition 7.3.** [12, Lemma 5.2] Let \( f \in \text{FP} \). There exists a finite restriction \( \mathcal{R}_f \not\subseteq R_B \) such that \( \mathcal{R}_f \) computes \( f \).

We arrive at our completeness result.

**Theorem 7.4.** For every \( f \in \text{FP} \) there exists an orthogonal predicative recursive TRS \( \mathcal{R}_f \) that computes \( f \).

**Proof.** Take the TRS \( \mathcal{R}_f \not\subseteq R_B \) from Proposition 7.3 that computes \( f \). Obviously \( \mathcal{R}_f \) is orthogonal hence confluent, it remains to verify that \( \mathcal{R}_f \) is compatible with some instance \( \triangleright_{\text{pop}} \). To define \( \triangleright_{\text{pop}} \), we use the separation of normal from safe argument positions as indicated in the rules. To define the precedence underlying \( \triangleright_{\text{pop}} \), we first define a mapping \( \text{lh} \) from the signature of \( \mathcal{F}_B \) into the natural numbers as follows:

- \( \text{lh}(f) := 0 \) if \( f \) is one of \( \epsilon, S_0, S_1, C, P, I_j^{k,l} \) or \( O^{k,l} \);
- \( \text{lh}(\text{SC}[h, \bar{r}, \bar{s}]) := 1 + \text{lh}(h) + \sum_{r \in \bar{r}} \text{lh}(r) + \sum_{s \in \bar{s}} \text{lh}(s) \);
- \( \text{lh}(\text{SRN}[g, h_1, h_2]) := 1 + \text{lh}(g) + \text{lh}(h_1) + \text{lh}(h_2) \).
Finally for each pair of function symbol \( f \) and \( g \) occurring in \( \mathcal{R}_f \) set \( f > g \) if and only if \( \text{lh}(f) > \text{lh}(g) \). Then \( > \) defines an admissible precedence. It is straightforward to verify that \( \mathcal{R}_f \subseteq \succ_{\text{pop}_*} \), where \( \succ_{\text{pop}_*} \) is based on the precedence \( > \) and the safe mapping as indicated in Definition 7.2.

By Theorem 7.1 and Theorem 7.4 we thus obtain a precise characterisation of the class polytime computable functions.

**Corollary 7.5.** The class of confluent (or orthogonal) predicative recursive TRSs define exactly \( \text{FP} \).

### 8. A Non-Trivial Closure Property of the Polytime Computable Functions

Bellantoni already observed that the class \( \mathcal{B} \) is closed under *predicative recursion on notation with parameter substitution* (scheme \( \text{(SRN}_{\text{ps}}) \)). Essentially this recursion scheme allows substitution on *safe* argument positions. More precise, a new function \( f \) is defined by the equations

\[
\begin{align*}
    f(0, \bar{x}; \bar{y}) &= g(\bar{x}; \bar{y}) \\
    f(2z + i, \bar{x}; \bar{y}) &= h_i(z, \bar{x}; \bar{y}; f(z, \bar{x}; \bar{p}(\bar{x}; \bar{y}))), \quad i \in \{1, 2\}.
\end{align*}
\]

Notably closure of \( \mathcal{B} \) under parameter substitution has been proven also been Beckmann and Weiermann [12] based on rewriting techniques. In the following we introduce a polynomial path order beyond MPO, the *polynomial path order with parameter substitution* \( (\text{POP}_{\text{ps}}^*) \). The next definition introduces \( \text{POP}_{\text{ps}}^* \). It is a variant of \( \text{POP}^* \) where clause \( \succ_{\text{pop}_*} \) has been modified and allows computation at safe argument positions.

**Definition 8.1.** Let \( s, t \in T(F, V) \) such that \( s = f(s_1, \ldots, s_k; s_{k+1}, \ldots, s_{k+l}) \). Then \( s \succ_{\text{pop}_*}^* t \) with respect to the precedence \( \succ \) and safe mapping *safe* if either

1. \( s_i \succ_{\text{pop}_*}^* t \) for some \( i \in \{1, \ldots, k+l\} \), or
2. \( f \in D, t = g(t_1, \ldots, t_m; t_{m+1}, \ldots, t_{m+n}) \) where \( f > g \) and the following conditions hold:
   - \( s \succ_{\text{pop}_*}^* t_j \) for all normal argument positions \( j = 1, \ldots, m \);
   - \( s \succ_{\text{pop}_*}^* t_j \) for all safe argument positions \( j = m+1, \ldots, m+n \);
   - \( t_j \notin T(F^{<\text{Fun}(s)}, V) \) for at most one safe argument position \( j \in \{m+1, \ldots, m+n\} \);
3. \( f \in D, t = g(t_1, \ldots, t_m; t_{m+1}, \ldots, t_{m+n}) \) where \( f \sim g \) and the following conditions hold:
   - \( \{s_1, \ldots, s_k\} \succ_{\text{pop}_*}^* \{t_1, \ldots, t_m\} \);
   - \( s \succ_{\text{pop}_*}^* t_j \) and \( t_j \notin T(F^{<\text{Fun}(s)}, V) \) for all safe argument positions \( j = m+1, \ldots, m+n \).

Here \( \succ_{\text{pop}_*}^* := \succ_{\text{pop}_*} \cup \succ_\ast \).

We adapt the notion of predicative recursive TRS to \( \text{POP}_{\text{ps}}^* \) in the obvious way. It is not difficult to see that \( \text{POP}_{\text{ps}}^* \) extends the analytic power of \( \text{POP}^* \).

**Lemma 8.2.** For any underlying admissible precedence \( \succ, \succ_{\text{pop}_*} \subseteq \succ_{\text{pop}_*}^* \).

Note that \( \text{POP}_{\text{ps}}^* \) is strictly more powerful than \( \text{POP}^* \), as witnessed by the following example.
Example 8.3. Consider the TRS $\mathcal{R}_{rev}$ defining the reversal of lists in a tail recursive fashion:

$$rev(xs) \rightarrow rev(\text{nil}) \quad rev([\ ]; ys) \rightarrow ys \quad rev(\text{cons}(x, xs); ys) \rightarrow rev(\text{cons}(x, ys)).$$

Then $\mathcal{R}_{rev} \subseteq \succ_{\text{pop}_\ast}$ with precedence $\succ_{\text{pop}_\ast} \succ \text{nil} \sim \text{cons}$. Note that orientation of the last rule with $\succ_{\text{pop}_\ast}$ breaks down to $\text{cons}(x, xs) \succ_{\text{pop}_\ast} xs$ and $\text{rev}(\text{cons}(x, xs); ys) \succ_{\text{pop}_\ast} \text{cons}(x, ys)$. On the other hand, $\succ_{\text{pop}_\ast}$ fails as the corresponding clause $\succ_{\text{pop}_\ast}$ requires $ys \succ_{\text{pop}_\ast} \text{cons}(x, ys)$.

The order $\text{POP}^*_\ast$ is complete for the class of polytime computable functions. To show that it is sound, we prove that $\text{POP}^*_\ast$ induces polynomially bounded runtime complexity in the sense of Theorem 3.7. The crucial observation is that the embedding of $\downarrow_{\mathcal{R}}$ into $\bullet_k$ does not break if we relax compatibility constraints to $\mathcal{R} \subseteq \succ_{\text{pop}_\ast}$.

Lemma 8.4. Suppose $s = f(s_1, \ldots, s_k; s_{k+1}, \ldots, s_{k+l}) \in T_p$, $t \in T(F, V)$ and $\sigma : V \rightarrow T(C, V)$. Then for predicative interpretation $P \in \{P_3, P_7\}$ we have

$$s \succ_{\text{pop}_\ast} t \implies P(s) \bullet_{2|t|} P(t).$$

Proof. First one verifies that Lemma 8.4 holds even if we replace $\succ_{\text{pop}_\ast}$ by $\succ_{\text{pop}_\ast}^\ast$. In particular, the assumptions give

$$\text{len}(P_2(t\sigma)) \leq 2 \cdot |t| + \max\{\text{norm}(s_1\sigma), \ldots, \text{norm}(s_k\sigma), \text{norm}(s\sigma)\}. \quad (8.1)$$

The proof proceeds then in correspondence to Lemma 8.2 by induction on $\succ_{\text{pop}_\ast}$. We cover only the new case. Let $s, t, \sigma$ be as given in the lemma.

Case $s \succ_{\text{pop}_\ast}^\ast t$: Then $t = g(t_1, \ldots, t_m; t_{m+1}, \ldots, t_{m+n})$ where $f \sim g$. Further, the assumption gives $\{s_1, \ldots, s_k\} \succ_{\text{pop}_\ast}^\ast \{t_1, \ldots, t_m\}$. As $t \notin T(F^{\text{Fun}(s)}, V)$ it suffices to verify Property 1 and Property 3 from Lemma 8.2. Exactly as in the corresponding case of Lemma 8.2 we see

$$f_n(P_3(s_1\sigma), \ldots, P_3(s_k\sigma)) \bullet_{2|t| - 1} g_n(P_3(t_1\sigma), \ldots, P_3(t_m\sigma)). \quad (8.2)$$

As by assumption $s \succ_{\text{pop}_\ast} t_j$ and $t_j \in T(F^{\text{Fun}(s)}, V)$, induction hypothesis gives

$$f_n(P_3(s_1\sigma), \ldots, P_3(s_k\sigma)) \succ_{2|t| - 1} P_3(t_j\sigma). \quad (8.3)$$

As $\text{len}(P_3(t\sigma)) \leq |t|$ by Lemma 8.4(1), we obtain $f_n(P_3(s_1\sigma), \ldots, P_3(s_k\sigma)) \bullet_{2|t|} P_3(t\sigma)$ from Equation (8.2) and Equation (8.3). Likewise, from this Assertion 3 follows by $\bullet_{k}$ using additionally $f_n(P_3(s_1\sigma), \ldots, P_3(s_k\sigma)) \succ_{2|t| - 1} \bullet$ and

$$\text{len}(P_3(t\sigma)) \leq 2 \cdot |t| + \max\{\text{norm}(s_1\sigma), \ldots, \text{norm}(s_k\sigma), \text{norm}(s\sigma)\} \quad \text{by Equation (8.1)}$$

$$\leq 2 \cdot |t| + \text{wd}(f_n(P_3(s_1\sigma), \ldots, P_3(s_k\sigma)) \sim \text{norm}(s\sigma)).$$

\qed
Following the Proof of Theorem 3.7 but replacing Lemma 6.2 by Lemma 8.4 we obtain:

**Theorem 8.5.** Let \( \mathcal{R} \) be predicative recursive TRS (in the sense of Definition 8.1). Then the innermost derivation height of any basic term \( f(\vec{u}; \vec{v}) \) is bounded by a polynomial in the maximal depth of normal arguments \( \vec{u} \). The polynomial depends only on \( \mathcal{R} \) and the signature \( F \).

Using this theorem, Proposition 2.3 states that \( \text{POP}_F^* \) is sound for the polytime computable functions. Lemma 8.2 together with Theorem 7.4 shows completeness of \( \text{POP}_F^* \) for the polytime computable functions.

**Corollary 8.6.** The class of confluent (or orthogonal) predicative recursive TRSs (in the sense of Definition 8.1) define exactly \( \text{FP} \).

### 9. Automation of Polynomial Path Orders

In this section we present an automation of polynomial path orders, for brevity we restrict our efforts to the order \( \succ_{\text{pop}} \). Consider a constructor TRS \( \mathcal{R} \). Checking whether \( \mathcal{R} \) is predicative recursive is equivalent to guessing a precedence \( \succeq \) and partitioning of argument positions so that \( \mathcal{R} \subseteq \succ_{\text{pop}}^* \) holds for the induces order \( \succ_{\text{pop}}^* \). As standard for recursive path orders [39, 45], this search can be automated by encode the constraints imposed by Definition 8.5 into propositional logic. To simplify the presentation, we extend language of propositional logic with truth-constants \( \top \) and \( \bot \) in the obvious way. In the constraint presented below we employ the following atoms.

**Propositional Atoms.** To encode the separation of normal from safe arguments, we introduce \( f \in D \) and \( i = 1, \ldots, \text{ar}(f) \) the atoms \( \text{safe}_{f,i} \) so that \( \text{safe}_{f,i} \) represents the assertion that the \( i^{th} \) argument position of \( f \) is safe. Further we set \( \text{safe}_{f,i} := \top \) for \( n \)-ary \( f \in C \) and \( i = 1, \ldots, n \) which reflects that argument positions of constructors are always safe.

One verifies that predicative recursive TRSs are even compatible with \( \succ_{\text{pop}}^* \) as induced by an admissible precedence where constructors are equivalent, that is, polynomial path orders are blind on constructors. This is exploited in the propositional encoding of precedences, where we encode a precedence \( \succeq \) on the set of defined symbols \( D \) only: For each pair of symbols \( f, g \in D \), we introduce propositional atoms \( \succ_{f,g} \) and \( \sim_{f,g} \) so that \( \succ_{f,g} \) represents the assertion \( f > g \), and likewise \( \sim_{f,g} \) represents the assertion \( f \sim g \). Overall we define for function symbols \( f \) and \( g \) the propositional formulas

\[
\begin{align*}
\top & \quad \text{if } f \in D \text{ and } g \in C, \\
\bot & \quad \text{if } f \in C \text{ and } g \in C, \\
\succ_{f,g} & \quad \text{if } f \in D \text{ and } g \in C, \\
\sim_{f,g} & \quad \text{otherwise.}
\end{align*}
\]

To ensure that the variables \( \succ_{f,g} \) and respectively \( \sim_{f,g} \) encode a preorder on \( D \) we encode an order preserving homomorphism into the natural order \( > \). To this extend, to each \( f \in D \) we associate a natural number \( \text{rk}_f \) encoded as binary string with \( \lceil \log_2(|D|) \rceil \) bits. It is straightforward to define Boolean formulas \( \succ_{f,g} \text{ and } \sim_{f,g} \) (respectively \( \succ_{f,g} = \text{rk}_f > \text{rk}_g \) ) that are satisfiable iff the binary numbers \( \text{rk}_f \) and \( \text{rk}_g \) are decreasing (respectively equal) in the natural order. Using these we set

\[
\text{valid-precedence}(D) := \bigwedge_{f,g \in D} (\succ_{f,g} \rightarrow \text{rk}_f > \text{rk}_g) \land \bigwedge_{f,g \in D} (\sim_{f,g} \rightarrow \text{rk}_f = \text{rk}_g)
\]
We say that a propositional assignment \( \mu \) induces the precedence \( \succeq \) if \( \mu \) satisfies \('f > g'\) when \( f > g \) and \('f \sim g'\) when \( f \sim g \). The next lemma verifies that valid-precedence serves our needs.

**Lemma 9.1.** For any valuation \( \mu \) that satisfies valid-precedence(\( D \)), \( \mu \) induces an admissible precedence on \( F \). Vice versa, for any admissible precedence \( \succeq \) on \( F \), any valuation \( \mu \), satisfying \( \mu('f > g') \) iff \( f > g \) and \( \mu('f \sim g') \) iff \( f \sim g \), also satisfies the formula valid-precedence(\( D \)).

**Order Constraints.** For concrete pairs of terms \( s = f(s_1, \ldots, s_n) \) and \( t \), we define the order constraints

\[
's \gg_{\text{pop}} t' := 's \gg_{\text{pop}} t' \lor 's \gg_{\text{pop}} t' \lor 's \gg_{\text{pop}} t'
\]

which enforces the orientation \( f(s_1, \ldots, s_n) >_{\text{pop}} t \) using propositional formulations of the three clauses in Definition 3.5. To complete the definition for arbitrary left-hand sides, we set \('x >_{\text{pop}} t' := \bot \) for all \( x \in V \). Further weak orientation is given by

\[
's \gg_{\text{pop}} t' := 's \gg_{\text{pop}} t' \lor 's \gg t',
\]

where the constraint \('s \gg t'\) refers to a formulation of Definition 3.2 in propositional logic, defined as follows. For \( s = t \) we simply set \('s \gg t' := \top \). Consider the case \( s = f(s_1, \ldots, s_n) \) and \( t = g(t_1, \ldots, t_n) \). Then \( s \gg t \) if \( f \sim g \) and moreover \( s_i \gg t_{\pi(i)} \) for all \( i = 1, \ldots, n \) and some permutation \( \pi \) on argument positions that takes the separation of normal and safe positions into account. To encode \( \pi(i) = j \), we use fresh atoms \( p_{i,j} \) for \( i, j = 1, \ldots, n \). The propositional formula permutation(\( \pi, n \)) := \bigwedge_{i=1}^{n} \text{one}(\pi_i, 1, \ldots, \pi_i, n) \) is used to assert that the atoms \( p_{i,j} \) reflect a permutation on \( \{1, \ldots, n\} \). Here one(\( \pi_i, 1, \ldots, \pi_i, n \)) expresses that exactly one of its arguments evaluates to \( \top \). We set

\[
's \gg t' := 'f \sim g' \land \text{permutation}(\pi, n) \land \left( \bigwedge_{i=1}^{n} p_{i,j} \rightarrow 's_i \gg t_j' \land (\text{safe}_{f,i} \leftrightarrow \text{safe}_{g,j}) \right).
\]

To complete the definition, we set \('s \gg t' := \bot \) for the remaining cases.

**Lemma 9.2.** Suppose \( \mu \) induces an admissible precedence \( \succeq \) and satisfies \('s \gg t'\). Then \( s \gg t \) with respect to the precedence \( \succeq \). Vice versa, if \( s \nless t \) then \('s \gg t'\) is satisfiable by assignments \( \mu \) that induce the precedence underlying \( \less \).

We now define the encoding for the different cases underlying the definition of \( >_{\text{pop}} \). Assuming that \('s_i >_{\text{pop}} t'\) enforces \( s_i >_{\text{pop}} t \) clause \( >_{\text{pop}} \) is expressible as

\[
'f(s_1, \ldots, s_n) >_{\text{pop}} t' := \bigvee_{i=1}^{n} 's_i >_{\text{pop}} t'
\]

in propositional logic. For clause \( >_{\text{pop}} \), we use propositional atoms \( \alpha_i \) (\( i = 1, \ldots, m \)) to mark the unique argument position of \( t = g(t_1, \ldots, t_m) \) that allows \( t_i \notin T(F^{\text{Fun}}(s), V) \). The propositional formula zero-or-one(\( \alpha_1, \ldots, \alpha_m \)) expresses that zero or one \( \alpha_i \) evaluates to \( \top \). Further, we introduce the auxiliary constraint

\[
'g(t_1, \ldots, t_m) \in T(F^{\text{Fun}}(s), V) := \bigvee_{i=1}^{m} 'f > g' \land \bigwedge_{j=1}^{m} 't_j \in T(F^{\text{Fun}}(s), V)'.
\]
and \(\forall x \in T(F^n, V) \mapsto \top\) for \(x \in V\). Using these, clause \(\gamma_{\text{pop}}\) becomes expressible as

\[
f(s_1, \ldots, s_n) >_{\text{pop}}^\gamma g(t_1, \ldots, t_m) \mapsto \forall f \in D \wedge \forall g > g^\gamma \\
\wedge \bigwedge_{j=1}^m (\text{safe}_{g,j} \rightarrow s >_{\text{pop}} t_j^\gamma) \\
\wedge \bigwedge_{j=1}^m (\neg \text{safe}_{g,j} \rightarrow \neg s >_{\text{pop}} t_j^\gamma) \\
\wedge \text{zero-or-one}(\alpha_1, \ldots, \alpha_m) \wedge \bigwedge_{j=1}^m (\neg \alpha_j \rightarrow \neg t_j \in T(F^n, V) \mapsto \top).
\]

Here \(\forall f \in D = \top\) if \(f \in D\) and otherwise \(\forall f \in D = \bot\). The propositional formula \(\forall s >_{\text{pop}} t^\gamma\) expresses the orientation with the \(>_{\text{pop}}\) and is given by

\[
f(s_1, \ldots, s_n) >_{\text{pop}} t^\gamma := f(s_1, \ldots, s_n) >_{\text{pop}} t^\gamma \lor f(s_1, \ldots, s_n) >_{\text{pop}} t^\gamma
\]

and otherwise \(\forall x >_{\text{pop}} t^\gamma = \bot\), where

\[
f(s_1, \ldots, s_n) >_{\text{pop}} t^\gamma := \bigvee_{i=1}^m (f(s_i) >_{\text{pop}} t^\gamma \lor s_i \not>_{\text{pop}} t^\gamma) \wedge (\forall f \in D \rightarrow \neg \text{safe}_{f,i})
\]

\[
f(s_1, \ldots, s_n) >_{\text{pop}} t^\gamma := \begin{cases} 
(f \in D \land \forall g > g^\gamma) & \text{if } t = g(1, \ldots, t_m) \\
\bigwedge_{j=1}^m f(s_1, \ldots, s_n) >_{\text{pop}} t_j^\gamma & \text{if } t \in V.
\end{cases}
\]

This concludes the propositional formulation of clause \(\gamma_{\text{pop}}\).

The main challenge in formulating clause \(\gamma_{\text{pop}}\) is to deal with the encoding of multiset-comparisons. We proceed as in [41] and encode the underlying multiset cover.

**Definition 9.3.** Let \(\gamma_{\text{mul}}\) denote the multiset extension of a binary relation \(\succeq = \triangleright \sqsubseteq \sim\). Then a pair of mapping \((\gamma, \varepsilon)\) where \(\gamma: \{1, \ldots, m\} \rightarrow \{1, \ldots, n\}\) and \(\varepsilon: \{1, \ldots, n\} \rightarrow \{\top, \bot\}\) is a multiset cover on multisets \(\{a_1, \ldots, a_n\}\) and \(\{b_1, \ldots, b_m\}\) if the following holds for all \(j \in \{1, \ldots, m\}\):

1. if \(\gamma(j) = i\) then \(a_i \succeq b_j\), in this case we say that \(a_i\) covers \(b_j\);
2. if \(\varepsilon(j) = \top\) then \(s_{\tau(j)} \sim t_j\) and \(\tau\) is injective on \(\{j\}\), i.e., \(a_{\tau(j)}\) covers only \(b_j\).

The multiset cover \((\gamma, \varepsilon)\) is said to be **strict** if at least one cover is strict, i.e., \(\varepsilon(j) = \bot\) for some \(j \in \{1, \ldots, m\}\).

It is straightforward to verify that multiset covers characterise the multiset extension of \(\succeq\) in the following sense.

**Lemma 9.4.** We have \(\{a_1, \ldots, a_n\} >_{\text{mul}} \{b_1, \ldots, b_m\}\) if and only if there exists a multiset cover \((\gamma, \varepsilon)\) on \(\{a_1, \ldots, a_n\}\) and \(\{b_1, \ldots, b_m\}\). Moreover, \(\{a_1, \ldots, a_n\} >_{\text{mul}} \{b_1, \ldots, b_m\}\) if and only if the cover is strict.

Consider the orientation \(f(s_1, \ldots, s_n) >_{\text{pop}}^\gamma g(t_1, \ldots, t_m)\). Then normal arguments are strictly, and safe arguments weakly decreasing with respect to the multiset-extension of \(\gamma_{\text{pop}}\). Since the partitioning of normal and safe argument is not fixed, in the encoding of \(\gamma_{\text{pop}}\) we formalise a multiset-comparison on all arguments, where the underlying multiset-cover \((\gamma, \varepsilon)\) will be restricted so that if \(s_i\) covers \(t_j\), i.e., \(\gamma(i) = j\), then both \(s_i\) and \(t_j\) are safe or respectively normal. To this extend, for a specific multiset cover \((\gamma, \varepsilon)\) we introduce
variables $\gamma_{i,j}$ and $\varepsilon_i$, where $\gamma_{i,j} = \top$ represents $\gamma(j) = i$ and $\varepsilon_i = \top$ denotes $\varepsilon(i) = \top$ ($1 \leq i \leq n$, $1 \leq j \leq m$). We set

$$f(s_1, \ldots, s_n) \succ_{\text{pop}*}^g(t_1, \ldots, t_m) := \langle f \in D \rangle \land \langle f \succ g \rangle$$

$$\land \bigwedge_{i=1}^{n} \bigwedge_{j=1}^{m} \left( \gamma_{i,j} \rightarrow \varepsilon_i \rightarrow \langle s_i \succ t_j \rangle \land \neg \varepsilon_i \rightarrow \langle s_i \succ_{\text{pop}*}^g t_j \rangle \land \left( \text{safe}_{f,i} \Leftrightarrow \text{safe}_{g,j} \right) \right)$$

$$\land \bigwedge_{j=1}^{m} \text{one}(\gamma_{1,j}, \ldots, \gamma_{n,j}) \land \bigwedge_{i=1}^{n} \left( \varepsilon_i \rightarrow \text{one}(\gamma_{i,1}, \ldots, \gamma_{i,m}) \right) \land \bigvee \left( \neg \text{safe}_{f,i} \land \neg \varepsilon_i \right).$$

Here the first line establishes the Condition 9.3 (1), where $\text{safe}_{f,i} \leftrightarrow \text{safe}_{g,j}$ additionally enforces the separation of normal from safe arguments. The final line formalises that $\gamma$ maps $\{1, \ldots, m\}$ to $\{1, \ldots, n\}$, Condition 9.3 (2) as well as the strictness condition on normal arguments. This completes the encoding of $\succ_{\text{pop}*}^g$.

**Lemma 9.5.** Suppose $\mu$ induces an admissible precedence $\succeq$ and satisfies $\langle f \succ_{\text{pop}*}^g t \rangle$. Then $s \succ_{\text{pop}*}^g t$ with respect to the precedence $\succeq$. Vice versa, if $s \succ_{\text{pop}*}^g t$ then $\langle f \succ_{\text{pop}*}^g t \rangle$ is satisfiable assignments $\mu$ that induce the precedence underlying $\succ_{\text{pop}*}^g$.

As a predicative recursive TRS $R$ is a constructor TRS compatible with some polynomial path order $\succ_{\text{pop}*}^g$, putting the constraints together we get the following theorem.

**Theorem 9.6.** Let $R$ be a constructor TRS. The propositional formula

$$\text{predicative-recursive}(R) := \text{valid-precedence}(D) \land \bigwedge_{l \rightarrow r \in R} \langle l \succ_{\text{pop}*}^g r \rangle$$

is satisfiable if and only if $R$ is predicative recursive.

We have implemented this reduction to SAT in our complexity analyser TCT. As underlying SAT-solver we employ the open source solver MiniSat [19]. On the example from the introduction, TCT outputs the following result in a fraction of a second.

The input was oriented with 'POP*' as induced by the precedence

member $\succ$ if, member $\succ$ eq, guess $\succ$ choice, consistent $\succ$ if, consistent $\succ$ member, consistent $\succ$ neg, sat $\succ$ guess, sat $\succ$ sat', sat' $\succ$ if, sat' $\succ$ consistent.

Oriented rules in predicative notation are as follows.

sat'(cnf, assign;) -> if(; consistent(assign;), assign, unsat())

sat(cnf;) -> sat'(cnf, guess(cnf;))

consistent(cons(; 1, ls;)) ->
  if(; member(ls; neg(1;)), ff(), consistent(ls;))

consistent(nil;();) -> tt()

guess(nil;();) -> nil()

guess(cons(; c, cs;)) ->
  cons(; choice(c;), guess(cs;))

choice(cons(; a, nil;();)) -> a

choice(cons(; a, cons(; b, bs;));) -> a

choice(cons(; a, cons(; b, bs;));) -> choice(cons(; b, bs;))

neg(1(; x);) -> 0(; x)

neg(0(; x);) -> 1(; x)

eq(1(; y); 1(; x)) -> eq(y; x)
Efficiency Considerations. The SAT-solver MiniSat requires its input in CNF. For a concise translation of \texttt{predicative-recursive}(\mathcal{R}) to CNF we use the approach of Plaisted and Greenbaum\cite{PlaistedGreenbaum} that gives an equisatisfiable CNF linear in size. Our implementation also eliminates redundancies resulting from multiple comparisons of the same pair of term \(s, t\) by replacing subformulas ‘\(s \succ_{\text{pop}} t\)’ with unique propositional atoms \(\delta_{s,t}\). Since ‘\(s \succ_{\text{pop}} t\)’ occurs only in positive contexts, it suffices to add \(\delta_{s,t} \rightarrow \neg s \succ_{\text{pop}} t\), resulting in an equisatisfiable formula. Also during construction of \texttt{predicative-recursive}(\mathcal{R}) our implementation performs immediate simplifications under Boolean laws.

10. Experimental Assessment

In this section we present an empirical evaluation of polynomial path orders. We selected two testbeds: Testbed TC constitutes of 597 terminating constructor TRSs, obtained by restricting the innermost runtime complexity problemset from the termination problem database (TPDB for short), version 8.0 to known to be terminating constructor TRSs. Termination is checked against the full run of the complexity competition from December 2011 Testbed TCO, containing 290 examples, results from restricting Testbed TC to orthogonal systems. Unarguably the TPDB is an imperfect choice as examples were collected primarily to assess the strength of termination provers, but it is at the moment the only extensive source of TRSs. Since the creation of the dedicated complexity categories in 2008 the situation, although slowly, changes to the better.

Experiments were conducted with TCT version 1.9.\footnote{Available from \url{http://cl-informatik.uibk.ac.at/software/tct/projects/tct/archive/}.} on a laptop with 4Gb of RAM and Intel\textsuperscript{\textregistered} Core\textsuperscript{\texttrademark} i7–2620M CPU (2.7GHz, quad-core). We assess the strength of POP\textsuperscript{*} and POP\textsuperscript{*PS} in comparison to its predecessors MPO and LMPO. The implementation of MPO and LMPO follows the line of polynomial path orders as explained in Section\footnote{Full evidence available at \url{http://cl-informatik.uibk.ac.at/software/tct/experiments/popstar}.} We contrast these syntactic techniques to interpretations as implemented in our complexity tool TCT. The last column show result of constructor restricted matrix interpretations \cite{Kurz} (dimension 1 and 3) as well as polynomial interpretations \cite{Kurz} (degree 2 and 3), run in parallel on the quad-core processor. We employ interpretations in their default configuration of TCT, noteworthy coefficients (respectively entries in coefficients) range between 0 and 7, and we also make use of the \textit{usable argument positions} criterion \cite{Usable} that weakens monotonicity constraints. Table\ref{tab:experiments} shows totals on systems that can respectively cannot be handled. To the right of each entry we annotate the average execution time, in seconds.

It is immediate that syntactic techniques cannot compete with the expressive power of interpretations. In Testbed TC there are in fact only three examples compatible with POP\textsuperscript{*PS} where TCT could not find interpretations. There are additionally four examples compatible with LMPO but not so with interpretations, including the TRS \(\mathcal{R}_{\text{bin}}\) from Example\ref{ex:bin}. All but one (noteworthy the merge-sort algorithm from Steinbach and Kühlers collection\cite{Steinbach}, Example 2.43) of these do in fact admit exponential runtime-complexity, thus a priori they are not compatible to the restricted interpretations. We emphasise that parameter substitution significantly increases the strength of
Table 1: Empirical Evaluation, comparing syntactic to semantic techniques.

<table>
<thead>
<tr>
<th>TC</th>
<th>MPO</th>
<th>LMPO</th>
<th>POP*</th>
<th>POP\textsubscript{PS}</th>
<th>interpretations</th>
</tr>
</thead>
<tbody>
<tr>
<td>compatible</td>
<td>76{0.33</td>
<td>57{0.20</td>
<td>43{0.18</td>
<td>56{0.19</td>
<td>139{2.77</td>
</tr>
<tr>
<td>incompatible</td>
<td>521{0.58</td>
<td>540{0.47</td>
<td>554{0.42</td>
<td>541{0.43</td>
<td>272{6.47</td>
</tr>
<tr>
<td>timeout</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>186{25.0</td>
</tr>
</tbody>
</table>

| TCO compatible | 40\{0.29 | 29\{0.16 | 24\{0.14 | 29\{0.15 | 75\{2.81 |
| incompatible   | 250\{0.33 | 261\{0.27 | 266\{0.26 | 261\{0.27 | 133\{6.12 |
| timeout        | —    | —    | —    | —                    | 82\{25.0 |

POP\textsuperscript{*}, 13 examples are provable by POP\textsubscript{PS} but neither by POP\textsuperscript{*} nor LMPO. LMPO could benefit from parameter substitution, we conjecture that the resulting order is still sound for FP.

On Testbed TCO, containing only orthogonal TRSs, in total 75 systems (26% of the testbed) can be verified to encode polytime computable functions, 35 (12% of the testbed) can be verified polytime computable by only syntactic techniques. It should be noted that not all examples appearing in our collection encode polytime computable functions, the total amount of such systems is unknown.

Table 1 clearly illustrates one of our main motivations for investigating syntactic techniques. Our complexity analyser T\textsuperscript{C\textsubscript{F}} recursively decomposes complexity problems using various complexity preserving transformation techniques, discarding those problems that can be handled by basic techniques as contrasted in Table 1. Certificates are only obtained if finally all subproblems can be discarded, above all it is crucial that subproblems can be discarded quickly. POP\textsubscript{PS} succeeds on average 14 times faster than polynomial and matrix interpretations run parallel, it can be safely proposed to interpretations, speeding up the overall procedure. Note that the difficulty of implementing interpretations efficiently is also reflected in the total number of timeouts.

11. Conclusion and Future Work

We propose a new order, the polynomial path order POP\textsuperscript{*}. The order POP\textsuperscript{*} is a syntactical restriction of multiset path orders, with the distinctive feature that the (innermost) runtime complexity of compatible TRSs lies in $O(n^d)$ for some $d$. Based on POP\textsuperscript{*}, we delineate a class of rewrite systems, dubbed systems of predicative recursion, so that the class of functions computed by these systems corresponds to FP, the class of polytime computable functions. We have shown that an extension of POP\textsuperscript{*}, the order POP\textsubscript{PS} that also accounts for parameter substitution, increases the intensionality of POP\textsuperscript{*}. In contrast to interpretations, POP\textsuperscript{*} is partly lacking in intensionality but surpluses in verification time.

In our complexity prover T\textsuperscript{C\textsubscript{F}}, we do not intend to replace semantic techniques, but rather propose them by POP\textsubscript{PS}, in order to improve T\textsuperscript{C\textsubscript{F}} both in analytic power and speed. With T\textsuperscript{C\textsubscript{F}} we are in particular interested in obtaining asymptotically tight bounds. Although we could estimate the degree of the witnessing bounding function for POP\textsuperscript{*} and POP\textsubscript{PS}, a bound extracted from our proof yields unnecessarily an overestimation, compare Theorem 5.1 and particular the preceding construction of the degree $d_{k,p}$. Partly this is due to the underlying multiset extension. Future investigations will certainly include establishing tighter bounds.

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References


