Design and Tuning of Reduced Order H-Infinity Feedforward Compensators for Active Vibration Control
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Abstract—This brief presents a procedure for design and tuning of reduced order $H_{\infty}$ feedforward compensators for active vibration control systems subject to wide band disturbances. The procedure takes account the inherent “positive” feedback coupling between the compensator system and the measurement of the image of the disturbance. It also takes advantage of the availability of reliable models obtained by system identification. A controller order reduction technique is proposed for reducing the complexity of the nominal $H_{\infty}$ controller without degrading the performance. Experimental results obtained on an active vibration control system for a flexible mechanical structure will illustrate the procedure.

Index Terms—External disturbance rejection, feedforward compensator, $H_{\infty}$ control, inertial actuators.

I. INTRODUCTION

REDUCTION of the effect of external disturbances is crucial in many mechanical systems (in particular those which features flexibilities). Passive dynamic vibration absorbers [3], [6], [9] can be used but their performance is limited. Therefore active vibration control (AVC) can be used for enhancing rejection of the disturbances. The following situations can be distinguished in practice in terms of the nature of the disturbances:

• narrow band disturbances;
• wide band disturbances;

and in terms of the structure of the AVC:

• feedback structure with measurement of the residual acceleration (or force) [1], [2], [12];
• feedforward compensation structure using an additional transducer for measuring an image of the disturbance (i.e., a measurement correlated with the disturbance) [17];
• combined feedback-feedforward structures [4], [16].

A survey of these situations for ANC systems is presented in [11].

This brief is dedicated to the design and tuning of feedforward compensators for AVC systems subject to wide band disturbances and equipped with an additional transducer providing a signal highly correlated with the disturbance.1

While for narrow band disturbances adaptive feedback solutions seems to be very efficient [1], [2], [12], for the case of wide band disturbances, the feedforward compensation is mandatory since attenuation in feedback is limited by the Bode integral (“water bed” effect).

It is important to emphasize that in most of AVC systems, there is an internal “positive” feedback coupling between the compensator system and the measurement of the image of the disturbance [7], [8], [10], [19]. This is clearly illustrated in Figs. 1 and 2 representing an AVC system using a measurement correlated with the disturbance (an image of the disturbance) and an inertial actuator for reducing the residual acceleration (see Section II). The presence of this inherent “positive” feedback around the compensator filter transforms the original “feedforward compensation problem” in a “feedback” design.

Since a model based design approach will be used, identification of the various transfer function involved is crucial. It is important to remark, that the problems encountered are similar to those encountered in active noise control (ANC) systems.

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important to remark that in the absence of the feedforward filter it is possible to identify the models of the various paths required for design.

In [17], a $H_2$ approach for the design of the feedforward compensator has been considered. Close related approaches are considered in [4], [16]. Here, the $H_\infty$ approach is considered since it seems more appropriate for this problem. The $H_\infty$ norm of a LTI system is equal to the maximum energy amplification for all signals of finite energy. For this reason $H_\infty$ control design allows us to express the control problem as a mathematical optimization problem around a particular band of frequencies. In our context, the disturbance is considered as a wide band signal amplified at the output of the system around particular band of frequencies. A careful selection of the weighting functions will be done in order to ensure good performances as well as good robustness.

The resulting $H_\infty$ controller has a large dimension and a controller reduction technique [13] based on the identification in closed loop of a reduced controller will be used (after appropriate modifications required by the “positive” feedback structure). Finally an on site tuning procedure of the reduced controller is proposed using real data.

This brief is organized as follows. In Section II, the active vibration control system using an inertial actuator and control problem statement are presented. Section III is dedicated to the parametric identification of the various paths. The design of the proposed feedforward compensator using $H_\infty$ control technique is presented in Section IV. Controller order reduction by identification in closed loop is done in Section V. Section VI presents the results obtained in real-time on the active vibration control system. Some conclusions are given in Section VII.

II. AVC SYSTEM USING AN INERTIAL ACTUATOR

A general view of the whole system equipment is shown in Figs. 1 and 2. This structure is a relevant representation of some configurations encountered in practice for the active vibration control. It consists of three mobile metallic plates M1, M2, and M3 (1.8 kg each) connected by springs. These plates are also connected by springs to the rigid part of the system formed by two other metallic plates connected themselves rigidly. The upper and lower mobile plates (M1 and M3) are equipped with inertial actuators (inertial actuators uses a similar principle as loudspeakers). The first on the top serves as disturbance generator (inertial actuator I in Fig. 1), and the second one at the bottom serves for disturbance compensation (inertial actuator II in Fig. 1). It will create vibrational forces which can counteract the effect of the vibrational disturbances. The system is equipped with a measure of the residual acceleration (on plate M3) and a measure of the image of the disturbance made by an accelerometer posed on plate M1. The unknown disturbance is the position of the mobile part of the upper inertial actuator. The control input is the position of the mobile part of the lower inertial actuator.

A. Basic Equations and Notations

The objective is to compute a feedforward filter $N(q^{-1})$ such that the measured residual acceleration be minimized. The description of the various blocks will be made with respect to Fig. 3.

The optimal feedforward filter is defined by

$$N(q^{-1}) = \frac{B_N(q^{-1})}{A_N(q^{-1})}$$

(1)

where

$$B_N(q^{-1}) = b_{n0} + b_{n1}q^{-1} + \cdots + b_{mnB_N}q^{-mn_{B_N}}$$

(2)

$$A_N(q^{-1}) = 1 + a_{n1}q^{-1} + \cdots + a_{mnA_N}q^{-mn_{A_N}}$$

$$= 1 + q^{-1}A_N^*(q^{-1}).$$

(3)

The transfer operator between the measure of the image of the disturbance $d(t)$ (in the absence of the compensator system) and the residual acceleration $\nu(t)$ called the primary path is given by

$$D(q^{-1}) = q^{-d_D} \frac{B_D(q^{-1})}{A_D(q^{-1})},$$

(4)

The path between the output of the feedforward compensator filter $u(t)$ and the residual acceleration $\nu(t)$ called the secondary path is described by

$$G(q^{-1}) = q^{-d_G} \frac{B_G(q^{-1})}{A_G(q^{-1})},$$

(5)

The positive feedback coupling is characterized by the asymptotically stable transfer operator

$$M(q^{-1}) = q^{-d_M} \frac{B_M(q^{-1})}{A_M(q^{-1})},$$

(6)

This positive coupling occurs when the compensator system is active. The actuator acts upon the residual acceleration, but also upon the measurement of the image of the disturbance. It is a positive feedback because the measured quantity is the sum of the disturbance signal $d(t)$ and of the effect of the actuator.

\(^2\)The complex variable $z^{-1}$ will be used for characterizing the system’s behavior in the frequency domain and the delay operator $q^{-k}$ will be used for describing the system’s behavior in the time domain.
Similar structures occur also in feedforward active noise control [10], [19].

The disturbance $s(t)$ is assumed to be an unmeasurable wide band signal. In a stationary Gaussian context, all the signals $s(t)$, $d(t)$, and $\nu(t)$ can be viewed as being generated by a white noise $e(t)$ passed through a filter. The signal $d(t)$ can be viewed as a signal generated by the white noise $e(t)$ filtered by $W$. This signal is further filtered by $D$ to get the residual acceleration in the absence of the compensator system.

The spectral densities of $d(t)$ and $\nu(t)$ are assumed to be available and a parametric model for the disturbance $d(t)$ will be identified ($W(q^{-1})$).

The input of $G(q^{-1})$, $M(q^{-1})$ being a position and the output a force, these transfer functions have a double differentiator behavior.

### B. Control Problem Statement

The control problem is to design a filter $N(q^{-1})$ such that the impact of disturbance $s(t)$ on the output $\nu(t)$ be attenuated. In addition, the feedforward filter $N(q^{-1})$ has also to guarantee the stability of the internal positive feedback formed by $N(z^{-1})$ and $M(z^{-1})$.

The physical parameters of the system are in general not available. Then, the system has to be considered as a “black box” and the corresponding models ($M$, $G$, $D$) used for control design should be identified, as well as the disturbance model relating the signal $d(t)$. As it will be shown later the different paths in the absence of the disturbances will contain a number of low damped vibration modes as well as low damped complex zeros (anti-resonance). They will be taken into account during the control synthesis.

## III. System Identification

To be able to design a good feedforward control it is necessary to have the mathematical models of the transfer functions $D$, $G$, and $M$. In this brief the models of the different paths of the system will be obtained by parametric system identification techniques as in [5], [13], [15]. In such techniques, time domain data from dedicated experiments are used to determine the parametric models. The system is considered as a “black box”.

The mathematical model $W(z^{-1})$ of the measured disturbance $d(t)$ will be obtained by spectral analysis of the open loop data. It is possible to reconstruct the parametric model of $W(z^{-1})$ from the spectrum of $d(t)$ assuming that $e(t)$ is a white noise signal.

The structure of a linear time invariant discrete time model of the different paths of the plant is as in (4)–(6), given by

\[
X(q^{-1}) = q^{-d_X} \frac{B_X(q^{-1})}{A_X(q^{-1})}
\]

where $d_X$ stands for the pure time delay of system $X$ in number of sampling periods, and

\[
A_X(z^{-1}) = 1 + a_1z^{-1} + \ldots + a_{n_{A_X}}z^{-n_{A_X}}
\]

\[
B_X(z^{-1}) = b_1z^{-1} + \ldots + b_{n_{B_X}}z^{-n_{B_X}} = q^{-d_X}B^*_X(z^{-1})
\]

are polynomials in the complex variable $z^{-1}$ and $n_{A_X}$, $n_{B_X}$ represent their orders. In this application, a sampling frequency of 800 Hz has been used.

The secondary path $G(z^{-1})$ between control signal $u(t)$ and the output $\nu(t)$ has been identified in open-loop operation. The excitation signal was a pseudo random binary sequence (PRBS) generated with a shift register with $N = 10$ and a frequency divider of $p = 4$. The estimated order of the model is $n_{BG} = 15$, $n_{AG} = 13$, $d_G = 0$. The best results in terms of model validation were obtained with Recursive Extended Least Square method (see [14], [15]).

The frequency characteristic of the secondary path $G(z^{-1})$ is shown in Fig. 4 (solid line). There exist several very low damped vibration modes. The first vibration mode is at 46.56 Hz with a damping of 0.013, the second at 83.9 Hz with a damping of 0.011 and the third one at 116 Hz has a damping of 0.014. There are two zeros on the unit circle corresponding to a double differentiator behavior and also a pair of low damped complex zeros at 108 Hz with a damping of 0.021.

The reverse path $M(z^{-1})$ has been identified by the same PRBS excitation ($N = 10$ and a frequency divider of $p = 4$) applied into $u(t)$ and measuring output signal of the primary transducer $y(t)$. The order of the obtained model is $n_{BM} = 15$, $n_{AM} = 13$, $d_M = 0$. The best results in terms of model validation were also obtained with Recursive Extended Least Square method. The frequency characteristic of the reverse path $M(z^{-1})$ is presented in Fig. 4 (dashed line). Similarly, there exist several very low damped vibration modes at 46.20 Hz with a damping of 0.045, at 83.9 Hz with a damping of 0.01 and at 115 Hz with a damping of 0.014 plus some modes in high frequencies. There are two zeros on the unit circle corresponding to the double differentiator behavior.

The primary path $D(z^{-1})$ has been identified from the spectrum of the input/output data $d(t)$ and $\nu(t)$ obtained in open loop.

The obtained frequency characteristic of the parametric model is presented in Fig. 5 (dashed line) together with the frequency characteristic of the non-parametric model (solid line) of $D(z^{-1})$ obtained by spectral density analysis.

Figs. 4 and 5 show that the secondary path has a pair of low damped zeros close to the unit circle at 108 Hz, while the pri-
IV. CONTROLLER SYNTHESIS

To formulate the design of a fixed-gain feedforward controller, an optimal control design that takes into account the positive feedback between the compensator \( N \) and the mechanical coupling \( M \), was considered. The optimal control design approach makes a tradeoff between control energy, broadband disturbance rejection and mechanical coupling. This is achieved by choosing appropriate weighting functions \( W_{\nu} \) and \( W_{\nu'} \), and a good model \( W_n \) of the sensor noise \( n \) (measured on \( y(t) \)).

in the block diagram shown in Fig. 3 producing the new block diagram presented in Fig. 7.

The weighting functions are used to specify the control objectives for the design of the feedforward compensator. The role of these weighting functions can be explained as follows.

- \( W_n \) is used to shape the process input (control) \( u(t) \) to the desired value (to have a low enough input level to avoid saturations for a given magnitude of the disturbance).
- \( W_{\nu'} \) is used to describe the frequency areas where the controller has to minimize the disturbance effect on the output \( \nu(t) \).
- \( W_{\nu} \) is used to take into account the presence of measurement noise \( n(t) \) at the output \( y(t) \). It has been determined from open loop experiments.

Once the models \( D, W, G, M \), are available and the weighting functions \( W_{\nu}, W_{\nu'}, W_n \) are computed, the standard plant description presented in Fig. 8 can be constructed from Fig. 7. The input in Fig. 8 reflects the external disturbances \( e(t) \) and the measurement noise \( n(t) \).

The performance vector \( \nu_c \) is built up from the combination of the filtered output error signal \( \nu_w = W_{\nu'} \nu \) and the filtered signal \( \nu_w = W_{\nu} \nu \). The disturbance effect on this performance vector \( \nu_c \) needs to be minimized. Referring to Fig. 7, the disturbance signal \( e(t) \) passes through \( W(z^{-1}) \) so as to generate the acceleration input \( d(t) \). The signal \( y(t) \) is the measured acceleration that will act as an input to the feedforward compensator and \( u(t) \) is the input to the actuator used for compensation.

Using basic algebraic manipulations it can be shown that the generalized plant \( P \) is given by

\[
\begin{bmatrix}
\nu_w \\
\nu_w \\
y
\end{bmatrix} =
\begin{bmatrix}
W_{\nu'} \cdot W \cdot D & 0 & W_{\nu'} \cdot G \\
0 & 0 & W_{\nu} \\
W & W_n & M
\end{bmatrix}
\begin{bmatrix}
e \\
n \\
u
\end{bmatrix}.
\]

(8)

Once the generalized plant has been formulated, a feedforward compensator can be computed via a standard \( H_{\infty} \) control design [20].

Fig. 5. Frequency characteristics of the primary path \( D(z^{-1}) \) “parametric and non-parametric model”.

Fig. 6. Frequency characteristics of \( W(z^{-1}) \) “disturbance model”.

Fig. 7. Design model structure for the feedforward compensation system.

Fig. 8. Generalized plant with feedforward compensator.
A. Mixed-Sensitivity $H_\infty$ Control Problem

The objective of the feedforward control design is to be able to control the vibrations of the third mass ($M_3$) of the AVC system shown in Figs. 1 and 2. The signal $\nu(t)$ to be minimized in the $H_\infty$ feedforward design procedure is therefore the acceleration of the third mass of the system. According to Fig. 7, let us define the following sensitivity functions to be shaped:

- disturbance-output sensitivity function
  \[ S_{\nu e} = W \cdot \left( D + G \cdot \left( \frac{N}{1 - NM} \right) \right); \]  \hfill (9)

- disturbance-input sensitivity function
  \[ S_{\nu u} = W \cdot \left( \frac{N}{1 - NM} \right); \]  \hfill (10)

- noise-input sensitivity function
  \[ S_{\nu m} = W_n \cdot \left( \frac{N}{1 - NM} \right). \]  \hfill (11)

The $H_\infty$ control problem is to find a stabilizing feedforward compensator $N$ which minimizes $\gamma$ [20] such that

\[
\begin{bmatrix}
  W_{nu} \cdot S_{\nu e} \\
  W_{nu} \cdot S_{\nu u} \\
  W_{nu} \cdot S_{\nu m}
\end{bmatrix}_{\infty} < \gamma.
\]  \hfill (12)

The obtained compensator $N$ will have the same number of state variables as $P$.

The weighting function $W_{nu}$ was chosen to be constant (high value $\geq$15–20). This choice is motivated by the presence of a low frequency measurement noise (from the primary transducer) and a low level of the actuator saturations (around $\pm$0.5 V).

One can also take $W_n$ equal to the stable inverse of $D$ multiplied by a gain, to ensure that the controller should input energy so as to cancel the resonance peaks of $D$ and should not input energy where $D$ has no gain.

$W_{nu}$ was chosen to be equal to the fourth order approximation of $G$, since it is interesting to minimize the error signal $\nu(t)$ where $G$ has enough gain.

$W_n$ was characterized as a fourth order low pass filter according to some open loop experiments (spectral analysis of measurement noise on $y(t)$).

The resulting controller has following orders: $n_{B_N} = 70$; $n_{A_N} = 70$.

The obtained sensitivity functions are close to the desired ones. See Figs. 9–11.

From Figs. 9 and 10, it results that $|S_{ue}|$ and $|S_{um}|$ are above $|1/W_{nu}|$ which is very small (for performance reasons), but it is very important to see that both of them are less than 0 dB. Therefore, the feedforward compensator attenuates disturbances and noise on the secondary path input.

From Fig. 11, one concludes that $N$ gives very good results for the disturbance rejection over the output $\nu(t)$ around frequency regions where the secondary path has enough gain.

B. Alternative Formulation

Instead of minimizing the effect of $s(t)$ (more exactly of $e(t)$ since $s(t)$ is the result of $e(t)$ passed through an unknown filter) on $\nu(t)$, it is possible to consider the minimization of the effect of signal $d(t)$ upon the output $\nu(t)$.

According to this problem, the sensitivity functions to be shaped become as follows:

- disturbance-output sensitivity function
  \[ S_{\nu d} = D + G \cdot \left( \frac{N}{1 - NM} \right); \]  \hfill (13)

- disturbance-input sensitivity function

Fig. 9. Disturbance-input sensitivity function $S_{ue}$ with $H_\infty$ feedforward compensator.

Fig. 10. Noise-input sensitivity function $S_{um}$ with $H_\infty$ feedforward compensator.

Fig. 11. Disturbance-output sensitivity function $S_{ue}$ with $H_\infty$ feedforward compensator.
In this case $W$ will be included in the weighting functions. The new $H_\infty$ problem to solve, is finding a feedforward compensator $\hat{N}$ which minimizes $\gamma$ such that

$$
\| (W \cdot W_P \cdot S_{dh}) (W \cdot W_u \cdot S_{ad} ) \|_{\infty} < \gamma.
$$

(15)

Mathematically the minimization problem is the same and the feedforward compensator obtained is exactly the same as for the first problem.

V. FEEDFORWARD CONTROLLER ORDER REDUCTION BY IDENTIFICATION IN CLOSED LOOP

As it was shown in [13] and [15], controller order reduction can be cast in a closed-loop identification problem. The closed loop formed by $N$ and $M$ will be considered.

The objective is the minimization of the error signal between the nominal closed loop using the $H_\infty$ controller synthesized in Section IV and the loop using the reduced order controller. To do this one can use simulated and/or real data. The algorithm has been modified in order to take into account that in this context, there is a positive feedback loop.

Let consider the diagram of Fig. 12. For the “input matching” objective, one has to minimize the following norm:

$$
\| S_{wr} - \hat{S}_{wr} \| = \left\| \frac{N}{1 - NM} - \frac{\hat{N}}{1 - NM} \right\|
$$

(16)

where $\hat{S}_{wr}$ is the sensitivity function between $r$ and $u$.

Therefore the optimal reduced order controller will be given by

$$
\hat{N} = \arg \min_N \| S_{wr} - \hat{S}_{wr} \| = \arg \min_N \| S_{wr}(N - \hat{N})\hat{S}_{gr} \|
$$

(17)

with

$$
S_{gr} = \frac{1}{1 - NM}.
$$

(18)

Equation (17) shows that a weighted norm of $N - \hat{N}$ will be minimized.

A. Algorithm for Direct Closed-Loop Identification of Reduced-Order Feedforward Controller

Let us consider the block diagram of Fig. 12.

The parametric adaptation algorithm which will be used to identify the parameters of a reduced order controller is presented in [13] as “closed-loop input matching” algorithm and it is very similar to the “closed-loop output error” algorithm used for plant model identification in closed loop [14].

Consider the upper part of Fig. 12. The simulated reverse path is operated in closed loop with the nominal $H_\infty$ controller obtained in Section IV.

$u(t)$ is the output of $N$ as well as the reverse path input, $y(t)$ is the input to $N$, and $r(t)$ is the external excitation signal.

The output of the reduced controller (feedforward compensator) is given by

$$
\dot{\hat{u}}(t+1) = -\hat{A}_N^* (q^{-1}) \dot{\hat{u}}(t) + B_N(q^{-1}) \dot{\hat{y}}(t+1) = \hat{\theta}^T \psi(t)
$$

(19)

where

$$
\dot{\hat{y}}(t+1) = r(t+1) + \hat{c}(t+1)
$$

(20)

$$
\hat{c}(t+1) = -\hat{A}_M \hat{c}(t) + \hat{B}_M \hat{u}(t-d)
$$

(21)

$$
\psi(t) = [-\hat{u}(t), \cdots, -\dot{\hat{u}}(t-n_{AS} + 1), \dot{\hat{y}}(t+1), \cdots,\dot{\hat{y}}(t-n_{BS}+1)]
$$

(22)

$$
\hat{\theta}^T = [\hat{\alpha}_n, \cdots, \hat{\alpha}_{mnA_N}, \hat{\beta}_n, \cdots, \hat{\beta}_{mnB_N}]
$$

(23)

To estimate the reduced-order filters, in this brief, only the “CLIM” algorithm presented in [13] is used.

B. Reduced-Order Feedforward Controller for the Active Vibration Control

The complexity of the $H_\infty$ feedforward filter obtained in Section IV is $n_{BS} = 70$; $n_{AS} = 70$.

To reduce the controller order, data obtained in simulation (see Fig. 12) and/or in real time are used. The real data are obtained by replacing the simulated loop $(\hat{M}, N)$ of Fig. 12 by the real system $(M, N)$.

In order to estimate a reduced order controller using real data, it is assumed that the system is not subject to disturbance ($s(t) \equiv 0$), and a PRBS excitation signal is applied at the input of the feedforward compensator. $r(t)$ and $u(t)$ are used in the “CLIM” algorithm.

The excitation signal was a PRBS generated with a shift register with $N = 10$ and a frequency divider of $p = 2$.

To reduce the controller order in simulation, the reverse path model $\hat{M}$ obtained in closed-loop identification by using the $H_\infty$ feedforward filter obtained in Section IV, and the same PRBS.

Good results were obtained with reduced feedforward filters of order $n_{BS} = 19$, $n_{AS} = 20$, $d_N = 0$ either using real or simulated input/output data.

In order to compare the nominal $H_\infty$ feedforward compensator with the reduced controllers, Figs. 13–15 present the new
sensitivity functions obtained with the reduced order controllers and the nominal one.

Figs. 13–15 show that sensitivity functions obtained with the nominal controller and the reduced order controllers are very close.

Fig. 16, shows the frequency characteristics of the obtained feedforward compensators using $H_\infty$ control techniques and the reduced ones.

VI. EXPERIMENTAL RESULTS

The performance of the system for rejecting broadband disturbances will be illustrated using the feedforward compensator scheme with the various designed controllers.

A PRBS excitation is applied to the inertial actuator as a disturbance (i.e., the mechanical structure is disturbed by an almost white noise filtered by the transfer function of the inertial actuator).

Time domain results obtained with and without feedforward compensator scheme are shown in Figs. 17 and 18. Almost the same level of the residual acceleration is obtained using the nominal or the reduced controllers. The variance of the residual acceleration in open loop is: $\variancesq(\nu(t)) = 0.0328$. With feedforward compensation, the variance is around: $\variancesq(\nu(t)) = 0.0079$ in the three cases (nominal and reduced controllers). This corresponds to a global attenuation of 12.37 dB.

Fig. 19 shows the spectral densities of the residual acceleration obtained in open loop and with feedforward compensation. One can remark a strong attenuation in the frequency regions where the secondary path $G$ has a high gain.

These results are summarized in Table I, where $N_{hi}$ is the nominal feedforward compensator, $N_f$ and $N_a$ are the reduced
count the mechanical coupling (positive feedback) between the feedforward compensation and the measurement of the image of the external disturbance. The feedforward compensator design is formulated as an $H_{\infty}$ optimal control design.

Since the nominal designed compensator has a large dimension, a procedure for compensator order reduction and on site tuning has been proposed and applied successfully. The proposed approach is illustrated by an experimental study on an AVC system attached to a flexible structure.

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